Properties and Performance Bounds for Closed Free Choice Synchronized Monoclass Queueing Networks

Javier Campos*, Giovanni Chiola†, and Manuel Silva*

Abstract

Several proposals exist for the introduction of synchronization constraints into Queueing Networks (QN). We show that many monoclass QN with synchronizations can naturally be modelled with a subclass of Petri Nets (PN) called Free Choice nets (FC), for which a wide gamut of qualitative behavioural and structural results have been derived. We use some of these net theoretic results to characterize the ergodicity, boundedness and liveness of closed Free Choice Synchronized Queueing Networks (FCSQN). Moreover we define upper and lower throughput bounds based on the mean value of the service times, without any assumption on the probability distributions (thus including both the deterministic and the stochastic cases). We show that monotonicity properties exist between the throughput bounds and the parameters of the model in terms of population and service times. We propose (theoretically polynomial and practically linear complexity) algorithms for the computation of these bounds, based on linear programming problems defined on the incidence matrix of the underlying FC net. Finally, using classical laws from queueing theory, we provide bounds for mean queue lengths and response time.

1 Introduction

Product Form Queueing Networks (PFQN) [1] have long been used for the performance evaluation of computer systems. Their success has been due to their capability of naturally expressing sharing of resources and queueing, that are typical situations of traditional computer systems, as well as to their efficient solution algorithms, of polynomial complexity on the size of the model. Unfortunately, the introduction of synchronization constraints usually destroys the product form solution, so that general concurrent and distributed systems are not easily studied with this class of models.

Timed and stochastic Petri nets constitute an adequate model for the evaluation of performance measures of concurrent and distributed systems (see, e.g., [2, 3, 4]). Nevertheless, one of the main problems in the actual use of these models for the evaluation of large systems is the explosion of the computational complexity of the analysis algorithms. Structural computation (i.e., based on the net structure and not on its state space) of exact performance measures is only possible for some subclasses of nets, such as Jackson networks [5] and totally open systems of sequential processes [6]. In the general case, efficient computation methods for the performance measures are still needed.

From the Petri net perspective, the computation of (upper and lower) bounds for the steady-state performance of timed and stochastic free choice nets is considered in this paper. In particular, we study the *throughput of transitions*, defined as the average number of firings per unit time. For this measure we compute upper and lower bounds in polynomial time on the size of the net model (number of nodes). The model is completely specified by the Petri net structure together with its initial marking, the firing rule, the average transition firing times, and the conflicts resolution policy [7]. In the case of free choice nets, the conflict resolution policy can be completely defined at the structural level, using a *preselection* policy.

*Departamento de Ingeniería Eléctrica e Informática, Universidad de Zaragoza, María de Luna 3, E-50015 Zaragoza, Spain. This work was supported by project PA86-0028 of the Spanish Comisión Interministerial de Ciencia y Tecnología and ESPRIT-BRA 3148 DEMON.

†Dipartimento di Informatica, Università di Torino, corso Svizzera 185, I-10149 Torino, Italy. Part of this work was performed while G. Chiola was visiting the University of Zaragoza with the financial support of the Caja de Ahorros de la Inmaculada de Zaragoza under project Programa Europa of CAI-DGA.

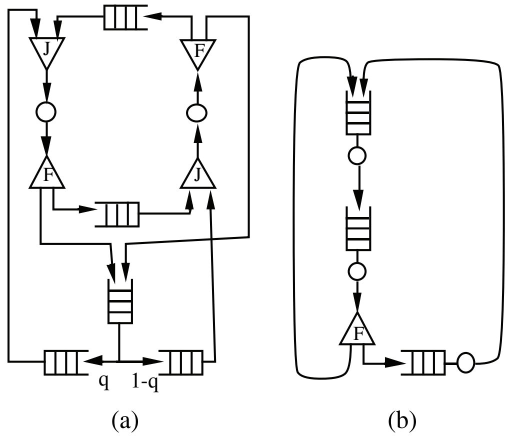

Figure 1: Pathological cases of free choice synchronized queueing networks. (a) A deadlock will be reached sooner or later, even for $q = 1/2$. (b) Any infinite behaviour will lead to an infinite number of customers in the QN.

Under this assumption, the computation of bounds is independent of the type of probability distribution functions associated with transitions; only their mean values are relevant.

The particular case of strongly connected Marked Graphs (MGs) has been studied in [8]. The bounds obtained for this subclass of Petri nets are computable in polynomial time on the size of the net model. Moreover, both upper and lower bounds are tight, in the sense that for any MG model it is possible to define families of stochastic timings such that the steady-state performances of the timed Petri net models are arbitrarily close to either bound.

An extension of strongly connected MGs is studied in [9], where *mono-T-semiflow nets* are introduced. A characteristic of these nets is the existence of a unique consistent firing count vector. They are either decision-free or such that the decision policy at effective conflicts is not relevant for our computation of performance bounds (mono-T-semiflow nets allow concurrency and decision, in a particular way). Both the upper and lower bounds are independent of any assumption on the probability distribution of the delay associated with transitions, and their values can be computed based on the knowledge of the averages.

Free Choice nets (FC nets, for short) [10] are a well-known subclass of Petri nets that constitute an alternative interplay between concurrency and decisions. They are rich enough to be non-trivial but restricted enough to allow a number of interesting results that do not hold in general and that constitute a quite elegant theory (see, e.g., [10, 11, 12]).

The results presented in this paper are an extension to Live and Bounded FC nets (LBFC nets) of those in [8] and [9]. The idea is that several consistent firing count vectors can be reproduced in steady-state, but the decisions, freely done at certain places, are completely governed by the stochastic interpretation of the net. Therefore, the steady-state "average firing count vector" can be defined independently of the marking.

From a different perspective the obtained results can be applied to the analysis of queueing networks extended with some synchronization schemes [13]. Bounds for the performance measures of a particular case of such models (essentially *fork-join* queues) have been studied in [14] using stochastic ordering theory and recursive equations. We propose an alternative approach based on structural analysis of stochastic Petri nets and basic queueing laws. Many monoclass queueing networks can be mapped on stochastic FC Petri nets. On the other hand, FC nets can be interpreted as monoclass queueing networks augmented with some form of synchronization primitives [15] (preserving the free choice decision scheme). In this paper we consider strongly connected (i.e., "closed") FC synchronized QNs (closed FCSQN). The reader may notice that "unclever" use of synchronizations in the free choice synchronized queueing networks can lead to pathological cases as unbounded number of customers or complete stop (*deadlock*) of activity (see Figure 1), that need to be carefully studied.

The paper is organized as follows. Section 2 presents the connection between (synchronized) queueing networks and free choice stochastic Petri nets. In Section 3 various behavioural and structural properties of FC nets are considered. Ergodicity of FCSQN is characterized in Section 4. In Sections 5 and 6 the upper and lower bounds of transition throughputs are defined for LBFC nets (closed FCSQN) in terms of Linear Programming Problems (LPPs) set up on the incidence matrix of the net. In Section 7 upper and lower bounds for other performance indexes are derived using classical laws of queueing theory. Section 8 contains some concluding remarks and considerations on possible extensions of the work.

2 Queueing networks and stochastic Petri nets

2.1 Extended queueing networks

Many extensions have been proposed to introduce synchronization primitives into the QN formalism, in order to allow the modelling of distributed asynchronous systems: passive resources, fork and join, customer splitting, etc. Some very restricted forms of synchronization, such as some special use of passive resources [16, 17], preserve the local balance property [1] that allows efficient algorithms to be used for the computation of exact product form solution. In general, however, these extensions destroy the local balance property so that extended queueing models with synchronization are used mainly as system descriptions for simulation experiments [13]. Even the computation of bounds for these classes of models is not yet well developed.

In [15] a comparison has been proposed between synchronized QNs and stochastic Petri nets, showing that *the two formalisms are roughly equivalent from a modelling point of view.* However, no special computation based on the Petri net structure has yet been proposed, to motivate the use of a Petri net formalism. In the following of this paper we show some connections between structural analysis of Petri net models and some interesting performance-oriented questions on distributed systems.

2.2 Stochastic Petri nets

2.2.1 Some terminology related to PNs

Petri nets are a well-known formalism for describing concurrent discrete event dynamic systems with synchronizations (see [18] for a nice recent survey; [19] and [20] are textbooks, while [21] is the text of an advanced course). We assume that the reader is familiar with the structure, firing rules, and basic properties of net models. The purpose of this section is to make some notations precise since they will be extensively used in the sequel.

A Petri net is a bipartite directed graph, in which the nodes are called *places* and *transitions*.

Net structure. A Petri net is a 4-tuple $\mathcal{N} = \langle P, T, Pre, Post\rangle$, where P is the set of places ($|P| = n$), T is the set of transitions ($|T| = m$, $P \cap T = \emptyset$), Pre ($Post$) is the pre- (post-) incidence function representing the input (output) arcs $Pre\colon P \times T \to \mathbb{N} = \{0, 1, 2, \ldots\}$ ($Post\colon P \times T \to \mathbb{N}$).

Ordinary nets are Petri nets whose pre and post incidence functions take values in $\{0, 1\}$.

The *pre-* and *post-set* of a transition $t \in T$ are defined respectively as ${}^\bullet t = \{p | Pre(p, t) > 0\}$ and $t^\bullet = \{p | Post(p, t) > 0\}$. The *pre-* and *post-set* of a place $p \in P$ are defined respectively as ${}^\bullet p = \{t | Post(p, t) > 0\}$ and $p^\bullet = \{t | Pre(p, t) > 0\}$.

The *incidence matrix* of the net $C = [c_{ij}]$ ($1 \le i \le n$, $1 \le j \le m$) is defined by $c_{ij} = Post(p_i, t_j) - Pre(p_i, t_j)$. Similarly the pre- and post-incidence matrices are defined as $PRE = [a_{ij}]$ and $POST = [b_{ij}]$, where $a_{ij} = Pre(p_i, t_j)$ and $b_{ij} = Post(p_i, t_j)$.

A transition t such that $|{}^\bullet t| > 1$ is called *synchronization.* If $t_1, t_2 \in p^\bullet$ we say that t_1 and t_2 are in *structural conflict.*

Token game. A function $M\colon P \to \mathbb{N}$ (usually represented in vector form) is called *marking.* Markings represent (distributed) states. A *marked Petri net* or *net system* $\langle \mathcal{N}, M_0\rangle$ is a Petri net $\mathcal{N}$ with an *initial marking* M_0.

A transition $t \in T$ is *enabled* in marking M iff $\forall p \in P$ $M(p) \ge Pre(p, t)$. A transition t enabled in M can *fire* yielding a new marking M' (*reached* marking) defined by $M'(p) = M(p) - Pre(p, t) + Post(p, t)$ (it is denoted by $M[t\rangle M'$).

A sequence of transitions $\sigma = t_1 t_2 \dots t_n$ is a *firing sequence* of $\langle \mathcal{N}, M_0 \rangle$ iff there exists a sequence of markings such that $M_0[t_1\rangle M_1[t_2\rangle M_2 \dots [t_n\rangle M_n$. In this case, marking M_n is said to be *reachable* from M_0 by firing σ, and this is denoted by $M_0[\sigma\rangle M_n$. $M[\sigma\rangle$ denotes a firable sequence σ from marking M. The function $\vec{\sigma}: T \to \mathbb{N}$ is the *firing count vector* of the firable sequence σ, i.e., $\vec{\sigma}(t)$ represents the number of occurrences of $t \in T$ in σ. If $M_0[\sigma\rangle M$, then we can write in vector form $M = M_0 + C \cdot \vec{\sigma}$, which is referred to as the *linear state equation* of the net. The *reachability set* $R(\mathcal{N}, M_0)$ is the set of all markings reachable from the initial marking.

Basic properties. A place $p \in P$ is said to be k–bounded iff $\forall M \in R(\mathcal{N}, M_0)$, $M(p) \leq k$. A marked net $\langle \mathcal{N}, M_0 \rangle$ is said to be (marking) k–bounded iff each of its places is k–bounded, and it is bounded iff it is k–bounded for some $k \in \mathbb{N}$. A marked net is said to be *safe* iff it is 1–bounded. A net $\mathcal{N}$ is *structurally bounded* iff $\forall M_0$ the marked nets $\langle \mathcal{N}, M_0 \rangle$ are bounded.
Given an initial marking, an *implicit* place [22] is one which never is the only place that restricts the firing of its output transitions. Let $\mathcal{N}$ be any net and $\mathcal{N}_p$ be the net resulting from adding an implicit place p to $\mathcal{N}$. Therefore, the firing sequences in $\langle \mathcal{N}, M_0 \rangle$ and $\langle \mathcal{N}_p, M_0 \cup m_0(p) \rangle$ are identical.
A transition $t \in T$ is *live* in $\langle \mathcal{N}, M_0 \rangle$ iff $\forall M \in R(\mathcal{N}, M_0)$, $\exists M' \in R(\mathcal{N}, M)$ such that M' enables t. The marked net $\langle \mathcal{N}, M_0 \rangle$ is live iff all its transitions are live (i.e., liveness of the net guarantees the possibility of an infinite activity of all transitions). A net $\mathcal{N}$ is *structurally live* iff $\exists M_0$ such that the marked net $\langle \mathcal{N}, M_0 \rangle$ is live. The marked net $\langle \mathcal{N}, M_0 \rangle$ is *deadlock-free* iff $\forall M \in R(\mathcal{N}, M_0)$ $\exists t \in T$ such that M enables t. A marked net has a *deadlock* iff it is not deadlock-free.
A *consistent component* (or *T-semiflow*) is a function (vector) $X: T \to \mathbb{N}$ such that $X \neq 0$ and $C \cdot X = 0$. A *conservative component* (or *P-semiflow*) is a function (vector) $Y: P \to \mathbb{N}$ such that $Y \neq 0$ and $Y^T \cdot C = 0$. The *support* of (T- and P-) semiflows is defined by $||X|| = \{t \in T | X(t) > 0\}$ and $||Y|| = \{p \in P | Y(p) > 0\}$. A (T- or P-) semiflow I has *minimal support* iff there exist no other semiflow I' such that $||I'|| \subset ||I||$. A (T- or P-) semiflow is *canonical* iff the greatest common divisor of its components is 1. A (T- or P-) semiflow is *elementary* iff it is canonical and has minimal support.
A net $\mathcal{N}$ is *consistent* if there exists a T-semiflow $X \geq \vec{1}$ (where $\vec{1}$ is a vector with all entries equal to 1). A net $\mathcal{N}$ is *conservative* if there exists a P-semiflow $Y \geq \vec{1}$.

Net subclasses. The following are classical ordinary net subclasses, characterized by local structural properties:

- State machines (SM) are ordinary nets such that $\forall t \in T : |{}^\bullet t| = |t^\bullet| = 1$. State machines allow the modelling of decisions (conflicts) and concurrency (when $\sum_{p \in P} M_0(p) \geq 2$) but not synchronization.

- Marked graphs (MG) are ordinary nets such that $\forall p \in P : |{}^\bullet p| = |p^\bullet| = 1$. Marked graphs allow the modelling of concurrency and synchronization, but not of conflict.

- Free choice (FC) nets are ordinary nets such that $\forall p \in P : |p^\bullet| > 1 \Rightarrow {}^\bullet(p^\bullet) = \{p\}$. Free choice nets (see, e.g., [10, 11, 12]) allow both synchronization and conflict but in a restricted and disciplined way. In an FC net, if a place has a shared output transition then it is the only output transition of this place. And, equivalently, if a transition has a shared input place then it is the only input place of this transition. FC nets do not allow the modelling of mutual exclusion semaphores. Throughout the paper we consider live and bounded FC nets (LBFC nets).

- Simple nets are ordinary nets such that each transition has at most one shared input place, i.e., $\forall t \in T, |\{p \in {}^\bullet t : |p^\bullet| > 1\}| \leq 1$. Simple nets allow the modelling of decisions, concurrency, synchronization, and shared resources (mutual exclusion schemes), but they do not allow coupled shared resources.

The following is a net subclass characterized by global structural properties:

- Mono-T-semiflow nets [9] are structurally bounded nets with a unique minimal T-semiflow X, that contains all transitions. Thus, they verify $rank(C) = m - 1$, with C the incidence matrix of the net and $m = |T|$.

2.2.2 On stochastic Petri nets

In the original definition, Petri nets did not include the notion of time, and tried to model only the logical behaviour of systems by describing the causal relations existing between events. This approach showed its power in the specification and analysis of concurrent systems in a non-interleaved way, i.e., in a primitive way independent of the concept of time. Nevertheless the introduction of timing specification is essential if we want to use this class of models for an evaluation of the performance of distributed systems.

Timing and firing process. Since Petri nets are bipartite graphs, historically there have been two ways of introducing the concept of time in them, namely, associating a time interpretation with either places [23] or transitions [24]. Since transitions represent activities that change the state (marking) of the net, it seems natural to associate a duration with these activities (transitions). The latter has been our choice.

In order to solve conflicts among transitions, two alternatives have been proposed: either a "timed firing" of transitions in three phases (which changes the firing rule of Petri nets introducing a timed phase in which the transition is "working" after having removed tokens from the input places and before adding tokens to the output arcs) or a "timed enabling" followed by an atomic firing (which does not affect the usual Petri net firing rule). A more detailed discussion of the timing and firing process can be found in [7]. These different timing interpretations have different implications on the resolution of conflicts. Since in the context of this work we are considering FC nets, any conflict can be resolved in a local way by specifying the routing rates of tokens at places with several output transitions; thus we are not forced to choose one particular firing mechanism.

We consider both timed and immediate transitions. Timed transitions model *services* while immediate transitions are used to model *decisions* (routing rates are associated with them). Both timed and immediate transitions can be used to model synchronizations. Even if the following constraint can be relaxed, for simplicity it is assumed that there do not exist circuits containing only immediate transitions. For each $p \in P$ with more than one output transition: $p^\bullet = \{t_1, ..., t_k\}$, we assume that these transitions are *immediate* (i.e., they fire in zero time); the constants $r_1, ..., r_k \in \mathbb{N}^+$ are explicitly defined in the net interpretation in such a way that when $t_1, ..., t_k$ are enabled, transition t_i $(i = 1, \ldots, k)$ fires with probability (or with long run rate, in the case of deterministic conflicts resolution policy) $r_i/(\sum_{j=1}^{k} r_j)$. Note that the routing rates are assumed to be strictly positive, i.e., all possible outcomes of any conflict have a non-null probability of firing. This fact guarantees a *locally fair* behaviour for the non-autonomous Petri nets that we consider (a marked net is said to be locally fair iff all output transitions of a shared place that are simultaneously enabled infinitely many times will fire infinitely often).

Concerning the transitions that are neither synchronizations nor in conflict (i.e., $t \in T$ such that ${}^\bullet t = \{p\}, p^\bullet = \{t\}$), an (almost surely) finite non-null time is associated with each one of them (enabling time). The absence of conflict for these transitions assures a *persistent* service, i.e., no customer can leave an initiated service (preemption is not considered).

Single versus multiple server semantics. Another possible source of confusion in the definition of the timing interpretation of a Petri net model is the concept of "degree of enabling" of a transition (or re-entrance). In the case of timing associated with places, it seems quite natural to define an unavailability time which is independent of the total number of tokens already present in the place, and this can be interpreted as an "infinite server" policy from the point of view of queueing theory. In the case of time associated with transitions, it is less obvious a-priori whether a transition enabled k times in a marking should work at conditional throughput 1 or k times the one it would work in the case it was enabled only once. In the case of stochastic Petri nets with exponentially distributed firing times associated with transitions, the usual implicit hypothesis is to have "single server" semantics (see, e.g., [25, 26]), and the case of "multiple server" is handled as a case of firing rate dependent on the marking; this trick cannot work in the case of other probability distributions. This is the reason why people working with deterministic timed transition Petri nets prefer an infinite server semantics (see, e.g., [27, 28]). Of course an infinite server transition can always be constrained to a "k–server" behaviour by just reducing its

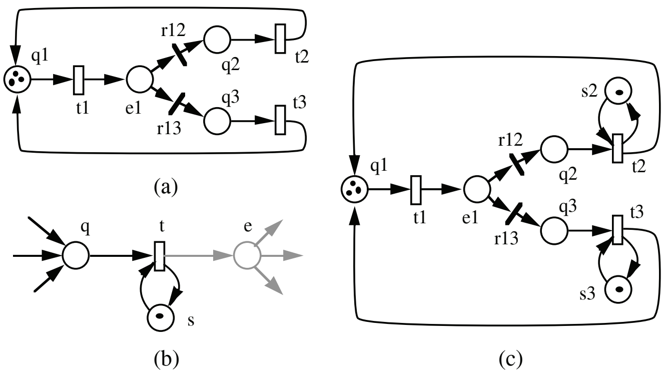

Figure 2: (a) An example of state machine representing a network of delay stations. (b) A PN representation of a monoclass single-server queue. (c) An FC net representation of a queueing network.

enabling bound to k, as we will see later.
Therefore the infinite server semantics appears to be the most general one, and for this reason it is adopted in this work.

2.3 Mapping between monoclass QN with synchronizations and SPN

Here we show how the FC net models defined in the previous section can be represented with a Queueing network formalism. In particular we define the class of Free Choice Synchronized Queueing Networks (FCSQN) as the queueing representation counterpart of LBFC nets.

An "infinite-server" queue [29] (i.e., with a pure delay node) can be represented by a Petri net containing one place to model the number of customers in the system and a timed transition connected with the place through an input arc to model departures. A queueing network containing only pure delay nodes can be modelled, as depicted in the example in Figure 2.a, by a state machine. Persistent timed transitions represent service times of the nodes, while (free choice) conflicting immediate transitions model the routing of customers moving from one node to the other.

A monoclass "single-server" station [29] can be modelled by a subnet of the type depicted in Figure 2.b. Monoclass queueing networks containing both delay and finite-server nodes are thus naturally modelled by FC nets of the type depicted in the example of Figure 2.c (t_1 is a delay, while t_2 and t_3 are single server stations). Also in this more general context conflicting immediate transitions model the routing of customers among the stations, while persistent timed transitions model the service times.

On the other hand, FC nets can assume forms much more complex than the one illustrated in the example of Figure 2.c in which transitions have at most two input and output arcs, connected to three places as shown in the pattern in Figure 2.b. Figure 3 illustrates a more general strongly connected FC net that cannot be mapped onto a product-form queueing network. In fact this net can be mapped on an extended queueing network, a closed FCSQN, in which such constructs as "fork and join" and "passive resources" are used to map the effect of the pairs of transitions t_2–t_7 and t_9–t_{10}, respectively. These examples show how, using a PN formalism, extensions of product-form queueing networks are represented with an analogous level of structural complexity of (single-server) Jackson networks.

From the point of view of the computation of performance bounds, much work has been devoted to the analysis of product-form queueing networks, but little is known for the case of extended queueing networks with fork and join, passive resources, or customer duplications. Borrowing results from PN theory, and applying them to the proposed stochastic interpretation of FC nets, we contribute to the knowledge of extended queueing networks.

3 Relationships between qualitative behaviour and structure

There exists a large body of theory concerning the relationships between the qualitative behaviour and structure for PNs (see, e.g., [11, 12, 18, 19, 20, 21, 30, 31]). This section briefly recalls some well-established results concerning these relationships. A few very strong statements for the subclass of FC

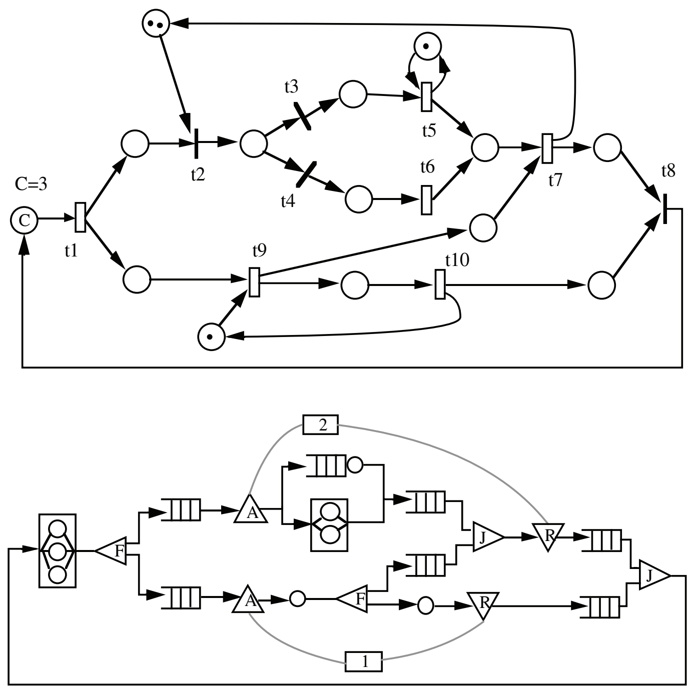

Figure 3: A more general FC net and the corresponding FCSQN.

nets are grouped in Section 3.2. Section 3.1 recalls two general results on PNs plus one devoted to the subclass of mono-T-semiflow.

3.1 Three relationships

Structural boundedness has a nice algebraic characterization (of polynomial time complexity):

Theorem 3.1 [21, 30, 18] *$\mathcal{N}$ is structurally bounded iff $\exists Y \in (\mathbb{N}^+)^n$ such that $Y^T \cdot C \leq 0$.*

Obviously, if $\mathcal{N}$ is conservative (i.e., $\exists Y \in (\mathbb{N}^+)^n, Y^T \cdot C = 0$) then it is structurally bounded. The following is a sufficient condition for consistency, conservativity, and strong connectivity.

Theorem 3.2 [18, 20] *Let $\mathcal{N}$ be a structurally live and structurally bounded Petri net, then $\mathcal{N}$ is consistent and conservative. Moreover, if $\mathcal{N}$ is connected, it is strongly connected.*

The last statement of this section concerns mono-T-semiflow nets (that are bounded by definition):

Theorem 3.3 [9] *Let $\mathcal{N}$ be a mono-T-semiflow net.*

1. *Deadlock-freeness and liveness are equivalent properties (i.e., either* all *transitions are live or* none *of them is live).*

2. *$\mathcal{N}$ is strongly connected.*

3.2 A brief review of structural theory for LBFC nets

This section introduces a minimum of qualitative results from the large body of FC net theory [10, 11, 12, 32, 33, 34, 35]. Additional qualitative results are derived from the quantitative/performance based approach introduced in this paper. This fact clearly points out the interest of interleaving the qualitative and quantitative theories.

Let $\mathcal{N} = \langle P, T, Pre, Post \rangle$ be a Petri net and $P' \subseteq P$. $\mathcal{N}' = \langle P', T', Pre', Post' \rangle$ is called a *P-component* of $\mathcal{N}$ iff $\mathcal{N}'$ is the subnet of $\mathcal{N}$ generated by P' (i.e., $T' \subseteq T$ and $Pre', Post'$ are the restrictions of $Pre, Post$ to P' and T') and $\forall t \in T' : |{}^\bullet t \cap P'| \leq 1 \wedge |t^\bullet \cap P'| \leq 1$. The next result follows from the well-known Hack's Theorem [10].

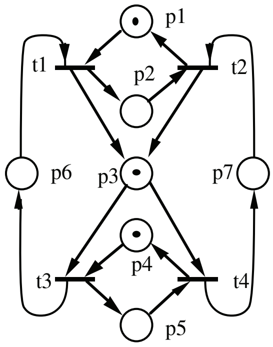

Figure 4: A live and bounded simple net: the addition of a token to p_5 kills the net (sequence $\sigma = t_4$ leads to a deadlock).

Theorem 3.4 (A first liveness characterization) [32] *Let $\langle \mathcal{N}, M_0 \rangle$ be a marked FC net. $\langle \mathcal{N}, M_0 \rangle$ is live and bounded iff: (a) $\mathcal{N}$ is structurally live and structurally bounded, and (b) every P-component of $\mathcal{N}$ is marked at M_0.*

Structure theory of FC nets assures [35] that each minimal P-semiflow of a structurally live and structurally bounded FC net generates a P-component. Therefore, the next result can be derived.

Theorem 3.5 (An algebraic characterization of liveness) *Let $\mathcal{N}$ be a structurally live and structurally bounded FC net. $\langle \mathcal{N}, M_0 \rangle$ is a live marked net iff all its P-semiflows are marked (i.e., $\forall Y \geq 0$ such that $Y^T \cdot C = 0, Y^T \cdot M_0 > 0$).*

Proof. If $\langle \mathcal{N}, M_0 \rangle$ is live, since it is also bounded then all its P-components are marked (by Theorem 3.4). If Y is a P-semiflow of $\mathcal{N}$, its support includes (or is equal to) the support of a minimal P-semiflow, thus it includes the places of a P-component, hence Y is marked. Conversely, if all (and, in particular, the minimal) the P-semiflows are marked, then all the P-components are marked and the net is live (Theorem 3.4). $\diamondsuit$

Corollary 3.1 (Liveness monotonicity) *If $\langle \mathcal{N}, M_0 \rangle$ is an LBFC net and $M_0' \geq M_0$ then $\langle \mathcal{N}, M_0' \rangle$ is live.*

The above corollary is a direct consequence of Theorem 3.5. Nevertheless, it must be pointed out that it holds also without assuming boundedness [12] (as a consequence of Commoner's Theorem). Liveness monotonicity does not hold for more general classes of nets (e.g., simple nets, Figure 4).

Given a place p of a marked net, the maximum number of tokens at this place over all reachable markings is called the *marking bound* of p (denoted $B(p)$). The structural counterpart of this concept can be defined in terms of a linear programming problem (LPP) as follows:

Definition 3.1 (Structural marking bound, SB) *Let $\langle \mathcal{N}, M_0 \rangle$ be a marked Petri net, $\forall p \in P$*

$$SB(p) \stackrel{\text{def}}{=} \begin{array}{ll} \textit{maximize} & M(p) \\ \textit{subject to} & M = M_0 + C \cdot \vec{\sigma} \\ & M \geq 0, \vec{\sigma} \geq 0 \end{array} \qquad \text{(LPP1)}$$

It is clear that $M_0[\sigma\rangle M$ implies $M = M_0 + C \cdot \vec{\sigma} \geq 0$ with $\vec{\sigma} \geq 0$. Since the reverse is not true in general, $SB(p)$ is greater than or equal to $B(p)$. The structural marking bound can always be reached in an LBFC net:

Theorem 3.6 [32] *Let $\langle \mathcal{N}, M_0 \rangle$ be an LBFC net, then $\forall p \in P, B(p) = SB(p)$.*

Corollary 3.2 *A live FC net is bounded iff it is structurally bounded.*

The importance of the above result lies in the fact that structural boundedness can be algebraically characterized (Theorem 3.1).

A home state is a marking that may be reached from all other reachable markings. Vogler proved that an LBFC net has at least one home state:

Theorem 3.7 (Home state existence)[33] *Let $\langle \mathcal{N}, M_0 \rangle$ be an LBFC net. Then $\langle \mathcal{N}, M_0 \rangle$ has a home state.*

The importance of this result from the performance evaluation point of view is stressed in the next section.

4 Ergodicity of closed free choice synchronized QNs

In order to speak of steady-state performance we have to assume that some kind of "average behaviour" can be estimated on the long run of the system we are studying. The usual assumption in this case is that the system models must be *ergodic*, meaning that at the limit when the observation period tends to infinity, the estimates of average values tend (almost surely) to the theoretical expected values of the (usually unknown) probability distributions that characterize the performance indexes of interest.

This assumption is very strong and difficult to verify in general; moreover, it creates problems when we want to include the deterministic case as a special case of a stochastic model [9]. Thus we also use the concept of *weak ergodicity* that allows the estimation of long run performance even in the case of deterministic models.

Definition 4.1 (Ergodicity) *Let X_τ be a stochastic process (or deterministic as a special case), where τ represents the time.*

i) X_τ is said to be weakly ergodic (or measurable in long run) iff the following limit exists:

$$\overline{X} \stackrel{\text{def}}{=} \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau X_s\, ds < \infty$$

ii) X_τ is said to be strongly ergodic iff the following condition holds:

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau X_s\, ds = \lim_{\tau\to\infty} E[X_\tau] < \infty, \;\; a.s.$$

For stochastic Petri nets, weak ergodicity of the marking and the firing processes can be defined in the following terms:

Definition 4.2 *The marking process of a stochastic marked net is weakly ergodic iff the following limit exists:*

$$\overline{M} \stackrel{\text{def}}{=} \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau M_s\, ds < \vec{\infty}$$

and $\overline{M}$ is called the limit average marking.

The firing process of a stochastic marked net is weakly ergodic iff the following limit exists:

$$\vec{\sigma}^* \stackrel{\text{def}}{=} \lim_{\tau\to\infty} \frac{\vec{\sigma}_\tau}{\tau} < \vec{\infty}$$

and $\vec{\sigma}^*$ is the limit firing flow vector (in both cases, the initial marking M_0 is a given deterministic vector and τ represents the time).

The usual (strong) ergodicity concepts [26] are defined in the obvious way taking into consideration Definition 4.1.ii.

Theorem 4.1 *Let $\langle \mathcal{N}, M_0 \rangle$ be a stochastic LBFC net.*

1. *Both the marking and the firing processes of $\langle \mathcal{N}, M_0 \rangle$ are weakly ergodic.*
2. *If $\langle \mathcal{N}, M_0 \rangle$ is semi-Markovian, then the marking and the firing processes are strongly ergodic.*

Proof. For LBFC nets, the existence of home state is assured (Theorem 3.7). Then, after a possible transient phase, the system state is always trapped in a unique strongly connected finite subset of the state space (*terminal class*). Thus, the marking and firing processes are weakly ergodic.

If *semi-Markovian* LBFC nets are considered (stochastic LBFC nets whose marking process is semi-Markov) strong ergodicity of the marking and firing processes is assured. This is because the existence of home state implies that only one proper closed subset of the state space exists and it has finite cardinality. Therefore, the Markov chain restricted to that subset is irreducible and positive recurrent, hence strongly ergodic. ♢

In other words, for LBFC nets it makes sense to speak of a unique steady-state behaviour and to compute bounds for the performance of this steady-state.

5 Upper bounds for the throughput of LBFC nets

In this Section, upper bounds for the throughput of LBFC nets are presented. First we derive some general results from the structural theory of nets, and then we specialize the problem to MGs and FC nets.

5.1 General approach and MGs case

Let us take into account just the first moments of the Probability Distribution Functions (PDFs, for short) associated with transitions. In the following, let θ_i be the mean value of the random variable associated with the firing of transition t_i. The limit firing flow vector per time unit (under weak ergodicity assumption) is $\vec{\sigma}^* = \lim_{\tau\to\infty} \vec{\sigma}_\tau/\tau$ and the mean time between two consecutive firings of a selected transition t_i (*mean cycle time* of t_i), $\Gamma_i = 1/\vec{\sigma}^*_i$.

In what follows, the *relative firing frequency vector* or vector of *visit ratios to transitions* (i.e., the limit firing flow vector $\vec{\sigma}^*$ normalized for having the i^{th} component equal 1) is denoted by

$$F_i \stackrel{\text{def}}{=} \Gamma_i \vec{\sigma}^*$$

Obviously, the above definition makes no sense if a deadlock is reachable. In this case $\vec{\sigma}^* = 0$ or, in other words, the cycle time Γ_i is infinite for all transitions.

The following Little's formula [29] for stochastic Petri nets [26] holds under weak ergodicity assumption:

$$\overline{M}(p_i) = PRE(p_i) \cdot \overline{R}(p_i)\vec{\sigma}^*$$

where $\overline{M}(p_i)$ is the limit mean marking of place p_i, $PRE(p_i)$ is the i^{th} row of the pre-incidence matrix, and $\overline{R}(p_i)$ is the mean response time at place p_i (i.e., the mean sojourn time of tokens: sum of the waiting time and the service time). The response times at places are unknown but can be lower-bounded from the knowledge of the mean firing time associated with transitions:

$$\overline{M} \geq PRE \cdot D \cdot \vec{\sigma}^* \tag{1}$$

where D is the diagonal matrix with elements $\theta_i, i = 1, \ldots, m$. From this inequality, a lower bound Γ_i^{lb} for the mean cycle time associated with transition t_i can be derived. We take into account that Γ_i^{lb} must be such that inequality (1) holds for every place p_j:

$$\Gamma_i^{lb} \geq \frac{PRE(p_j) \cdot D \cdot F_i}{\overline{M}(p_j)} \tag{2}$$

Since the vector $\overline{M}$ is unknown, (2) cannot be solved. However, taking the product with a P-semiflow Y for any reachable marking M:

$$Y^T \cdot M_0 = Y^T \cdot M = Y^T \cdot \overline{M} \tag{3}$$

Now, from (1) and (3):

$$\Gamma_i Y^T \cdot M_0 \geq Y^T \cdot PRE \cdot D \cdot F_i$$

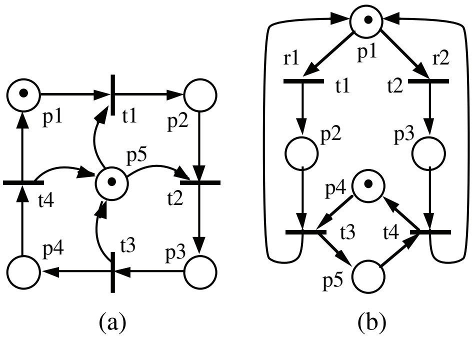

Figure 5: (a) Structurally live and structurally bounded mono-T-semiflow net which has all its P-semiflows marked but is not live. (b) Structurally bounded FC net with all its P-semiflows marked which is not live (it is not structurally live).

And a lower bound for the mean cycle time in steady state is:

$$\Gamma_i^{lb} = \max_{Y \in \{P-semiflow\}} \frac{Y^T \cdot PRE \cdot D \cdot F_i}{Y^T \cdot M_0}$$

Of course, an upper bound for the throughput of t_i is $1/\Gamma_i^{lb}$.

The previous lower bound for the mean cycle time can be formulated in terms of a fractional programming problem and later, after some considerations, transformed into a linear programming problem:

Theorem 5.1 (Throughput upper bound) [9] *For any net, a lower bound for the mean cycle time of transition t_i can be computed by the following LPP:*

$$\begin{aligned} \Gamma_i^{lb} = \quad & \textit{maximize} \quad Y^T \cdot PRE \cdot D \cdot F_i \\ & \textit{subject to} \quad Y^T \cdot C = 0 \\ & \qquad\qquad\quad Y^T \cdot M_0 = 1; \; Y \geq 0 \end{aligned} \tag{LPP2}$$

The basic advantage of the above statement lies in the fact that the simplex method for the solution of LPPs has almost linear complexity in practice, even if it has exponential worst case complexity. In any case a discussion on algorithms of polynomial worst case complexity can be found in [36].

Corollary 5.1 *Assuming that $F_i > 0$ and that there do not exist circuits containing only immediate transitions, the problem (LPP2) has unbounded solution iff $\exists Y \geq 0, Y \neq 0$ such that $Y^T \cdot M_0 = 0$ and $Y^T \cdot C = 0$.*

If the solution of problem (LPP2) is unbounded, since it is a lower bound for the mean cycle time of a transition, the non-liveness can be assured (infinite cycle time). This result has the following interpretation: *if the problem (LPP2) is unbounded then there exists an unmarked P-semiflow, and the net is non-live.* The converse is not true in general. The mono-T-semiflow net depicted in Figure 5.a is structurally live and structurally bounded, all its P-semiflows are marked, but it is not live. On the other hand, the FC net of Figure 5.b is structurally bounded and all its P-semiflows are marked but is is not live (even more, it is not structurally live).

The upper bound for the steady-state throughput of a transition obtained from (LPP2) in Theorem 5.1 is valid for any net. But in the general case, F_i may depend on the net structure, on the timing interpretation, and on the initial marking, i.e., $F_i = F_i(\mathcal{N}, \Lambda, M_0)$, where Λ denotes the timing interpretation (including firing times of transitions and routing rates at conflicts). However, the marking independence of F_i has been shown (if liveness is assumed) for important subclasses of nets such as strongly connected MGs and mono-T-semiflow nets [9]. These net subclasses can be recognized in polynomial time, thus the computation of F_i for them has polynomial complexity. For LBFC nets, the marking independence of F_i is proved in Section 5.2. The relative firing frequency vector for an LBFC net is completely determined by the net structure and the stochastic interpretation, and can be computed in polynomial time. The previous considerations are summarized in the next table:

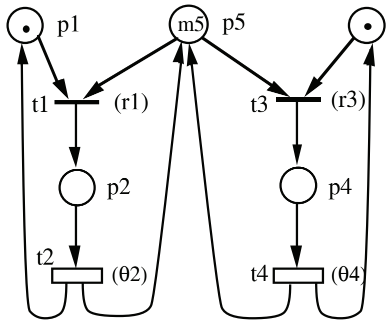

Figure 6: A simple net. The relative firing frequency vector depends on the initial marking.

strongly connected MGs	$F_i = \vec{1}$ (constant)
mono-T-semiflow nets	$F_i = F_i(\mathcal{N})$
LBFC nets	$F_i = F_i(\mathcal{N}, \Lambda)$
simple nets	$F_i = F_i(\mathcal{N}, \Lambda, M_0)$

The above conclusions are not valid for simple nets. For these nets, the relative firing frequency vector depends also on the initial marking. As an example, let us consider the net in Figure 6. Transitions t_1 and t_3 are immediate (i.e., enabling time equal 0). The constants $r_1, r_3 \in \mathbb{N}^+$ define the conflict resolution policy, i.e., when t_1 and t_3 are simultaneously enabled, t_1 fires with relative frequency $r_1/(r_1+r_3)$ and t_3 with $r_3/(r_1+r_3)$. Let θ_2 and θ_4 be the mean enabling delays associated with t_2 and t_4. If $m_5 = 1$ (initial marking of p_5) then p_1 and p_3 are implicit [30], hence they can be deleted (without affecting the behaviour!); a state machine is obtained and the relative firing frequency vector for transition t_1 can be computed: $F_1^{(m_5=1)} = (1, 1, r_3/r_1, r_3/r_1)^T$. If $m_5 = 2$ (different initial marking for p_5) then p_5 is implicit, hence it can be deleted; two isolated state machines (cycles) are obtained and $F_1^{(m_5=2)} = (1, 1, \theta_2/\theta_4, \theta_2/\theta_4)^T$. Obviously $F_1^{(m_5=1)} \neq F_1^{(m_5=2)}$.

For strongly connected MGs, the bound derived from Theorem 5.1 is the same as that obtained for the deterministic case by other authors (see, e.g., [24, 37, 38]), but here it is considered in a practical LPP form. For deterministic timed nets, the reachability of this bound has been shown [24, 37]. Since deterministic timing is just a particular case of stochastic timing, the reachability of the bound is assured for our purposes. Even more, the next result shows that the previous bound cannot be improved only on the base of the knowledge of the coefficients of variation for the transition firing times.

Theorem 5.2 (Reachability of the bound for strongly connected MGs) [8] *For strongly connected MGs with arbitrary values of mean and variance for transition firing times, the lower bound for the mean cycle time obtained from (LPP2) cannot be improved.*

As a byproduct, we obtain the classical characterization of liveness for a strongly connected MG (see, for example, [18]). It will be live iff the optimal value of (LPP2) is finite (see Corollary 5.1). Then, it is possible to decide in polynomial time on the net structure about liveness of a given strongly connected MG.

Corollary 5.2 [8] *A strongly connected MG is non live iff there exists an unmarked circuit, and this can be decided in polynomial time looking for the finiteness of the problem (LPP2).*

The above statement holds in general for any MG [18]. It is equivalent to decide if the graph obtained removing the marked places for M_0 is acyclic.

5.2 Upper bounds for LBFC nets

In this section, an efficient method for computing the relative firing frequency vector for LBFC nets is presented, and bounds for the throughput of transitions are derived from (LPP2) for this subclass of marked Petri nets. Some important qualitative properties are derived from the approach.

Without loss of generality, let us consider LBFC nets with *binary* decisions (i.e., $|p^\bullet| \leq 2$). Given a stochastic interpretation Λ and considering only the relative firing frequencies at conflicts (i.e., independently of the service times), let us construct a related Petri net $\mathcal{N}_R$ with the same relative firing frequency vector as the original net. For each pair of transitions in conflict in $\mathcal{N}$, t_1 and t_2, let $r_1, r_2 \in \mathbb{N}^+$ be the constants defining the resolution policy, in such a way that when t_1 and t_2 are enabled, the transition t_i fires with probability (or long run rate) $r_i/(r_1+r_2), i = 1, 2$. In $\mathcal{N}_R$, this resolution policy is summarized by the following regulation circuit: $\rho_1, \rho_2 \in P_R$ such that ${}^\bullet\rho_1 = \rho_2^\bullet = \{t_2\}$, $\rho_1^\bullet = {}^\bullet\rho_2 = \{t_1\}$, and $Pre(\rho_1, t_1) = Post(\rho_2, t_1) = r_2$, $Post(\rho_1, t_2) = Pre(\rho_2, t_2) = r_1$. Even if the structural conflict between t_1 and t_2 remains in $\mathcal{N}_R$, the added regulation circuit assures the same relative firing frequency of both transitions in the limiting behaviour, therefore it reduces the non-determinism of the original net.

Lemma 5.1 *Let $\langle \mathcal{N}, M_0 \rangle$ be a live and structurally bounded FC net and Λ a stochastic interpretation of $\mathcal{N}$.*

1. *The net $\mathcal{N}_R$, defined above from $\mathcal{N}$, is mono-T-semiflow.*

2. *With a sufficiently large number of tokens in the places of the regulation circuits (i.e., enough number of tokens for making the regulation circuits live in isolation) and the initial marking in the rest of places equal M_0, the marked net $\langle \mathcal{N}_R, M_0^R \rangle$ is live.*

Proof. 1) $\mathcal{N}$ is structurally live and structurally bounded, thus it is strongly connected, consistent, and conservative (Theorem 3.2). By construction, $\mathcal{N}_R$ is strongly connected, consistent and conservative.

$\mathcal{N}_R$ has at most one consistent component because all the output transitions of a conflict in $\mathcal{N}$ must belong to a unique consistent component in $\mathcal{N}_R$. Since $\mathcal{N}_R$ is consistent, it has at least a consistent T-semiflow, thus, $\mathcal{N}_R$ is mono-T-semiflow.

2) For a given live marked net, any local scheduling at conflicts preserves deadlock-freeness. Then the net $\langle \mathcal{N}_R, M_0^R \rangle$ is deadlock-free. Finally, for mono-T-semiflow nets, liveness and deadlock-freeness coincide (Theorem 3.3). ♢

Theorem 5.3 (Marking independence of F_i for structurally bounded and structurally live FC nets) *Let $\mathcal{N}$ be a structurally bounded and structurally live FC net and Λ a stochastic interpretation of $\mathcal{N}$. For all M_0 making the net live, $F_i = F_i(\mathcal{N}, \Lambda)$ and $F_i(t) > 0, \forall t \in T$.*

Proof. Let us consider the related net $\langle \mathcal{N}_R, M_0^R \rangle$. It has the same relative firing frequency vector as the original net (since the net is live, transitions t_1 and t_2 can be fired infinitely often, and their relative frequency is defined by the rates r_1 and r_2). It is mono-T-semiflow (Lemma 5.1) and its limiting behaviour for the firing count process is defined by the unique T-semiflow $X_R > 0$ (see [9]).

Finally, $F_i = X_R/X_R(t_i)$, and since $X_R > 0$ and it depends only on $\mathcal{N}_R$, i.e., on $\mathcal{N}$ and Λ (in fact, on the conflict resolution rates), then $F_i = F_i(\mathcal{N}, \Lambda) > 0$. ♢

The relative firing frequency vector F_i for a given transition t_i of the structurally bounded net $\langle \mathcal{N}_R, M_0^R \rangle$ must be a consistent component [31]:

$$C_R \cdot F_i = 0, \quad F_i > 0, \quad F_i(t_i) = 1 \tag{4}$$

But $C_R = (C^T | R^T | -R^T)^T$, where R is a matrix with $a - n$ rows ($a = \sum_{p \in P, t \in T} Pre(p, t)$; i.e., the number of arcs in Pre) derived from the conflicts resolution policy: each row of R gives an independent relation between the throughput of two transitions in free conflict. And equation (4) is equivalent to:

i) $C \cdot F_i = 0$ (i.e., F_i is a consistent component of C; n equations)

ii) $R \cdot F_i = 0$ (i.e., the routing rates are respected; $a - n$ equations)

iii) $F_i > 0$ (i.e., the relative firing frequency between any pair of transitions $F_i(t_j)/F_i(t_k)$ is finite)

iv) $F_i(t_i) = 1$ (i.e., the vector is normalized for having the i^{th} component equal 1)

The above system can be rewritten in a more compact way:

$$\begin{pmatrix} C \\ R \end{pmatrix} F_i = 0, \quad F_i > 0, \quad F_i(t_i) = 1 \tag{5}$$

The following observations about the previous system can be done:

1. The system (5) has at most one solution (Lemma 5.1).

2. If (5) has no solution then the net is structurally non-live and sooner or later it will reach a deadlock. This follows from the fact that if system (4) has no solution, then $\mathcal{N}_R$ has no consistent component and the net cannot have any infinite behaviour [31]. See, for example, the net in Figure 5.b. For $r_1 = 1, r_2 = 2$, the system (5) has no solution, and the net is structurally non-live.

3. The existence of solution for system (5) is a necessary but non-sufficient condition for the structural liveness of $\mathcal{N}$. This can be easily checked using once again the net in Figure 5.b. For $r_1 = r_2 = 1$, there exists a solution of system (5): $F_i = (1, 1, 1, 1)^T$, but the FC net is structurally non-live!

The open question from the previous considerations is: When a strongly connected and structurally bounded FC net is structurally live? The answer is given by the next theorem.

Theorem 5.4 (Algebraic characterization of structural liveness for strongly connected structurally bounded FC nets) *Let $\mathcal{N}$ be a strongly connected and structurally bounded FC net and C its incidence matrix.*

1. $\mathcal{N}$ is structurally live iff $rank(C) = m - 1 - (a - n)$.

2. $\mathcal{N}$ is structurally non-live iff $rank(C) \geq m - (a - n)$.

where $n = |P|, m = |T|$, and $a = \sum_{p \in P, t \in T} Pre(p, t)$.

Proof. 1) If $\mathcal{N}$ is strongly connected, structurally live, and structurally bounded then $\mathcal{N}_R$ is strongly connected, structurally live, and structurally bounded and has one T-semiflow, i.e., it is mono-T-semiflow. Thus $rank(C_R) = m - 1$. And this is true for all (locally fair) conflict resolution rates. Then $m - 1 = rank(C_R) = rank(C) + rank(R) = rank(C) + a - n$, i.e., $rank(C) = m - 1 - (a - n)$. Note that $rank(C_R) = rank(C) + rank(R)$ because none of the places of the regulation circuits can be implicit: if one of them was implicit, the choice in the original net would not be free, against the hypothesis.

If $rank(C) = m - 1 - (a - n)$, the system (5) has solution for all conflict resolution policies (the number of independent equations is $rank(C)$ plus $rank(R) = a - n$ plus one, for the normalization equation; the number of variables is m). This leads to claim that under any locally fair conflict resolution policy, infinite behaviours in which all transitions fire ($F_i > 0$) can always be obtained for a large enough initial marking and no deadlock can be reached. In other words, the net is structurally live.

2) If $\mathcal{N}$ is a strongly connected and structurally bounded FC net and C its incidence matrix, then $rank(C) \geq m - 1 - (a - n)$. This result follows considering once more the derived net $\mathcal{N}_R$. It is equivalent to see that $rank(C_R) \geq m - 1$, i.e., the net $\mathcal{N}_R$ has not more than one consistent component. And this is true because each pair of output transitions of a given place that could generate two consistent components are related with a regulation circuit, thus they should belong to the same consistent component.

Now, statement 2 of this theorem follows from statement 1 and from the fact that $rank(C) \geq m - 1 - (a - n)$. ◇

The classical duality result for free choice nets [10] can be derived from the previous theorem. The reverse dual of a net is obtained by changing places by transitions (dual) and reversing the arc orientation (reverse). The reader can check that the reverse dual of an FC net is also an FC net. If C (C_{rd}) is the incidence matrix of $\mathcal{N}$ ($\mathcal{N}_{rd}$), it is easy to verify that $C_{rd} = -C^T$. Therefore $rank(C) = rank(C_{rd})$.

Corollary 5.3 (Duality theorem) *Let $\mathcal{N} = \langle P, T, Pre, Post \rangle$ be an FC net. $\mathcal{N}$ is structurally live and structurally bounded iff the reverse-dual of $\mathcal{N}$, $\mathcal{N}_{rd} = \langle T, P, Post, Pre \rangle$, is structurally live and structurally bounded.*

Proof. If $\mathcal{N}$ is structurally live and structurally bounded then it is strongly connected, consistent, and conservative (Theorem 3.2). Then $\mathcal{N}_{rd}$ is strongly connected, consistent, and conservative, thus structurally bounded.

Finally $rank(C) = rank(C_{rd})$ and $m_{rd} - 1 - (a_{rd} - n_{rd}) = n - 1 - (a - m) = m - 1 - (a - n)$, i.e., if $\mathcal{N}$ is structurally live then $\mathcal{N}_{rd}$ is also structurally live. ◇

Two well-known results of structural theory of nets, can also be deduced from the previous results:

Corollary 5.4 (Structural liveness in FC net subclasses)

1. *Strongly connected MGs are structurally live nets.*
2. *Strongly connected state machines are structurally live nets.*

In fact, for the MG case, strong connection is not needed (an MG is live iff all circuits are marked, i.e., there always exists an initial marking making the MG live [18]).

Now, from Theorem 5.1 and from the fact that the relative firing frequency vector can be computed in polynomial time for LBFC nets (Theorem 5.3), the next result follows:

Corollary 5.5 (Polynomial complexity) *For strongly connected and structurally bounded FC nets, the computation of the lower bound for the mean cycle time of a transition given by Theorem 5.1 has worst case polynomial complexity on the net size.*

Proof. Step 1) Both the strong connectivity and the structural boundedness of a net can be characterized in polynomial time. FC nets are also characterized in polynomial time. Thus the subclass of nets which are referred to in the statement are characterized in polynomial time.

Step 2) For this subclass of nets, the computation of the relative firing frequency vector F_i is polynomial, solving the system (5).

Step 3) Finally, from the knowledge of F_i, the lower bound for the mean cycle time of transition t_i can be computed, solving the problem (LPP2), thus in polynomial time. ♢

As in the case of strongly connected MGs, a characterization of liveness for structurally live and structurally bounded FC nets can be derived.

Corollary 5.6 (Liveness characterization) *Assuming that $F_i > 0$ and that there do not exist circuits containing only immediate transitions, liveness of structurally live and structurally bounded FC nets can be decided in polynomial time, checking the boundedness of the problem (LPP2).*

Proof. For FC nets, both structural boundedness (Theorem 3.1) and structural liveness in structurally bounded nets (Theorem 5.4) are polynomial problems. The optimal value of (LPP2) is a lower bound for the mean cycle time. If this optimal value is infinite the mean cycle time is unbounded so the net is non live. If the optimal value of (LPP2) is finite, this means that for all $Y \geq 0$ such that $Y^T \cdot C = 0$, then $Y^T \cdot M_0 > 0$. In other words, all P-semiflows are marked, thus the net is live (Theorem 3.5). ♢

This result is nothing more but the "natural" generalization of the existence of an unmarked circuit for strongly connected MGs (Corollary 5.2). It does not hold for non-FC nets and for non live FC nets.

The throughput upper bound derived from (LPP2) is not reachable in general for LBFC nets. Several improvements and a reachable bound for the case of live and safe FC nets can be found in [34].

Linear programming problems give an easy way to derive results and interpret them. Just looking at the problem (LPP2) the following monotonicity property is obtained.

Corollary 5.7 (Performance bound monotonicity) *Let $\langle \mathcal{N}, M_0 \rangle$ be an LBFC net and θ the mean firing times vector.*

1. *For a fixed θ, if $M_0' \geq M_0$ (i.e., increasing the number of initial resources) then the throughput upper bound of $\langle \mathcal{N}, M_0', \theta \rangle$ is greater than or equal to the one of $\langle \mathcal{N}, M_0, \theta \rangle$ (i.e., $\Gamma^{lb'} \leq \Gamma^{lb}$).*
2. *For a fixed M_0, if $\theta' \leq \theta$ (i.e., for faster resources) then the throughput upper bound of $\langle \mathcal{N}, M_0, \theta' \rangle$ is greater than or equal to the one of $\langle \mathcal{N}, M_0, \theta \rangle$ (i.e., $\Gamma^{lb'} \leq \Gamma^{lb}$).*

We conjecture that the above monotonicity properties hold, in fact, for the exact throughput of LBFC nets. Nevertheless, the first is not true for live and bounded simple nets: remember that the addition of tokens (i.e., resources) to a live and bounded simple net can make it non-live (see Figure 4).

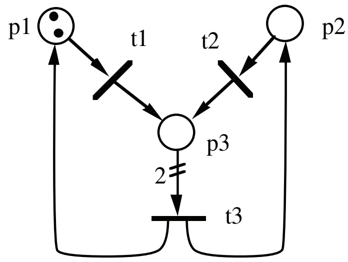

Figure 7: A net with enabling bound greater than liveness bound for transition t_1.

6 Lower bounds for the throughput of LBFC nets

In this section, lower bounds on throughput are proposed, independent of the higher moments of the firing delay PDFs, based on the computation of the transition *liveness bounds*. First we introduce the liveness bound concept as a generalization of the concept of liveness of a transition, and some related results.

Liveness bound is a measure of the maximum *degree of enabling* of a transition. The degree of enabling of transition t in a marking M is the minimum among all the input places $p \in {}^\bullet t$ of the integer part of $M(p)/Pre(p,t)$. It identifies the number of activities associated to the transition that could potentially progress concurrently, disregarding conflicts, at a given marking.

6.1 Additional liveness concepts and results

The performance of a model with *infinite server semantics* depends on the maximum degree of enabling of the transitions; and in particular, the steady-state performance depends on the maximum degree of enabling of transitions in steady-state, which in general can be different from the maximum degree of enabling of a transition during its evolution starting from the initial marking. For this reason we introduce here two concepts of degree of enabling of a transition t: the enabling bound $E(t)$, and the liveness bound $L(t)$. The last is obviously constrained to the steady-state. They allow to generalize the classical concepts of enabling and liveness of a transition.

Definition 6.1 (Enabling bound, E) *Let $\langle \mathcal{N}, M_0 \rangle$ be a marked Petri net and $t \in T$, $E(t) \stackrel{\text{def}}{=} \max\{k | \exists M \in R(\mathcal{N}, M_0) : M \geq kPRE(t)\}$.*

Definition 6.2 (Liveness bound, L) *Let $\langle \mathcal{N}, M_0 \rangle$ be a marked PN and $t \in T$, $L(t) \stackrel{\text{def}}{=} \max\{k | \forall M_1 \in R(\mathcal{N}, M_0), \exists M \in R(\mathcal{N}, M_1) : M \geq kPRE(t)\}$.*

From the above definitions it appears clear how it is possible to obtain a k–server transition from an infinite server one: adding one place that is both input and output for that transition and marking it with k tokens. The following is also obvious from the definition.

Property 6.1 *Let $\langle \mathcal{N}, M_0 \rangle$ be a marked PN, then $\forall t \in T$, $E(t) \geq L(t)$.*

A case of strict inequality in this Property can be interpreted as a generalization of the concept of non-liveness: there exist transitions containing "potential servers" that are never used in the steady-state; these additional servers might only be used in a transient phase, so they "die" during the evolution of the model. See, as an example, the net in Figure 7. It is decision-free but not free choice (it is not an ordinary net), and $E(t_1) > L(t_1)$ ($E(t_1) = 2$ and $L(t_1) = 1$). On the other hand it is not difficult to see that the condition $L(t) > 0$ is equivalent to the usual liveness condition for transition t.

Since for any reversible net (i.e., such that M_0 is a home state) the reachability graph (which is a directed labelled graph) is strongly connected, the following can be stated.

Property 6.2 *Let $\langle \mathcal{N}, M_0 \rangle$ be a reversible PN, then $\forall t \in T$, $E(t) = L(t)$.*

As a particular case, live MGs are reversible, so that enabling and liveness bounds are equal for them.

The definition of enabling bound refers to a behavioural property that depends on the reachability graph of a PN. Since we are looking for computational techniques at the structural level, we can also introduce the structural counterpart of the enabling bound concept. To do this, the reachability condition is substituted by the weaker (linear) constraint that markings satisfy the net state equation: $M = M_0 + C \cdot \vec{\sigma}$, with $M, \vec{\sigma} \geq 0$.

Definition 6.3 (Structural enabling bound, SE) *Let* $\langle \mathcal{N}, M_0 \rangle$ *be a marked Petri net and* $t \in T$,

$$SE(t) \stackrel{\text{def}}{=} \begin{array}{ll} \textit{maximize} & k \\ \textit{subject to} & M = M_0 + C \cdot \vec{\sigma} \geq kPRE(t) \\ & \vec{\sigma} \geq 0 \end{array} \qquad \text{(LPP3)}$$

Note that the definition of structural enabling bound is in fact an LPP. Its dual problem can be easily stated:

$$SE(t) = \begin{array}{ll} \text{minimize} & Y^T \cdot M_0 \\ \text{subject to} & Y^T \cdot C = 0; \quad Y^T \cdot PRE(t) = 1 \\ & Y \geq 0 \end{array} \qquad \text{(LPP4)}$$

Now let us remark the relation between behavioural and structural enabling bound concepts that follows from the implication $M \in R(\mathcal{N}, M_0) \Rightarrow M = M_0 + C \cdot \vec{\sigma} \wedge \vec{\sigma} \geq 0$.

Property 6.3 *Let* $\langle \mathcal{N}, M_0 \rangle$ *be a marked PN, then* $\forall t \in T$, $SE(t) \geq E(t)$.

An interesting property of LBFC nets, that allows an efficient computation of liveness bounds, is the following:

Theorem 6.1 *Let* $\langle \mathcal{N}, M_0 \rangle$ *be an LBFC net, then* $\forall t \in T, SE(t) = E(t) = L(t)$.

Proof. Let t_i be a given transition of $\mathcal{N}$. A new LBFC net $\langle \widehat{\mathcal{N}}, \widehat{M_0} \rangle$ is obtained by splitting transition t_i into a transition t_{i_1}, an unmarked place p_i, and another transition t_{i_2}. Then, for t_i and p_i: $SE(t_i) = \widehat{SB}(p_i)$ and $E(t_i) = \widehat{B}(p_i)$ (*hat* refers to the bounds in the derived net). Since for LBFC nets $\widehat{B}(p_i) = \widehat{SB}(p_i)$ (Theorem 3.6) then $E(t_i) = SE(t_i)$.

LBFC nets are structurally bounded (Corollary 3.2). Since live and structurally bounded nets are conservative (Theorem 3.2), the structural marking bound coincides with the bound obtained from a basis of P-semiflows [22]: $\widehat{SB}(p_i) = \max\{\widehat{M}(p_i) | \widehat{B}^T \cdot \widehat{M} = \widehat{B}^T \cdot \widehat{M_0}, \widehat{M} \geq 0\}$.

Let $\widehat{M_h}$ be a home state of $\langle \widehat{\mathcal{N}}, \widehat{M_0} \rangle$ (its existency is guaranteed by Theorem 3.7). Because $\widehat{M_h}$ is reachable from $\widehat{M_0}$, $\widehat{B}^T \cdot \widehat{M_h} = \widehat{B}^T \cdot \widehat{M_0}$. Considering as a new starting time that in which $\widehat{M_h}$ is reached for the first time: $\widehat{SB}(p_i) = \max\{\widehat{M}(p_i) | \widehat{B}^T \cdot \widehat{M} = \widehat{B}^T \cdot \widehat{M_h}, \widehat{M} \geq 0\}$. Thus $\widehat{SB}(p_i)$ is reached from a home state, and $E(t_i) = L(t_i)$. ♢

Now, from the previous theorem and taking into account that for any transition t the computation of the structural enabling bound $SE(t)$ can be formulated in terms of the problem (LPP4), the following monotonicity property of the liveness bound of a transition with respect to the initial marking is obtained:

Corollary 6.1 (Liveness bound monotonicity) *If* $\langle \mathcal{N}, M_0 \rangle$ *is an LBFC net and* $M_0' \geq M_0$ *then the liveness bound of* t *in* $\langle \mathcal{N}, M_0' \rangle$ *is greater than or equal to the liveness bound of* t *in* $\langle \mathcal{N}, M_0 \rangle$.

The previous result appears to be a generalization (stated for the particular case of bounded nets) of the classical liveness monotonicity property for FC nets stated in Corollary 3.1.

6.2 Lower bounds on throughput for strongly connected MGs

A trivial lower bound in steady-state performance for a live PN with a unique repetitive firing count vector [9] is of course given by the inverse of the sum of the firing times of all the transitions weighted by the firing count vector itself. Since the net is live all transitions must be fireable, and the sum of all firing times multiplied by the number of occurrences of each transition in the (unique) average cycle of the model corresponds to any complete sequentialization of all the activities represented in the model. This lower bound is always reached in an MG consisting of a single loop of transitions and containing a single token in one of the places, independently of the higher moments of the PDFs (this observation can be trivially confirmed by the computation of the upper bound, which in this case gives the same value).

This trivial lower bound has been improved in [8], based on the knowledge of the liveness bound $L(t)$ for all transitions t of the MG.

Theorem 6.2 (Throughput lower bound for strongly connected MGs and its reachability) [8] *For any live and bounded MG with a specification of the mean firing times θ_j for each $t_j \in T$ it is not possible to assign PDFs to the transition firing times such that the average cycle time is greater than*

$$\Gamma^{ub} = \sum_{j=1}^{m} \frac{\theta_j}{L(t_j)} = \sum_{j=1}^{m} \frac{\theta_j}{SE(t_j)}$$

independently of the topology of the net.

Moreover, this upper bound for the mean cycle time is reachable for any MG topology and for some assignement of PDFs to the firing delay of transitions (i.e., the bound cannot be improved).

MGs are a subclass of FC nets. According to Theorem 6.1, the liveness bound equals the structural enabling bound for each transition (see also [8]); thus the problem of the determination of the structural enabling bound can be characterized in terms of the problem (LPP4), which is known to be solvable in polynomial time. The optimum of the objective function is always achieved with elementary P-semiflows Y. In case of MGs, these elementary P-semiflows can only be elementary cycles, so that we can give the following interpretation of the LPP in net terms: the liveness bound for a transition t of a strongly connected MG is given by the minimum number of tokens contained in any cycle containing transition t.

6.3 Lower bounds on throughput for LBFC nets

The non-trivial lower bound for the throughput of MGs (dividing by the liveness bound) presented in Section 6.2 can be applied now in the following way: weighting the mean firing time of t_j, θ_j, with the component of the relative firing flow vector $F_i(t_j)$, for each transition.

Theorem 6.3 (Throughput lower bound for LBFC nets) *For any LBFC net with a specification of the mean firing times θ_j for each $t_j \in T$ it is not possible to assign PDFs to the transition firing times such that the average cycle time of transition t_i is greater than*

$$\Gamma_i^{ub} = \sum_{j=1}^{m} \frac{F_i(t_j)}{L(t_j)}\theta_j = \sum_{j=1}^{m} \frac{F_i(t_j)}{SE(t_j)}\theta_j$$

independently of the topology of the net, where F_i is the relative firing frequency vector with $F_i(t_i) = 1$.

Proof. Let us consider a deterministic conflicts resolution policy. A strongly connected MG with the same relative firing frequency vector can be constructed as follows (in fact, since for the MG $F_i = \vec{1}$, what can appear are several instances of transitions to get the F_i of the original net):

1. Steady-state markings must be home states. Let M_h be one of the home states (there always exist some for LBFC nets, according to Theorem 3.7), and substitute it to the initial marking (i.e., $\langle \mathcal{N}, M_h \rangle$ is reversible).
2. From the LBFC net, a safe marking can be derived preserving liveness, removing tokens from M_h.
3. Develop the process, resolving the conflicts with the deterministic given policy, until cyclicity appears (see [24]) and the relative firing frequency holds. A safe MG is obtained in which transitions appear according to their relative firing frequencies.
4. The rest of tokens at each place in M_h in the original LBFC net, can be added now in the corresponding place of the MG.

The actual cycle time of the original FC net (with deterministic conflicts resolution policy) is less than or equal to the one of the derived MG because the behaviour of the net has been constrained. Now, apply the bound obtained in Theorem 6.2. Different instances of a given transition are considered in the relative rate of the corresponding component in the relative firing frequency vector. Thus, the bound obtained for the derived MG applying Theorem 6.2 coincides with the bound obtained for the original net using the formula stated in this theorem. The theorem follows because $L(t_j) = SE(t_j)$ for LBFC nets (Theorem 6.1). $\diamondsuit$

Note that the structural enabling bound of a transition can be computed by means of an LPP, which is known to be solvable in polynomial time, thus the above lower bound for the throughput of LBFC nets can be computed in polynomial time on the net structure.

7 Bounds for other performance indexes

From the knowledge of upper and lower bounds for the steady-state throughput of transitions and from well-known queueing theory laws (such as Little's formula) [29] fast bounds for other performance indexes of interest can be derived.

7.1 Bounds for the mean length of queues

In this section, a fast computation of upper and lower bounds for the limit mean marking of places (i.e., length of queues including the customers in service) is proposed.

In Section 5.1 the following inequality was derived from Little's formula for stochastic PNs:

$$\overline{M} \geq PRE \cdot D \cdot \vec{\sigma}^*$$

where D is the diagonal matrix with elements θ_i (mean firing time associated with transition $t_i, i = 1, \dots, m$). Then, a lower bound for the mean marking of places in steady-state can be computed, from the knowledge of a lower bound for the throughput of transitions.

Theorem 7.1 *For any LBFC net with a specification of the mean firing times associated with transitions and of the conflict resolution policy, it is not possible to assign PDFs to the transition firing times such that the mean marking of places in steady-state is less than*

$$\overline{M}^{lb} = PRE \cdot D \cdot \vec{\sigma}^{lb}$$

where $\vec{\sigma}^{lb}$ is a lower bound for the throughput vector (i.e., $\vec{\sigma}^{lb}(t_i) = 1/\Gamma_i^{ub}, i = 1, \dots, m,$ with Γ_i^{ub} the upper bound for the mean cycle time of t_i computed in Theorem 6.3).

For the computation of an upper bound for the mean marking of a given place p_1 in steady-state, let us consider a P-semiflow $Y = (Y_1, \dots, Y_n)^T$ whose support includes this place (i.e., $Y_1 \neq 0$). We have

$$Y^T \cdot M_0 = Y^T \cdot \overline{M}$$

Therefore

$$Y^T \cdot M_0 \geq Y_1 \overline{M}(p_1) + (Y_2, \dots, Y_n) \cdot (\overline{M}^{lb}(p_2), \dots, \overline{M}^{lb}(p_n))^T$$
$$\overline{M}(p_1) \leq \overline{M}^{lb}(p_1) + \frac{1}{Y_1} Y^T \cdot (M_0 - \overline{M}^{lb})$$

and the same condition holds for each P-semiflow including place p_1. Then, the computation of an upper bound for the mean marking of places can be formulated in terms of an LPP as follows:

$$\begin{array}{rll} \overline{M}^{ub}(p) = & \text{minimize} & \overline{M}^{lb}(p) + Y^T \cdot (M_0 - \overline{M}^{lb}) \\ & \text{subject to} & Y^T \cdot C = 0; Y^T \cdot e_p = 1; Y \geq 0 \end{array}$$

where $e_p = (0, \dots, 0, \overset{p}{\smile}{1}, 0, \dots, 0)^T$, and the restriction $Y^T \cdot e_p = 1$ allows us to omit the denominator Y_p which is assumed to be non null.

The bound can also be computed from a dual version of the previous problem. Because LBFC nets are conservative, the dual problem is equivalent to the following one, that admits a nice direct interpretation.

Theorem 7.2 *For any live and bounded free choice net with a specification of the mean firing times associated with transitions and of the conflict resolution policy, it is not possible to assign PDFs to the transition firing times such that the mean marking of place p in steady-state is greater than*

$$\overline{M}^{ub}(p) = \begin{array}{ll} \textit{maximize} & M(p) \\ \textit{subject to} & B^T \cdot M = B^T \cdot M_0; \quad M \geq \overline{M}^{lb} \end{array}$$

where the rows of B^T are a basis of the left annullers of C.

In this problem, the maximum mean marking of place p is computed, subject to the following restrictions: the mean marking must satisfy the place invariant equations, and it must be greater than or equal to the lower bound computed in Theorem 7.1.

7.2 Maximum capacity of queues

An interesting information for the designer is the maximum capacity of queues that is needed for the execution of the processes from the fixed initial state. This information can be used for giving a correct dimension of the model implementation. For live and bounded free choice nets, it is possible to compute in polynomial time on the net size, the exact maximum marking that can be reached from the initial state in each place, solving an LPP. This is based on the fact that the behavioural bound of p, $B(p)$, is equal to the structural bound, $SB(p)$ (Theorem 3.6).

Because LBFC nets are conservative, the problem (LPP1) that defines $SB(p)$ can be easily rewritten leading to the following statement:

Theorem 7.3 *For LBFC nets, the reachable marking bound of places coincides with the structural marking bound obtained solving the following LPP:*

$$SB(p) = \begin{array}{ll} \textit{maximize} & M(p) \\ \textit{subject to} & B^T \cdot M = B^T \cdot M_0; \quad M \geq 0 \end{array}$$

with B^T a basis of the left annullers of C.

The reader is invited to compare the LPPs in Theorems 7.2 and 7.3. The first is more constrained ($M \geq \overline{M}^{lb} \geq 0$), therefore as expected, $\overline{M}^{ub}(p) \leq SB(p) = B(p)$.

7.3 Other computable bounds

Using fundamental laws of queueing theory [29], bounds for other performance figures can be computed. As an example, let us consider the computation of bounds for the mean response time at places.

The mean response time $\overline{R}(p_i)$ at a place p_i is the mean value of the sojourn time of a token in this place (i.e., sum of waiting plus service time). From the knowledge of upper and lower bounds for the throughput of transitions and for the mean marking of places, and applying Little's law, upper and lower bounds for the response time at places can be deduced as follows:

$$\begin{aligned} \overline{R}^{ub}(p_i) &= \frac{\overline{M}^{ub}(p_i)}{PRE(p_i) \cdot \vec{\sigma}^{lb}} \\ \overline{R}^{lb}(p_i) &= \frac{PRE(p_i) \cdot D \cdot F_k}{PRE(p_i) \cdot F_k} \end{aligned}$$

where $\vec{\sigma}^{lb}$ and $\overline{M}^{ub}$ are the bounds computed in previous sections.

8 Conclusions

Among the main achievements of this work the following can be stressed: (1) it is a starting point for a performance evaluation (i.e., quantitative) theory for free choice synchronized queueing networks, a model class that generalizes many proposals of QN extensions; (2) the theory is developed in a unified framework considering qualitative and quantitative properties; and (3) from the quantitative approach, classical and new qualitative fundamental properties of FC nets appear in a simple and straightforward way.

The extensive bibliography of this work may be surprising at a first glance, but it can be justified because one of our primary goals was to try to deeply bridge two active fields: Petri nets (in particular, free choice nets) qualitative theory and stochastic models (stochastic nets and extensions of queueing networks) theory. The benefits have been for both the qualitative and quantitative understanding of such models. From the qualitative point of view, some unexpected fundamental new results allow the linear algebra-based characterization of liveness in FC nets. These results strongly influenced the introduction of a new linear algebra based perspective of qualitative theory of LBFC nets [39]. An extension of the necessary condition in the rank theorem for general nets has been presented in [40]. From the quantitative (performance analysis) point of view, fast algorithms (polynomial complexity) allow to compute bounds for throughput for a class of synchronized QNs for which ergodicity is assured. From the above bounds and classical fundamental queueing theory laws, some derived performances (as queue bounds) can also be computed in polynomial time.

Among the "natural" extensions of this work, we can point out two, presently under consideration: relaxing the topology of nets (synchronized QNs) and considering open free choice synchronized monoclass queueing networks.

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