LINEAR ALGEBRAIC TECHNIQUES FOR THE ANALYSIS OF PETRI NETS

M. Silva, J.M. Colom, J. Campos

ABSTRACT

One of the indigenous techniques for the analysis of Petri Net system models is based on its non-negative state equation, bridging convex goemetry and linear programming theories to the theory of Petri Nets. This invited survey briefly overviews some recent developments in the use of linear algebraic techniques for the qualitative (i.e., logical) and quantitative (i.e., performance) analysis of Petri Net system models, dealing with properties like deadlock-freeness, structural liveness or throughput bounds.

Key Words: Convex geometry, linear algebra, linear programming, logical properties, performance properties, Petri nets, state equation.

1. MOTIVATION

Structural analysis of Petri Nets focuses on the relationship between the net structure and its behaviour. Net structure can be studied using *graph theory* arguments or through *linear algebra* based arguments, using the *incidence matrices* ($Pre = C^-$, $Post = C^+$, $C = C^+ - C^-$).

The behaviour of a net model is non-linear, nevertheless there exists a nice *linear relaxation* that allows interesting analytical studies. Let $\langle N, M_0 \rangle$ be a net system, and σ a firable sequence of transitions from M_0. The (integer) linear relaxation looks as follows:

$$M_0[\sigma\rangle M \Rightarrow M = M_0 + C \cdot \vec{\sigma} \geq 0, \;\; \vec{\sigma} \geq 0 \qquad (1)$$

where M is reachable from M_0 firing σ, $\vec{\sigma}$ is the Parikh (or firing count) vector of σ and C the incidence matrix of the net, N. The system (1) is known as the *state* or *fundamental equation* of the net system model. Two classical papers on the state equation based analysis of net models are [1] and [2].

Unfortunately the reverse of the above implication is not true. More precisely, the state equation has integer solutions, $(M, \overline{\sigma})$, not reachable on the net system. We call them *spurious* solutions. The existence of spurious solutions leads usually to *necessary* or *sufficient* conditions to study classical behavioural properties as boundedness, deadlock freeness, liveness or fairness. The following are two spurious

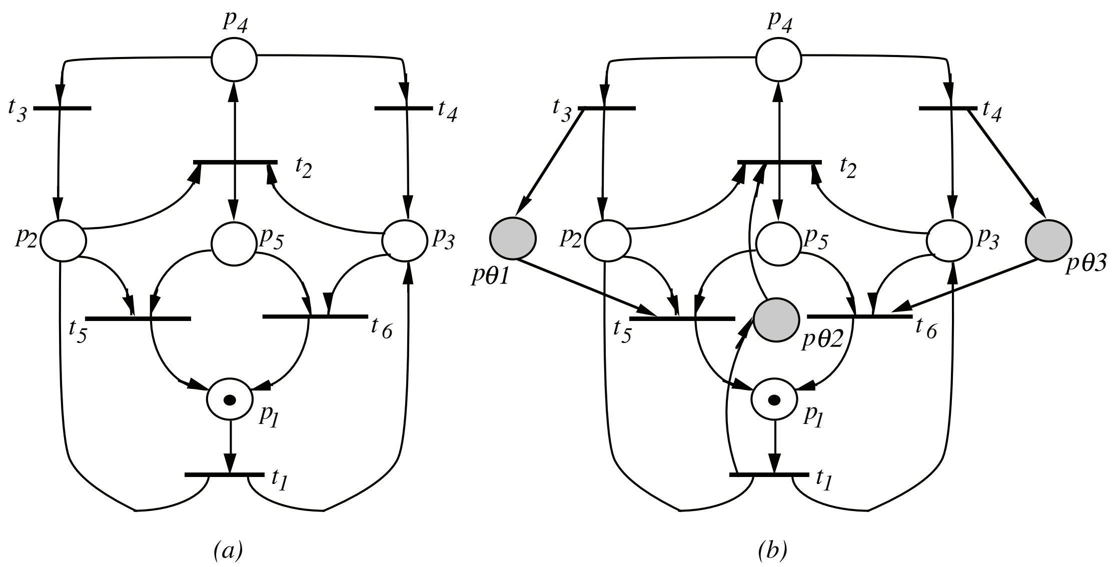

Fig. 1. Two 1-bounded and live Petri Nets.

solutions for the state equation of the net model in Fig. 1(a). The first, $\tilde{M}_1 = (00020)^T$, allows to say that p_4 is 2-bounded, while it is really 1-bounded (check). The second spurious solution, $\tilde{M}_2 = (02000)^T$, leads to a deadlock. Then using the state equation we cannot conclude that the net system in Fig. 1(a) is deadlock-free.

Spurious solutions can be removed using different approaches [3]. A place is said to be *implicit* if its deletion does not increase the firing possibilities. Adding implicit places, a new net system model with equivalent behaviour is obtained. If the implicit places are chosen carefully, the state equation of the new net system may have no integer spurious solution preventing to conclude on the bound of a place or the deadlock freeness of the system. The net system in Fig. 1(b) has been obtained adding places $p_{\theta 1}$, $p_{\theta 2}$ and $p_{\theta 3}$ to that in Fig. 1(a). The above mentionned spurious solutions, $\tilde{M}_1$ and $\tilde{M}_2$, are not solutions of the new state equation. Moreover, we can conclude now that the new (and original) net systems were 1-bounded for p_4 and deadlock-free.

Classical reasoning to prove logical properties uses *invariants* on the behaviour of a system. The right and left non negative annullers of the incidence matrix lead to two kinds of structural invariants:

i) $Y \geq 0,\ Y^T \cdot C = 0 \Rightarrow YT \cdot M = Y^T \cdot M_0$ (token conservation law)

ii) $X \geq 0,\ C \cdot X = 0 \Rightarrow X = \vec{\sigma}$ and $M[\sigma\rangle M' = M$ (cyclic behaviour)

The computation of minimal p-invariants (Y) and minimal t-invariants (X) has been extensively studied. In [4] the study is done merging traditional techniques in Convex Geometry with those developed within Petri Net field. From a conceptual point of view the consideration of invariants provides a decomposed view of the structure of the net model (see Fig. 2(a)). The decomposed view of a net is even useful to derive an *implementation*: for example, the net in Fig. 2(b) can be implemented using two sequential processes and three semaphores (S_R implements a mutual exclusion).

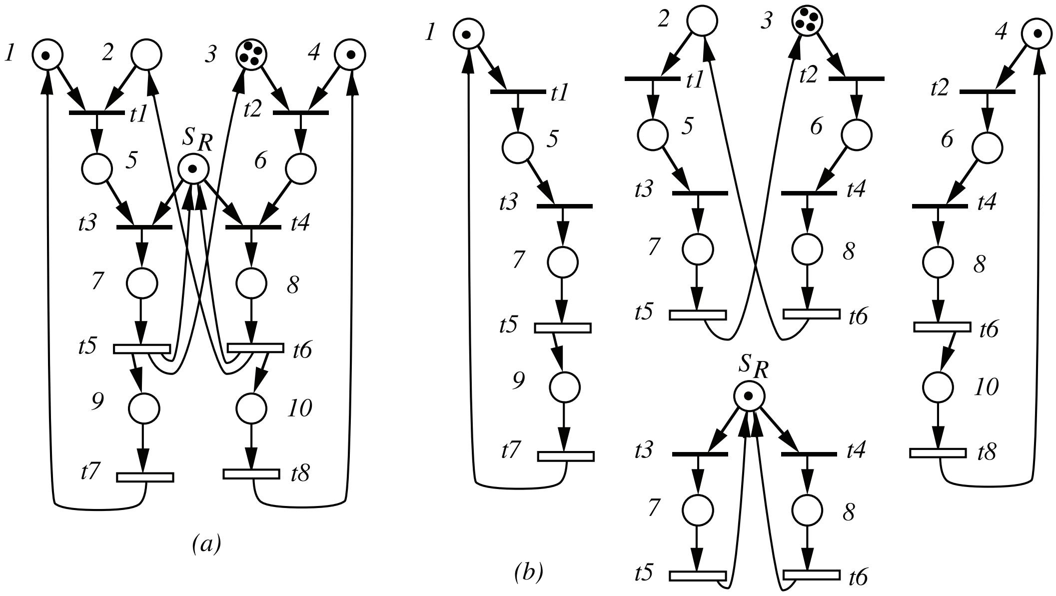

Fig. 2. A producer-consumer Petri net and its p-invariant decomposition.

The paper is structured as follows: Section 2 presents some basic ideas about the analysis of *safety* (in temporal logic sense) properties, while section 3 is devoted to *liveness* (in temporal logic sense) properties. Section 4 considers linear programming based computation of *throughput bounds*. Finally section 5 briefly highlights some of the advantages and limitations of the general approach.

2. SAFETY PROPERTIES

Safety properties express that "some bad thing" never happens in the system (e.g., synchronic properties [5]). In a structural analysis based on linear algebra, many properties can be characterized by means of a linear assertion in which only one kind of quantifier appears (i.e., $\forall$ or $\exists$). We illustrate the analysis method using one of the simplest properties. Let us consider the *k-boundedness* property of a place p: $\forall M$ reachable from M_0, $M[p] \leq k$. In order to verify this property we solve one of the dual linear programming problems (LPPs) given below (where the reachability condition has been relaxed by the net state equation, and e_p is an $n \times 1$ vector with $e_p[p_i] = 0$ for all $p_i \neq p$ and $e_p[p] = 1$).

$$\max\{e_p^T \cdot M \mid M = M_0 + C \cdot \vec{\sigma}; M \geq 0; \vec{\sigma} \geq 0\} \quad (2)$$

$$\min\{Y^T \cdot M_0 \mid Y^T \cdot C \leq 0; Y \geq e_p\} \quad (3)$$

The property is satisfied if any of the above LPPs is bounded and its optimal value is less than or equal to k (observe that the analysis gives a sufficient condition but, in general, non-necessary). The first condition on the boundedness of the LPPs holds if the LPP (3) is feasible: $\exists Y \geq e_p, Y^T \cdot C \leq 0$ (this is because LPP (2) always has a feasible solution: $M = M_0, \vec{\sigma} = 0$; and by the duality theorem of Linear Programming, LPP (3) is bounded if LPP (2) is bounded —and both have the same optimal value— or is non feasible if LPP (2) is unbounded). This algebraic condition is known as the structural boundedness property of place p, because the structure guarantees its boundedness for all M_0 (i.e., premultiplying the net state equation by the above vector Y we obtain a relation: $\forall M_0, Y^T \cdot M \leq Y^T \cdot M_0, Y \geq e_p$, expressing the boundedness of p for any M_0).

From the above discussion we conclude that the LPP approach for the boundedness property works in the case of structurally bounded places (sufficient condition). Nevertheless, we cannot conclude on the k-boundedness property if the optimal value of the LPPs is greater than k, because this value can be due to a spurious solution (remember the case of place $p4$ of Fig. 1).

The deadlock freeness property is another safety property that concerns the existence of some activity from any state. A deadlock in a net characterizes the existence of a marking from which none transition is firable. To analyze this property by means of linear algebra we must express the condition "transition t is not firable at marking M" in a linear form. This can be done very efficiently in nets where for all places the bound computed through the LPP (2) is equal to the weight of the output arcs of the place [8]. The linear condition is: $\sum_{p_i \in \bullet t} M[p_i] < \sum_{p \in P} C^-[p,t]$ (i.e., the amount of tokens in the input places of t is less than the needed). Therefore, a sufficient condition (because a spurious solution can be a deadlock, remember the case of Fig. 1) for a net to be deadlock free is that the following linear system has not solution (where C^- is the preincidence matrix and $\mathbb{1}$ is a vector with all entries equal to 1):

$$M = M_0 + C \cdot \vec{\sigma}; M \geq 0; \vec{\sigma} \geq 0; M^T \cdot C^- < \mathbb{1}^T \cdot C^- \qquad (4)$$

Under the name of synchronic properties, in [5] additional safety properties are introduced. The structural counterparts are presented in [6], while in [7] B-Fairness is studied with some detail.

3. LIVENESS PROPERTIES

Liveness properties express that "some good thing" eventually will happen in the system (e.g., transition liveness, home state, reversibility). Many properties can be characterized by means of an assertion in which there exist the two kinds of quantifiers ($\forall$ and $\exists$) affecting two different variables ranging over different domains. The direct application of a linearly based structural analysis fails, in the general case, when these properties are considered. This is because the linear relaxation of two domains referred to variables with different quantifiers (i.e., the obtained condition is neither necessary nor sufficient). The general approach to linearly verify these properties is based on the use of safety properties as necessary conditions for liveness properties together with an inductive proof method [8], [9].

For example, a classical necessary condition for a net to be structurally live (i.e., there exists an M_0 that makes live the net) and structurally bounded is its conservativeness (i.e., $\exists Y > 0,\ Y^T \cdot C = 0$) and consistency (i.e., $\exists X > 0,\ C \cdot X = 0$) of the net. This condition is improved in [8] by adding another constraint based on the rank of the incidence matrix of the net: rank(C) $\leq m - \delta - 1$; where m is the number of transitions and δ is the number of nonredundant conflicts of the net. If we apply this condition to the two conservative, consistent and structurally non live nets in Fig. 3 we obtain: (a) the net of Fig. 3(a) does not satisfy the rank condition ($4 = rank(C) > m - \delta - 1 = 5 - 1 - 1 = 3$) and we conclude that is not structurally live; (b) the net of Fig. 3(b) satisfies the rank condition ($4 = rank(C) \leq m - \delta - 1 = 7 - 2 - 1 = 4$) and we cannot conclude that is not structurally live (i.e., the rank property remains a necessary condition).

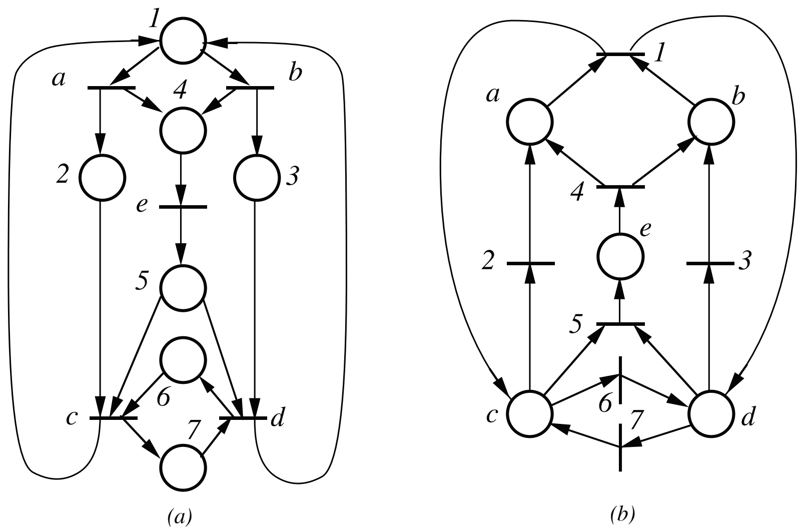

Fig. 3. Two consistent, conservative and structurally non live nets.

In order to illustrate inductive methods to prove liveness properties we analyze the reversibility (from any reachable marking M we can reach the initial one) of the net depicted in Fig. 2 (the method is a Liapunov-stability-like technique used in [9]).

Let W be a non-negative place weighting such that $W(i) = 0$ iff p_i is initially marked. The function $V(M) = W^T \cdot M$ has the properties: $V(M) \geq 0$ and $V(M_0) = 0$. For the net in Fig. 2(a) a stronger property holds: $V(M) = 0 \Leftrightarrow M = M_0$. This can be clearly seen because $W^T \cdot M = 0 \Leftrightarrow m_2 = m_5 = m_6 = m_7 = m_8 = m_9 = m_{10} = 0$ and the minimal p-invariants reduce to: $m_1 = 1$, $m_4 = 1$, $m_2 = 1$, $m_{SR} = 1$. Even more, it is easy to check the following: $M_0 \Leftrightarrow t_2$ is the unique firable transition. If there exists (warning: in Liapunov stability the universal quantifier is used!) a finite firing sequence (i.e., a finite trajectory per reachable marking such that $M_i[\sigma_k\rangle M_{i+1}$ and $V(M_i) > V(M_{i+1})$, in a finite number of transition firings $V(M) = 0$ is reached. If $V(M) = 0 \Leftrightarrow M = M_0$, a proof that M_0 is reachable from any marking has been obtained (i.e., the net model is reversible).

Premultiplying the state equation by W^T we obtain the following condition: If $\sigma_k = t_j$, $W^T \cdot M_{i+1} < W^T \cdot M_i \Leftrightarrow W^T \cdot C[t_j] < 0$. Now, removing in Fig. 2(a) the initially marked places (i.e. p_1, p_2, p_3, p_4 and S_R) and firable transitions (i.e., t_2) an acyclic net is obtained, so $W^T \cdot C[t_j] < 0$ exists $\forall j \neq 2$. For example, $W = (0400332211)^T$ leads to: $W^T \cdot C = (-1 + 3 - 1 - 1 - 1 - 1)$. Then reversibility is proven:

If $m_7 + m_8 + m_9 + m_{10} \geq 1$
  then $V(M)$ can decrease (firing t_5, t_6, t_7 or t_8)
  else **If** $m_5 + m_6 \geq 1$
    then $V(M)$ can decrease (firing t_3 or t_4)
    else $V(M)$ can decrease (firing t_1) **OR** t_2 is the unique firable transition ($\Leftrightarrow M_0$)

In [10] additional inductive methods to prove liveness and home-state properties are presented.

4. QUANTITATIVE ANALYSIS: PERFORMANCE BOUNDS

Timed and stochastic Petri nets constitute an adequate model for the evaluation of performance measures of concurrent and distributed systems. We consider nets with *timed transitions* having *general service time distributions* and *infinite server semantics* [11]. *Immediate* transitions (firing in zero time) are used for modelling decisions, by associating *routing rates* to them. For these nets, *Little's law* can be applied to each place under *weak ergodicity* assumption [11]. Bounding the average time spent by a token within the place by the average service time of the output transition (thus non considering the waiting time due to synchronizations), the following inequality can be derived [11]: $\Gamma^{(j)}\overline{M} \geq C^{-} \cdot \vec{D}^{(j)}$, where $\Gamma^{(j)} = 1/\sigma^{*}(t_j)$ is the *mean interfiring time* (inverse of the throughput) of transition t_j, $\overline{M}$ is the limit vector of average number of tokens at places, and $\vec{D}^{(j)}$ is the vector of average service demands for transitions (products of average service times and visit ratios), that can be computed at structural level for important net subclasses [12]. Premultiplying the above inequality by a P-semiflow, and after some algebraic considerations, it can be shown that for any net system and given $\vec{D}^{(j)}$, a lower bound for the mean interfiring time $\Gamma^{(j)}$ of transition t_j (or its inverse an upper bound for the throughput) can be computed by solving the following linear programming problem:

$$\Gamma^{(j)} \geq \max\{Y^T \cdot C^{-} \cdot \vec{D}^{(j)} \mid Y^T \cdot C = 0;\ Y^T \cdot M_0 = 1;\ Y \geq 0\} \tag{5}$$

The previous bound is based on the computation of the mean interfiring time of transitions of *subnets generated by P-semiflows considered in isolation* (i.e., with infinite servers at each transition). It has been shown to be reachable for any net topology and many service pdfs for the class of marked graphs. As an example, for the net in Fig. 2(a), if t_1, t_2, t_3, and t_4 are immediate, the application of the above theorem gives the bound: $\Gamma \geq \max\{s_5 + s_6, (s_5 + s_6)/4, s_5 + s_7, s_6 + s_8\}$, for the mean interfiring time of every transition (the net has only one T-semiflow thus all transitions have equal throughput).

We remark that the above bound is valid for arbitrary distributions of service times of transitions and that only the mean values of services times are used. An improvement of the bound is possible if we restrict ourselves to the case of *exponentially distributed* service times (*Markovian nets*). This improvement is based on the computation of the mean interfiring time of transitions of subnets generated by P-semiflows but in a "partial isolation", i.e., considering the maximum reentrance in steady-state, or *liveness bound* [11], of their transitions, allowed by the rest of the net. In this way, some queueing effects due to synchronizing transitions are taken into account [13]. An upper bound for the liveness bound can be efficiently computed at structural level solving the following linear programming problem:

$$SE(t) \stackrel{\text{def}}{=} \max\{k \mid M_0 + C \cdot \vec{\sigma} \geq kC^{-}[t];\ \vec{\sigma} \geq 0\} \tag{6}$$

In fact, this *structural enabling bound* of transition t has been shown to collapse with the liveness bound for important net subclasses [12]. Special interest has the case in which the subnets generated by P-semiflows (with the number of servers at each transition limited to its liveness bound) have *state machine* topology because, if exponential services are assumed, they can be seen as *product-form queueing networks* [13], and efficient algorithms for the computation of exact values or bounds of the throughput exist.

The bounds presented here are based on P-semiflows of the underlying net. They can be improved by adding some implicit places which increment the number of minimal P-semiflows of the net [14] or considering some *multisets of circuits* that capture more information about the behaviour of the timed model [15].

An absolute (i.e., for any pdf of the firing delays) upper bound for the mean interfiring time for live and bounded free choice nets is given by the following expression [12]: $\Gamma^{(j)} \leq \sum_{i=1}^{m} D^{(j)}(t_i)/SE(t_i)$.

5. CONCLUSIONS

The basic idea to be pointed out is that the approach allows the study of many important *qualitative* and *quantitative properties* using well known *polynomial complexity* algorithms (rank computation, linear programming solution, etc). The price payed is that in general *only* necessary or sufficient conditions are obtained.

Techniques for improving the linear study of net models exists [3, 14]. Nevertheless, there exist no full linear characterization of properties, but for some net subclasses. This happens, for example, for Free choice nets [16]. In this case even complete *top-down* and *botton up synthesis* theories can be derived from the linear perspective.

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