On liveness analysis through linear algebraic techniques

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DRAFT VERSION

Abstract

Proving properties of Place/Transition Nets through Linear Algebraic Techniques is very interesting because of the polynomial complexity of the algorithms used for this purpose. In this sense, many works have been devoted to the linear analysis of marking related properties (e.g. boundedness of the state space, mutual exclusions, etc.). Nevertheless, few results exist related to linear analysis of liveness properties. In this note, we investigate some applications of linear techniques to partial characterization of liveness properties. First, a necessary condition for *structural liveness* in structural bounded nets is presented. It is based on the rank of the incidence matrix. Finally, given an initial marking, some sufficient conditions for *dead transitions* and for *deadlock-freeness* are presented.

1   Introduction

The interest in parallel and distributed systems grows constantly according to the new domains of application of this kind of systems. One of the main problems arising from these systems is their complexity that implies a stressed necessity for analysis techniques of properties of good behaviour before the implementation.

Petri nets have been proved specially adequate to model parallel and distributed systems. Moreover, they have a well founded theory of analysis that allows to investigate a great number of properties of the system.

Two properties of a system that can be considered as paradigmatic are: *Boundedness*, related with the finiteness of the state space, and *Liveness* that concerns the infinite activity of all actions of the system from any reachable state.

The boundedness of a net system $\langle \mathcal{N}, M_0 \rangle$ characterizes the finiteness of the marking space and is a decidable property. Tipically, it is decided computing the *coverability graph* [FINK 90], but it is an exponential space hard problem. *Structural Boundedness (SB)* is a stronger condition in which all net systems that are built by defining an M_0 over $\mathcal{N}$ are asked to be bounded. Hoppefully, structural boundedness allows the following linear algebraic characterization [SILV 88]:

$$\mathcal{N} \text{ is SB } \Leftrightarrow \exists Y > 0, Y^T \cdot C \le 0$$

where C is the incidence matrix of the net.

Thus, *SB* can be characterized in *polynomial time* proving that there exists at least one solution of a linear system of inequalities. In order to prove this, Linear Programming techniques can be used by solving a *Linear Programming Problem (LPP)* that it is of polynomial time complexity [KARM 84]. Moreover, we can use the *simplex algorithm* to solve an *LPP* that even if it is of exponential complexity in theory, in practice it frequently becomes linear [SAKA 84]. For the above proposed system of linear inequalities, the existence of a solution can be proved through *phase I* of the simplex algorithm applied to the following *LPP*:

$$\begin{aligned} \text{maximize} \quad & 0^T \cdot Y \\ \text{subject to} \quad & Y^T \cdot C \le 0 \\ & Y \ge \mathbb{1} \end{aligned} \tag{1}$$

Phase I of the simplex works by computing (if there exists) a *basic feasible solution* of the set of constraints of the *LPP*.

The goal of this note is to improve the study of liveness property in structurally bounded nets by means of linear algebraic techniques. We firstly present a necessary condition computed through the rank of the incidence matrix of the net. After that, we present a sufficient condition for dead transitions and a sufficient condition for deadlock-freeness.

In this note we assume that the reader is familiar with concepts of P/T nets. The definitions and notations throughout the note follow [BRAM 83, SILV 85].

2 Previous Results

2.1 Conservativity and Consistency

In this subsection we present some classical necessary conditions for a net to be structurally live and structurally bounded. They will be used in the proofs of the results of section 3.

Theorem 2.1 (*Necessary condition for structural liveness*) *Let $\mathcal{N}$ be a connected P/T net.*

1) $\mathcal{N}$ is Structurally Live (SL) $\Rightarrow$ $\mathcal{N}$ Structurally Repetitive (SR).

2) $\mathcal{N}$ is Structurally Repetitive (SR) and Structurally Bounded (SB) $\Leftrightarrow$ $\mathcal{N}$ is Conservative (C_v) and Consistent (C_t).
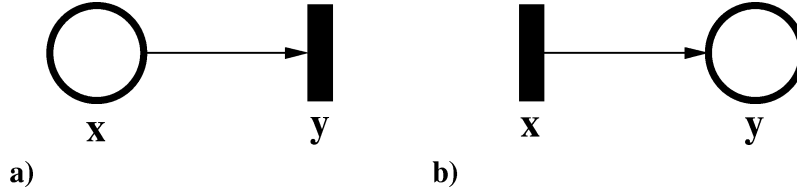
Figure 1: Illustrating the two cases in the proof theorem.

3) $\mathcal{N}$ is Conservative (C_v) and Consistent (C_t) $\Rightarrow \mathcal{N}$ is Strongly Connected (S_c).

Proof. Theorems 2.1.1 and 2.1.2 are classical and their proofs can be found in [BRAM 83, SILV 85], for example.

Proof of theorem 2.1.3. If $\mathcal{N}$ is not strongly connected it implies that there exist two nodes $x, y \in P \cup T$ such that x is connected to y by one directed arc but there is not a path from y to x. From this statement two alternative cases arise:

1) $x \in P$, $y \in T$ and $y \in x^{\bullet}$ but there is not a path from y to x (figure 1.a), or

2) $x \in T$, $y \in P$ and $y \in x^{\bullet}$ but there is not a path from y to x (figure 1.b).

The proof is done verifying that if $\mathcal{N}$ is not strongly connected then $\mathcal{N}$ is non-conservative (derived from Case 1) or $\mathcal{N}$ is non-consistent (derived from case 2). Therefore, the theorem is proved.

Let us consider Case 1 first. We try to build a vector $Y \geq 0$ such that $Y^T \cdot C = 0$ and $x \in \|Y\|$. In order to do that, we must combine the corresponding row of place x in the incidence matrix with each row of places belonging to $y^{\bullet}$ in such a way that the entry of the dot product $Y^T \cdot C$ corresponding to transition y is zero. In other words, to build the vector Y we proceed constructing all paths starting from x and ending in a place $p \in y^{\bullet}$. Iterating the scheme with respect to the new incidence matrix that contains the new rows obtained in the above combinations we obtain a vector $Y \geq 0$ such that $Y^T \cdot C = 0$ if there exists at least one path from y to x (since the entry of the dot product $Y^T \cdot C$ corresponding to transitions $^{\bullet}x$ will be zero if we go from y to x in the built path). But, this is not possible because the net is not strongly connected. Therefore the net is non-conservative since all vector $Y \geq 0$ such that $Y^T \cdot C = 0$ cannot include an entry corresponding to place x different from zero.

To prove Case 2 we can reason in a similar way to case 1 but now we proceed to build a vector $X \geq 0$ such that $C \cdot X = 0$ and $y \in \|X\|$. Therefore, we conclude that if the net is not strongly connected then the net is non-consistent. ∎

Assuming nets are connected, the former theorem can be rewritten in the following more compact way:

$$SL \text{ and } SB \Rightarrow SR \text{ and } SB \Leftrightarrow C_v \text{ and } C_t \Rightarrow S_c$$

The extremal parts of the above relations (i.e. SL and $SB \Rightarrow S_c$) represent a particular case of the proposition 11.4.7 in [SHIE 87]: A finite connected P/T net which

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is live and bounded for a marking M_0 is strongly connected. Nevertheless, this is non comparable with the second embedded statement: C_v and $C_t \Rightarrow S_c$.

The above theorem allows us to obtain the following obvious corollary.

Corollary 2.2 *Let* $\mathcal{N}$ *be a structurally live and structurally bounded net. Then* $rank(C) \leq min(m - 1, n - 1)$.

Example 2.3 shows that theorem 2.1 (and corollary 2.2) is a poor characterization of structural liveness and structural boundedness.

Example 2.3 *The four nets in figure 2 are conservative and consistent (thus they verify the rank formula of the corollary 2.2), but structurally non live.*

1) *Figure 2.a:* $rank(C) \leq min(n - 1, m - 1) = min(6, 4) = 4$ *and* $rank(C) = 4$

2) *Figure 2.b:* $rank(C) \leq min(n - 1, m - 1) = min(4, 6) = 4$ *and* $rank(C) = 4$

3) *Figure 2.c:* $rank(C) \leq min(n - 1, m - 1) = min(5, 3) = 3$ *and* $rank(C) = 3$

4) *Figure 2.d:* $rank(C) \leq min(n - 1, m - 1) = min(4, 4) = 4$ *and* $rank(C) = 4$

2.2   (Structurally) Implicit Places

In this subsection, implicit and structurally implicit place definitions are recalled in order to obtain some results which will be needed in section 3.

Let $\mathcal{N}$ be any net and $\mathcal{N}^p$ be the net resulting from adding a place p to $\mathcal{N}$. If M_0 is an initial marking of $\mathcal{N}$, M_0^p denotes the initial marking of $\mathcal{N}^p$. The incidence matrix of $\mathcal{N}$ is C and l_p is the incidence vector of place p.

Definition 2.4 (*Implicit place [SILV 85]*) *Given a net* $< \mathcal{N}^p, M_0^p >$, *the place* p *is Implicit (IP) iff* $L(\mathcal{N}^p, M_0^p) = L(\mathcal{N}, M_0)$ *(i.e. it preserves the firing sequences, thus liveness).*

Definition 2.5 (*Structurally implicit places [COLO 89b]*) *Given a net* $\mathcal{N}^p$, *the place* p *is Structurally Implicit (SIP) iff* $\forall M_0$, *there exists an* $M_0^p[p]$ *such that* p *is an IP in* $< \mathcal{N}^p, M_0^p >$.

The following theorem presents a necessary and sufficient condition for a place to be a SIP. The use of Convex Geometry Techniques allows a polynomial time computation algorithm to decide if a place is SIP (proofs and extensions can be found in [COLO 89b]).

Theorem 2.6 (*Linear Characterization of SIP [COLO 89b]*) *A place* p *is a SIP in* $\mathcal{N}^p$ *iff* $\exists Y \geq 0$ *such that* $Y^T \cdot C \leq l_p$.
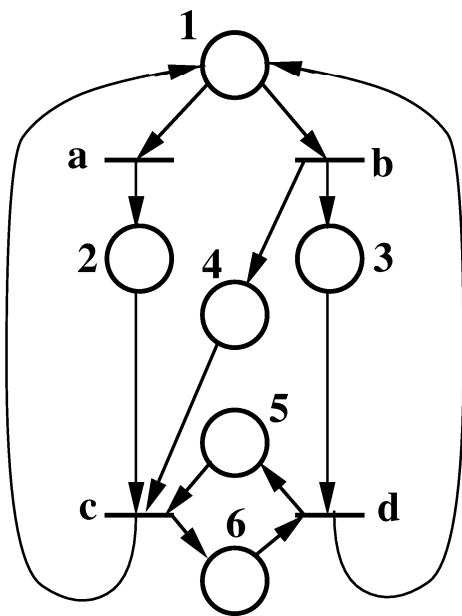
Marking structurally implicit places ($MSIP$) are a special class of $SIPs$. Their characterization is based on the existence of a vector $Y \geq 0$ such that the equality holds: $Y^T \cdot C = l_p$. The following results concern $MSIPs$ and they will be used in the proof of the main theorem of section 3.
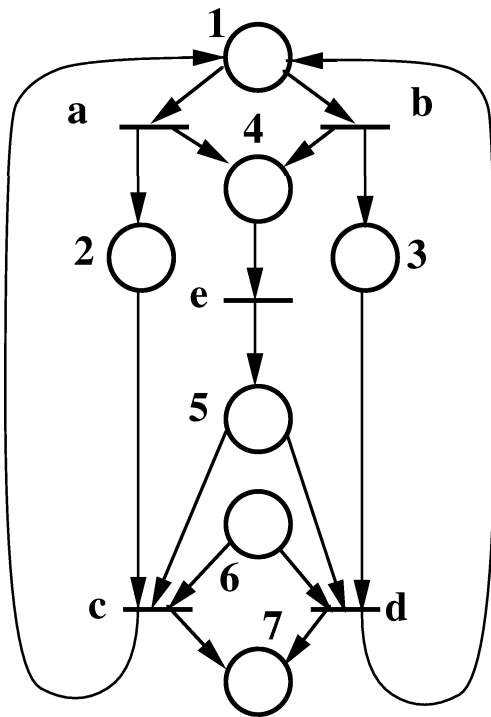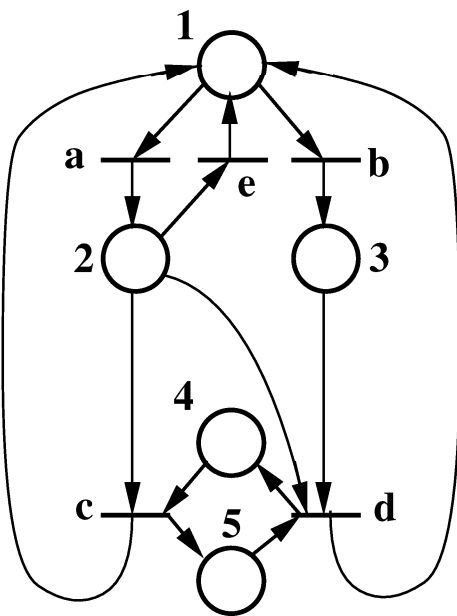
a)
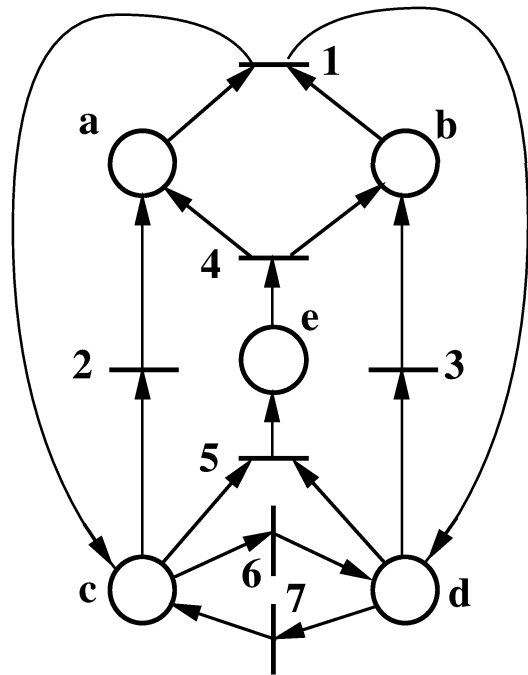
b)

c)

d)

Figure 2: Four consistent, conservative and structurally non live nets.

Corollary 2.7 *Let $\mathcal{N}$ be a conservative net. A place p is an MSIP in $\mathcal{N}^p$ iff it is a linear combination of other places: p is $MSIP \Leftrightarrow \exists Y$ such that $Y^T \cdot C = l_p$.*

> **Proof.** A place p is an $MSIP$ in $\mathcal{N}^p \Leftrightarrow \exists Y' \geq 0$ such that $Y'^T \cdot C = l_p$. The corollary follows because the net is conservative and then there exists a vector $Y_c > 0$ such that $Y_c^T \cdot C = 0$. Therefore, all vectors $Y = k \cdot Y_c + Y'$ with $k \in \mathbb{Z}$ verify $Y^T \cdot C = l_p$. ∎

Proposition 2.8 *If $\mathcal{N}$ is a conservative and structurally live net and p is a place such that $Y^T \cdot C = l_p$, then $\mathcal{N}^p$ is also conservative and structurally live.*

> **Proof.** If $\mathcal{N}$ is a conservative net then the place defined by the linear equality $Y^T \cdot C = l_p$ is an $MSIP$ (by corollary 2.7). Therefore, the addition of p preserves structural liveness. If $\mathcal{N}$ is a conservative net then there exists $Y_c \geq \mathbb{1}$ such that $Y_c^T \cdot C = 0$. We select a number $k \in \mathbb{N}$ such that $k \cdot Y_c - Y \geq 0$ (e.g. $k = max\{Y[i]|i = 1, \ldots, n\}$). From this last relation we obtain a p-semiflow of the net $\mathcal{N}^p$ that it includes the place p:

$$[(k \cdot Y_c - Y)^T | 1] \cdot \left[\begin{array}{c} C \\ l_p \end{array}\right] = 0, \text{ where } C^p = \left[\begin{array}{c} C \\ l_p \end{array}\right] \text{ is the incidence matrix of } \mathcal{N}^p$$

> Therefore the net $\mathcal{N}^p$ is also conservative. ∎

3   About structural liveness in structurally bounded nets

This section improves the necessary condition for a net to be structurally live and structurally bounded in theorem 2.1, adding an upper bound for the rank of the incidence matrix. The main stream for the proofs is a generalization of a scheme sketched in [CAMP 90]. Before presenting the main result we introduce a new concept.

Definition 3.1 *Two transitions t_a and t_b are said to be in equality conflict relation (ECR) iff $PRE[t_a] = PRE[t_b]$, where $PRE[t_a]$ and $PRE[t_b]$ are the pre-incidence functions of transitions t_a and t_b, respectively.*

Since ECR is based on the equality of vectors, it is an *equivalence relation* on the set of transitions. Each equivalence class will be called *equality conflict set (ECS)*. Let D be an ECS, the number $\delta_D = |D| - 1$ is called *number of non-redundant conflicts of D*. The *number of non-redundant conflicts of a net* is the sum of all δ_D corresponding to the $ECSs$ of the net. This number will be denoted as δ: $\delta = \sum_{D \in T/ECR} \delta_D$.

We present bellow the main theorem of this section.

Theorem 3.2 *Let $\mathcal{N}$ be a structurally live and structurally bounded net. Then $\mathcal{N}$ is conservative, consistent and $rank(C) \leq m - \delta - 1$; where C is the incidence matrix of $\mathcal{N}$, $m = |T|$, $n = |P|$ and δ is the number of non-redundant conflicts of the net.*
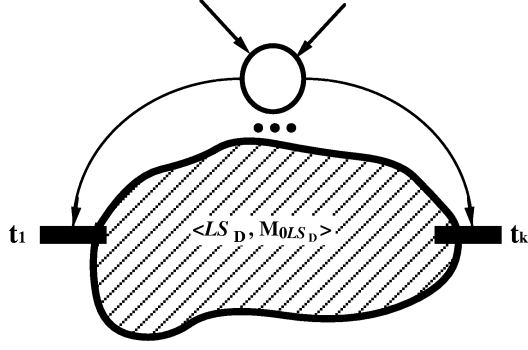
Figure 3: Introduction of a local scheduler at an *ECS*.

In order to prove the above theorem we previously present some lemmatas. The first lemma concerns the reduction of the non-determinism at equality conflicts, preserving the liveness property, by means of the merging of a special class of nets (*local schedulers*).

Let $D = \{t_i | i = 1, \ldots, \delta_D + 1\}$ be an *ECS* of the net $\mathcal{N}$. A *local scheduler for D* is a net, $\mathcal{LS}_D$, defined as (see figure 3):

$$\mathcal{LS}_D = (P_{\mathcal{LS}_D}, T_{\mathcal{LS}_D}, PRE_{\mathcal{LS}_D}, POST_{\mathcal{LS}_D})$$
$$T_{\mathcal{LS}_D} \cap T = D$$
$$T^{\bullet}_{\mathcal{LS}_D} \cup {}^{\bullet}T_{\mathcal{LS}_D} = P_{\mathcal{LS}_D}$$
$$P_{\mathcal{LS}_D} \cap P = \emptyset$$

Lemma 3.3 *Let $\mathcal{N}$ be a net and D an ECS of $\mathcal{N}$. Let $\mathcal{LS}_D$ be a local scheduler for D. If $\mathcal{N}$ and $\mathcal{LS}_D$ are structurally live in isolation, then the net obtained by merging the common transitions of $\mathcal{N}$ and $\mathcal{LS}_D$, $\mathcal{N}^{\mathcal{LS}_D}$, is structurally live.*

Proof. Let M_0 and $M_{0_{\mathcal{LS}_D}}$ be initial markings making live the nets $\mathcal{N}$ and $\mathcal{LS}_D$, respectively. Let $M_0^{\mathcal{LS}_D}$ be an initial marking of $\mathcal{N}^{\mathcal{LS}_D}$ such that its projection on P is M_0 and its projection on $P_{\mathcal{LS}_D}$ is $M_{0_{\mathcal{LS}_D}}$. Let $M^{\mathcal{LS}_D} \in R(\mathcal{N}^{\mathcal{LS}_D}, M_0^{\mathcal{LS}_D})$ and t a transition of $\mathcal{N}$. We prove that there exists a firing sequence, $\sigma^{\mathcal{LS}_D}$, in $< \mathcal{N}^{\mathcal{LS}_D}, M^{\mathcal{LS}_D} >$ that yields to a marking enabling t (i.e. the net $\mathcal{N}^{\mathcal{LS}_D}$ is live under $M_0^{\mathcal{LS}_D}$).

The projection of $M^{\mathcal{LS}_D}$ on P is a marking $M \in R(\mathcal{N}, M_0)$ from which there exists at least one $\sigma \in L(\mathcal{N}, M)$, yielding to a marking M' that enables t (because the net $\mathcal{N}$ is live). From this fact, three cases arise:

a) If σ does not contain any transition belonging to D then it is also firable in the net $\mathcal{N}^{\mathcal{LS}_D}$.

b) If σ contains one transition $t_a \in D$, that is $\sigma = \sigma_0 t_a \sigma_a$, then there exist $(\delta_D + 1)$ firable sequences from M of the form $\sigma_0 t_i \sigma_i$, $t_i \in D$, $i = 1, \ldots, \delta_D + 1$, that allow to reach a marking enabling t. This is because $\mathcal{N}$ is live and $M[\sigma_0\rangle M_D$; $\forall t_i \in D, M_D[t_i\rangle M_i \in R(\mathcal{N}, M_0)$ and $\forall i = 1, \ldots, \delta_D + 1$, $M_i[\sigma_i\rangle M_i'[t\rangle$. Therefore, at least one of the sequences $\sigma_0 t_i \sigma_i$ can be fired in $\mathcal{N}^{\mathcal{LS}_D}$: σ_0 and σ_i are firable according to the above case (a); at least one $t_i \in D$ is firable because $\mathcal{LS}_D$ is a live net (eventually, after the firing of some internal transitions of the local scheduler in order to enable t_i).
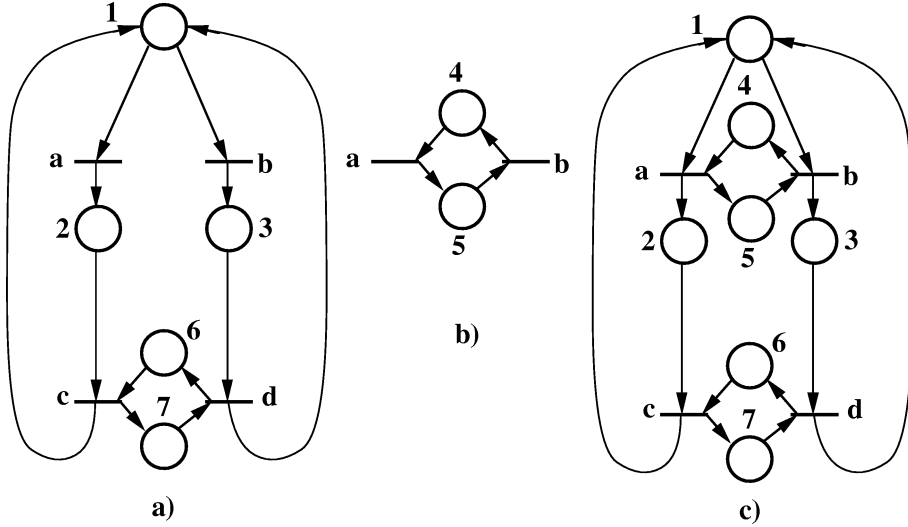
Figure 4: Counterexample to the converse of lemma 3.3.

c) If σ contains more than one transition of D we can find a firable sequence in $\mathcal{N}$ that it is firable in $\mathcal{N}^{\mathcal{LS}_D}$. This can be done by applying repeatedly the above case (b).

Liveness of transitions belonging to $\mathcal{LS}_D$ can be proved with similar arguments. Therefore, the net $\mathcal{N}^{\mathcal{LS}_D}$ is live under $M_0^{\mathcal{LS}_D}$ and then structurally live. ∎

Unfortunately, the converse of lemma 3.3 is not true. Let us consider, for instance, the structurally non live net in figure 4.a. The net of figure 4.b is a structurally live local scheduler for transitions a and b. The composition of the two nets is the net of figure 4.c that now is structurally live.

In the sequel, we consider a simple class of local schedulers called *regulation circuits*. These nets are used as a tool to proof the main theorem. Nevertheless, they are not the unique local schedulers that can be used for that purpose.

Let t_a and t_b be two transitions of $\mathcal{N}$ in equality conflict relation. A *regulation circuit for t_a, t_b* is a net $r_{ab} = < P_{r_{ab}}, T_{r_{ab}}, PRE_{r_{ab}}, POST_{r_{ab}} >$; where: $P_{r_{ab}} = \{p_{ab}, p_{ba}\}$, $T_{r_{ab}} = \{t_a, t_b\}$ and

$$^\bullet p_{ab} = \{t_a\}, p_{ab}^\bullet = \{t_b\}, PRE_{r_{ab}}[p_{ab}, t_b] = POST_{r_{ab}}[p_{ab}, t_a] = 1$$

$$^\bullet p_{ba} = \{t_b\}, p_{ba}^\bullet = \{t_a\}, PRE_{r_{ab}}[p_{ba}, t_a] = POST_{r_{ab}}[p_{ba}, t_b] = 1$$

The composition of $\mathcal{N}$ and r_{ab} (by merging the common transitions t_a, t_b) will be denoted as $\mathcal{N}^{r_{ab}}$. The incidence matrix of $\mathcal{N}^{r_{ab}}$ will be denoted as $C^{r_{ab}}$.

Let $\mathcal{N}$ be a net and $D = \{t_i | i = 1, \ldots, \delta_D + 1\}$ be an ECS. The net obtained from $\mathcal{N}$ by adding a regulation circuit per each pair of transitions $t_k, t_{k+1} \in D, k = 1, \ldots, \delta_D$ will be denoted $\mathcal{N}^{R_D}$ and its corresponding incidence matrix C^{R_D}. The net obtained by adding regulation circuits for all ECS as above will be denoted $\mathcal{N}^R$ and the corresponding incidence matrix C^R.

The following lemma presents some properties of $\mathcal{N}^{R_D}$ derived from the corresponding properties of $\mathcal{N}$.

Lemma 3.4 *Let $\mathcal{N}$ be a net and $D = \{t_i | i = 1, \ldots, \delta_D + 1\}$ be an ECS. If $\mathcal{N}$ is structurally live and structurally bounded then $\mathcal{N}^{R_D}$ is structurally live and structurally bounded.*

Proof. The set of regulation circuits added to $\mathcal{N}$ is a local scheduler for D. Let $r_{k,k+1}$ be a regulation circuit for $t_k, t_{k+1} \in D$. Because all pair of regulation circuits $r_{k,k+1}$ and $r_{k+1,k+2}$ share the transition t_{k+1}, the local scheduler is a strongly connected marked graph and therefore structurally live [MURA 89]. The net $\mathcal{N}$ is also structurally live and then, by lemma 3.3, the net $\mathcal{N}^{R_D}$ is structurally live.

All places of $\mathcal{N}$ are structurally bounded. Taking into account the definitions of $p_{t_k t_{k+1}}$ and $p_{t_{k+1} t_k}$ it is easy to see that $C^{R_D}[p_{t_k t_{k+1}}] + C^{R_D}[p_{t_{k+1} t_k}] = 0$ (i.e. the sum of the rows in the incidence matrix corresponding to these places is zero). Therefore all new places added to $\mathcal{N}$ are also structurally bounded and then $\mathcal{N}^{R_D}$ is structurally bounded. ∎

Lemma 3.5 *Let $D = \{t_i | i = 1, \ldots, \delta_D + 1\}$ be an ECS. If $\mathcal{N}$ is structurally live and structurally bounded, then*

$$min(m - 1, n + 2 \cdot \delta_D - 1) \geq rank(C^{R_D}) = rank(C) + \delta_D$$

Proof. $\mathcal{N}^{R_D}$ is structurally live and structurally bounded (lemma 3.4). Therefore, $rank(C^{R_D}) \leq min(m^{R_D} - 1, n^{R_D} - 1)$ (corollary 2.2); where $m^{R_D} = |T^{R_D}| = |T| = m$ and $n^{R_D} = |P^{R_D}| = |P| + |P_{r_{12}}| + \ldots + |P_{r_{kk+1}}| + \ldots + |P_{r_{\delta_D - 1 \delta_D}}| = n + 2 \cdot \delta_D$. So, we obtain: $rank(C^{R_D}) \leq min(m - 1, n + 2 \cdot \delta_D - 1)$.

Let $\mathcal{N}^{p_{t_2 t_1}}$ be a net obtained from $\mathcal{N}$ by adding the place $p_{t_2 t_1}$ belonging to the regulation circuit $r_{1,2}$. $\mathcal{N}^{p_{t_2 t_1}}$ is non-conservative because for all marking that enables the transitions of D we can decide to fire always the transition t_2 (i.e. the place $p_{t_2 t_1}$ is structurally unbounded). Taking into account the proposition 2.8 we conclude that there is not a vector Y such that $Y^T \cdot C = C^{p_{t_2 t_1}}[p_{t_2 t_1}]$ (i.e. the row vector $C^{p_{t_2 t_1}}[p_{t_2 t_1}]$ is linearly independent with respect to the row vectors of the incidence matrix of the net $\mathcal{N}$). Therefore, $rank(C^{p_{t_2 t_1}}) = rank(C) + 1$. If we add the place $p_{t_1 t_2}$ to the net $\mathcal{N}^{p_{t_2 t_1}}$ we obtain the net $\mathcal{N}^{r_{1,2}}$. This last net has the same rank that the net $\mathcal{N}^{p_{t_2 t_1}}$ because $C^{r_{1,2}}[p_{t_2 t_1}] = -C^{r_{1,2}}[p_{t_1 t_2}]$. Therefore, $rank(C^{r_{1,2}}) = rank(C) + 1$.

Let $\mathcal{N}^{R_{k-1}}$ be the net obtained from $\mathcal{N}$ by adding the regulation circuits $r_{1,2}, \ldots, r_{k-1,k}$. $\mathcal{N}^{R_{k-1}}$ verifies $rank(C^{R_{k-1}}) = rank(C) + (k - 1)$. We prove now that if we add the regulation circuit $r_{k,k+1}$ to the net $\mathcal{N}^{R_{k-1}}$ then $rank(C^{R_k}) = rank(C^{R_{k-1}}) + 1$.

We add the place $p_{t_{k+1} t_k}$ belonging to the regulation circuit $r_{k,k+1}$ to the net $\mathcal{N}^{R_{k-1}}$. This place is unbounded because for all marking that enables some transition of the set $\{t_1, \ldots, t_k\}$, t_{k+1} is also enabled at this marking and then we can decide to fire always the transition t_{k+1}. By proposition 2.8 the row vector $C^{R_k}[p_{t_{k+1} t_k}]$ is linearly independent with respect to the row vectors of the incidence matrix of the net $\mathcal{N}^{R_{k-1}}$. Therefore, $rank(C^{R_k}) = rank(C^{R_{k-1}}) + 1$ (because $C^{R_k}[p_{t_{k+1} t_k}] = -C^{R_k}[p_{t_k t_{k+1}}]$).

The number of added regulation circuits is δ_D, therefore $rank(C^{R_D}) = rank(C) + \delta_D$. ∎
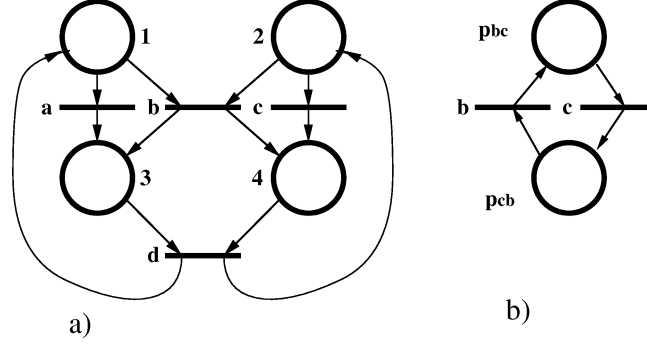
Figure 5: Structurally live and structurally bounded net

Now we prove the theorem 3.2.

Proof of theorem 3.2. $\mathcal{N}$ is conservative and consistent by theorem 2.1. If we add a local scheduler per each ECS of the net, we obtain a net denoted $\mathcal{N}^R$ that satisfies: $min(m-1, n+2\cdot\delta-1) \geq rank(C^R) = rank(C) + \delta$. Then, $rank(C) \leq min(m-\delta-1, n+\delta-1)$. Taking into account that the net $\mathcal{N}$ is structurally live and structurally bounded, by corollary 2.2, we also have that $rank(C) \leq min(m-1, n-1)$. Therefore, combining the two above upper bounds of $rank(C)$, we obtain: $rank(C) \leq min(m-\delta-1, n-1)$. But, $\mathcal{N}$ being conservative, $rank(C) \leq n-1$ and the theorem follows. ∎

The rank condition improves the result in theorem 2.1.2.

Example 3.6 *Let us consider the four nets in figure 2. Applying the rank condition of theorem 3.2, for three of them we conclude on non-liveness.*

1) *Figure 2.a: There exists only one conflict set: $D = \{a, b\}$; and $\delta = 1$. Taking into account that $rank(C) = 4$ we conclude: $4 = rank(C) > m - \delta - 1 = 3$ and then the net is structurally non live.*

2) *Figure 2.b: There exist two conflict sets: $D_1 = \{2, 6\}$, $\delta_1 = 1$; $D_2 = \{3, 7\}$, $\delta_2 = 1$ and $\delta = \delta_1 + \delta_2 = 2$. The application of the condition gives the following relations: $4 = rank(C) \leq m - \delta - 1 = 4$. Therefore, we cannot conclude about the structural non liveness (i.e. the condition is sufficient but non-necessary for structural non liveness).*

3) *Figure 2.c: $D = \{a, b\}$; and $\delta = 1$. Then, $3 = rank(C) > m - \delta - 1 = 2$. Therefore the net is structurally non live.*

4) *Figure 2.d: $D = \{a, b\}$; and $\delta = 1$. Then, $4 = rank(C) > m - \delta - 1 = 3$. Therefore the net is structurally non live.*

Example 3.6.2 shows that the proposed upper bound of the rank of the incidence matrix is not good enough for some nets. Nevertheless, if D is an ECS of $\mathcal{N}$, there is no other regulation circuit between transitions $t_i, t_j \in D, i < j$ and $i + 1 \neq j$ such that its addition to $\mathcal{N}^{R_D}$ increases the rank of its incidence matrix. In effect, the

place $p_{t_i t_j}$ is a linear combination of places belonging to existing regulation circuits: $l_{p_{t_i t_j}} = C^{R_D}[p_{t_i t_{i+1}}] + \ldots + C^{R_D}[p_{t_{j-1} t_j}]$. Therefore, the rank cannot be increased.

In order to improve this upper bound we must increase the constant δ. This can be eventually done by considering a class of conflicts greater than that of equality conflicts. Let us explore the (*dominating conflicts*), a "natural" generalization of equality conflicts. Two transitions t_a and t_b are said to be in dominating conflict relation (DCR) iff $PRE[t_a] \geq PRE[t_b]$ (i.e. t_a enabled $\Rightarrow t_b$ enabled; t_b dominates t_a).

Unfortunately, with this definition of conflicts is not possible to improve the rank condition in theorem 3.2. Effectively, the addition of a regulation circuit between two transitions in DCR does not preserves, in general, structural liveness (i.e. lemma 3.4 is not true, in general, with this kind of conflicts). To see this, we consider the net in figure 5.a. This net is $SB\&SL$ but if we add the regulation circuit of figure 5.b between the transitions b and c (both are in DCR), the resulting subnet is non-live. In effect, for any live marking of the original net and of the regulation circuit, in the composed net we can always apply a firing sequence such that $M[1] = M[4] = M[p_{bc}] = 0$:

$$
M_0 = \begin{pmatrix} M_0[1] \\ M_0[2] \\ M_0[3] \\ M_0[4] \\ M_0[p_{bc}] \\ M_0[p_{cb}] \end{pmatrix} [\sigma_1\rangle M' = \begin{pmatrix} 0 \\ M'[2] \\ M'[3] \\ M'[4] \\ 0 \\ M'[p_{cb}] \end{pmatrix} [(da)^{M'[4]}\rangle M'' = \begin{pmatrix} 0 \\ M''[2] \\ M''[3] \\ 0 \\ 0 \\ M'[p_{cb}] \end{pmatrix}
$$

and therefore a deadlock is reached. At this moment we do not know if the rank condition can be improved for general P/T nets.

The following result shows for *generalized extended free choice nets* (i.e. they are nets where ${}^\bullet t_i \cap {}^\bullet t_j \neq \emptyset \Leftrightarrow PRE[t_i] = PRE[t_j] \Leftrightarrow t_i, t_j \in ECR$) a stronger condition than that presented in theorem 3.2.

Theorem 3.7 *Let $\mathcal{N}$ be a structurally live and structurally bounded generalized extended free choice net. Then $\mathcal{N}$ is conservative, consistent and $rank(C) = m - \delta - 1$; where C is the incidence matrix of $\mathcal{N}$, $m = |T|$ and δ is the number of non-redundant conflicts of the net.*

   Proof. The rank equality condition holds if $\mathcal{N}^R$ has a unique (elementary) t-semiflow. So let us proof this condition.

   The number of t-semiflows of $\mathcal{N}^R$ is greater than or equal to 1 because it is consistent.

   We compute t-semiflows, $X \geq 0$ and $C \cdot X = 0$, applying the algorithm presented in [COLO 89a] to the net $\mathcal{N}^R$. To do so, we eliminate first the places $p_{t_i t_{i+1}}$ that connect two transitions in equality conflict relation (obviously, if we eliminate $p_{t_i t_{i+1}}$ we also eliminate $p_{t_{i+1} t_i}$ because $C^R[p_{t_i t_{i+1}}] = -C^R[p_{t_{i+1} t_i}]$). The elimination of $p_{t_i t_{i+1}}$ generates a unique new column that is a linear combination of the columns corresponding to t_i and t_{i+1}. In order to eliminate $p_{t_{i+1} t_{i+2}}$ we generate again a unique column that is a linear combination of the above added column and the column of t_{i+2}. If we repeat this procedure for all places $p_{t_i t_{i+1}}$ belonging to an ECS we obtain a unique new column in which all entries corresponding to places

of the local scheduler are zero. The non-null entries of this row are $^\bullet ECS \cup ECS^\bullet$. Applying this procedure for all ECS of the net we obtain a matrix in which there is a new column per ECS and all columns in the original net corresponding to transitions that do not belong to any ECS. This matrix can be interpreted as the incidence matrix of a structurally persistent net (there are no shared input places). Obviously, if a transition belongs to a t-semiflow, all output transitions of its ouput places must belong to the t-semiflow. Since the net is strongly connected (theorem 2.1.3) there exists at most one t-semiflow.

Therefore, applying the rank formula of lemma 3.5 with $rank(C^R) = m - 1$ we obtain: $rank(C) = m - \delta - 1$. ∎

Corollary 3.8 *Let $\mathcal{N}$ be a SL and SB free choice net. Then $\mathcal{N}$ is conservative, consistent and $rank(C) = m - 1 - (a - n)$; where C is the incidence matrix of $\mathcal{N}$, $m = |T|$, $n = |P|$ and a is the number of input arcs to transitions.*

> **Proof.** Let a_{ECS_i} be the number of input arcs to all transitions of the equality conflict set ECS_i and $n_{ECS_i} = |^\bullet ECS_i|$. Let a_t be the number of input arcs to a transition t that it does not belong to none ECS and $n_t = |^\bullet t|$.
>
> By the free choice property $\delta_{ECS_i} = |ECS_i| - 1 = a_{ECS_i} - n_{ECS_i}$. Therefore, $\delta = \sum_i (a_{ECS_i} - n_{ECS_i})$. For all transition t that does not belong to any ECS, $a_t - n_t = 0$. Therefore, $\delta = a - n$. Substituting $\delta = a - n$ in the formula of corollary 3.7 we obtain: $rank(C) = m - (a - n) - 1$. ∎

As the reader can easily check, the above formula does not apply even for extended free choice nets. In [CAMP 90] the statement of the above corollary is presented in a slightly different way: *Let $\mathcal{N}$ be a strongly connected structurally bounded free choice net; if $\mathcal{N}$ is structurally live then $rank(C) = m - (a - n) - 1$.* But for free choice nets, also the reverse can be proved. The importance of this result for free choice nets lies on the fact that several key results of free choice theory appear as corollaries. For example [CAMP 90, ESPA 90]: 1) the characterization of simultaneous structural liveness and structural boundedness in free choice nets is of polynomial complexity, and 2) the duality theorem.

4 About the liveness of a net system

In this section we consider the analysis of dead transitions and deadlock-freeness by means of linear techniques.

4.1 Dead transitions

In [MEMM 78, LASS 89] the following necessary condition for liveness is presented in the context of p-semiflows and analysis of Petri Nets: If a net system $\langle \mathcal{N}, M_0 \rangle$ is live then $\forall Y \geq 0$ such that $Y^T \cdot C = 0$, $Y^T \cdot (M_0 - PRE[t]) \geq 0, \forall t \in T$ is satisfied. Checking this property by observing p-semiflows is an exponential problem.

In this subsection we obtain a more general result that for conservative nets coincides with the above condition but in a form that can be checked in polynomial time. We

also obtain for this result an easy interpretation in net terms: If a net system $\langle \mathcal{N}, M_0 \rangle$ is live then all transitions are at least once firable.

Theorem 4.1 *Let $< \mathcal{N}, M_0 >$ be a marked P/T net. If the transition $t \in T$ is at least once firable then the following linear system has a solution:*

$$
\begin{aligned}
M &= M_0 + C \cdot \vec{\sigma} \\
M, \vec{\sigma} &\geq 0 \\
M &\geq PRE[t]
\end{aligned}
\tag{2}
$$

where PRE[t] is the column corresponding to transition t in the pre-incidence matrix.

    Proof. If t is at least once firable, there exists a marking $M \in R(\mathcal{N}, M_0)$ such that $M \geq PRE[t]$ (i.e. $\forall p \in {}^{\bullet}t, M[p] \geq PRE[p,t]$). Because M is a reachable marking, $M_0[\sigma > M \Rightarrow M = M_0 + C \cdot \vec{\sigma}, M, \vec{\sigma} \geq 0$.   ∎

The converse of the above theorem is not true because if the linear system has a solution, this can be non-reachable (i.e. it can be a spurious solution [COLO 89b]). Obviously, the negation of this theorem gives a sufficient condition for a transition t to be *dead*.

If the net is live then all transitions of the net are at least once firable, and for all t the above system has a solution. Therefore, the necessary condition for liveness requires in the worst case to solve $m = |T|$ linear systems.

The corollary presented below gives the dual system (in the Linear Programming sense) of the system in theorem 4.1. This alternative system is more general than the condition presented in [MEMM 78, LASS 89].

Corollary 4.2 *Let $< \mathcal{N}, M_0 >$ be a marked P/T net. If the transition $t \in T$ is at least once firable then the following linear system has no solution:*

$$
\begin{aligned}
Y^T \cdot C &\leq 0 \\
Y &\geq 0 \\
Y^T \cdot PRE[t] &> Y^T \cdot M_0
\end{aligned}
\tag{3}
$$

    Proof. The system of the corollary can be easily obtained from the system in theorem 4.1 by the direct application of the *Alternatives Theorem* [MURT 83].   ∎

If the above system has a solution this means that there exists a structurally bounded component (i.e. an invariant marking relation such as $Y^T \cdot M \leq Y^T \cdot M_0$) with less tokens than needed for firing the transition t from any reachable marking. If it has no solution it means that $\forall Y \geq 0$ such that $Y^T \cdot C \leq 0$, $Y^T \cdot PRE[t] \leq Y^T \cdot M_0$ is satisfied. This condition coincides with that in [MEMM 78, LASS 89] for conservative nets.

4.2 On deadlock-freeness

The *deadlock-freeness* property concerns the existence of some activity from any state. A deadlock in a net system characterizes the existence of a marking from which none transition is firable.

We build a linear necessary condition for the existence of a deadlock under a given marking M that contains $m = |T|$ "complex conditions", each one concerning the non-firability of the transitions of the net:

$$M = M_0 + C \cdot \vec{\sigma} \tag{4}$$
$$M \geq 0, \vec{\sigma} \geq 0$$
$$t_1 \text{ is not firable under } M$$
$$........$$
$$t_m \text{ is not firable under } M$$

Now, transition t_i is not firable under a marking M if the following condition holds:

$$(M[p_j] < PRE[p_j, t_i]) \text{ or ... or } (M[p_{j+k}] < PRE[p_{j+k}, t_i]) \text{ with } p_j, \ldots, p_{j+k} \in {}^{\bullet}t_i \tag{5}$$

Obviously, the above condition is non linear. If we consider the $m = |T|$ "complex conditions" in system (4) we need to solve in the worst case $|{}^{\bullet}t_1| \cdot |{}^{\bullet}t_2| \cdot \ldots \cdot |{}^{\bullet}t_{m-1}| \cdot |{}^{\bullet}t_m|$ linear systems. Each one of these systems is composed by the net state equation and $m = |T|$ inequalities of the form: $M[p_{i_j}] < PRE[p_{i_j}, t_i]$, $i = 1, \ldots, m$; where p_{i_j} is an input place of t_i.

Nevertheless, we can reduce the number of systems to solve considering the following rules:

1) If there exist two transitions t_1 and t_2 such that $PRE[t_1] \leq PRE[t_2]$, then all markings M that enable t_2 also enable t_1. Therefore, to verify that t_1 and t_2 are not firable from M is sufficient to verify that t_1 is not firable. Thus the number of systems to be solved is divided by $|{}^{\bullet}t_2|$.

   According with this rule, in system (4) we only need to consider one transition (the transition with the less number of input arcs) per each totally ordered set of transitions in dominating conflict relation.

2) Compute the structural marking bound (SB) of all places of the net by means of the following LPP [SILV 88]:

$$SB(p) = \begin{array}{ll} \text{maximize} & M[p] \\ \text{subject to} & M = M_0 + C \cdot \vec{\sigma} \\ & M \geq 0, \vec{\sigma} \geq 0 \end{array} \tag{6}$$

   If $SB(p) < PRE[p, t]$ then remove transition t from system (4) because it is a dead transition. In this case, the number of systems to solve is divided by $|{}^{\bullet}t|$.

3) If there exists a transition t such that $|{}^{\bullet}t| = k$ and $j < k$ places $p_j \in^{\bullet} t$ verify that $SB(p_j) = PRE[p_j, t]$, then the first j conditions in the "complex expression" (5) for t can be substituted by the following linear condition:

$$\sum_{j<k} M[p_j] < \sum_{j<k} PRE[p_j, t]$$

   This is because, $\forall M \geq 0$ such that $M = M_0 + C \cdot \vec{\sigma}, \vec{\sigma} \geq 0$, $\sum_{j<k} M[p_j] \leq \sum_{j<k} SB(p_j) = \sum_{j<k} PRE[p_j, t]$

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Therefore, the number of systems to solve is divided by $|^\bullet t| - (j - 1)$.

In the particular case in which all input places p of t verify $SB(p) = PRE[p, t]$, then the full complex condition can be substituted by the following unique linear inequality:

$$\sum_{p \in {}^\bullet t} M[p] < \sum_{p \in P} PRE[p, t]$$

In this case, the number of systems to solve is divided by $|^\bullet t|$.

In net systems where $\forall t \in T$, $\forall p \in {}^\bullet t$, $SB(p) = PRE[p, t]$ is satisfied, the existence of a deadlock can be done by means of a unique linear system of inequalities. In effect, from the above rule 3 each "complex condition" in system (4) is substituted by a linear inequality. Therefore, the resulting system is a unique linear system of inequalities: if there exists a deadlock in the net system $\langle \mathcal{N}, M_0 \rangle$, then the following linear system has a solution:

$$
\begin{aligned}
& M = M_0 + C \cdot \vec{\sigma} \qquad\qquad\qquad\qquad\qquad\qquad (7)\\
& M \geq 0, \vec{\sigma} \geq 0 \\
& \sum_{p \in {}^\bullet t_1} M[p] < \sum_{p \in P} PRE[p, t_1] \\
& \qquad \cdots\cdots\cdots \\
& \sum_{p \in {}^\bullet t_m} M[p] < \sum_{p \in P} PRE[p, t_m]
\end{aligned}
$$

or rewriting in a more compact way,

$$
\begin{aligned}
& M = M_0 + C \cdot \vec{\sigma} \qquad\qquad\qquad\qquad\qquad\qquad (8)\\
& M \geq 0, \vec{\sigma} \geq 0 \\
& PRE^T \cdot M < PRE^T \cdot \mathbb{1}
\end{aligned}
$$

Obviously, the non-existence of a solution for the above system is a sufficient condition to be the net deadlock-free. Because liveness and deadlock-freeness collapse in strongly connected structurally bounded free choice nets [BEST 87], system (8) gives a sufficient condition for liveness. Additionally, for structurally bounded nets with a unique elementary t-semiflow (mono-t-semiflow nets, [CAMP 89]), liveness and deadlock-freeness are equivalent, thus system (8) also gives a sufficient condition for liveness.

Therefore, in safe and strongly connected free choice nets it is a sufficient condition for liveness because deadlock-freeness is equivalent to liveness.

5   Conclusions

In this note we have illustrated the use Linear Algebraic Techniques to analyze some liveness properties.

A necessary condition for a net to be structurally live and structurally bounded has been presented. This condition can be applied to generalized Place/Transition nets and it is based on an upper bound of the rank of the incidence matrix of the net. As a by-product of the proof of this condition we have shown that the addition of a $SL\&SB$

local scheduler to an equality conflict set of a *SL&SB* net preserves structural liveness and structural boundedness. Unfortunately, the converse is not true.

Finally, sufficient conditions for dead transitions and for deadlock-freeness have been presented. In the first case it is based on a unique *LPP*. In the second, for some Place/Transition net systems we only need an *LPP*.

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On repairing lemma 3.3 from "Liveness Analysis through Linear Algebraic Techniques" by Campos, Colom, Silva.

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Definition 0.1 *A net* $\mathcal{N} =< T, P, PRE, POST >$ *is strongly reversible iff for every* $k \in \mathbb{N}$ *there exists a reversible marking* $M_R \in \mathbb{N}^P$ *such that* $M_R(p) \geq k$ *for every* $p \in P$.

Lemma 0.2 *Let* $\mathcal{N} =< T, P, PRE, POST >$ *be a structurally live net and* $D = \{t_1, \ldots, t_{\delta+1}\}$ *an ECS of* $\mathcal{N}$. *Let* $\mathcal{R}_D =< T_R, P_R, PRE_R, POST_R >$ *be a strongly reversible local scheduler for* D. *The net* $\mathcal{N} \cup \mathcal{R}_D$ *is structurally live.*

Proof. Take any live marking of $\mathcal{N}$, say $M_0 \in \mathbb{N}^P$. We shall construct a marking $M_R \in \mathbb{N}^{P_R}$ such that $M_0 \cup M_R$ is live in $\mathcal{N} \cup \mathcal{R}_D$.

Define $d(M, t) = min\{\#(D, \sigma) \mid M[\sigma t)\}$ for every $M \in [M_0]$ and $t \in T$. According to liveness of $\mathcal{N}$, $d(M, t)$ is well defined. Let $d(t) = max\{d(M, t) \mid M \in [M_0]\}$. The function $d(t)$ is well defined even if the net is unbounded, due to the following argumentation:

> Let's fix some $t \in T$. There are only finitely many minimal markings in the reachability set $[M_0]$ (Dickson's lemma). If $M' \geq M''$ then $d(M', t) \leq d(M'', t)$, so the maximum value of $d(M, t)$ is the value of some M_{min} from the finite set of minimal reachable markings. Hence $d(t)$ is finite.

So $d(t)$ is a bound precising that for every reachable marking $M \in [M_0]$, there exists a firing sequence σ such that $M[\sigma)$ activates t and there are at most $d(t)$ transitions of D in σ.

Define $M_{\mathcal{R}}^0 \in \mathbb{N}^{P_R}$: for each $p \in P$ we put $M_{\mathcal{R}}^0(p) = \sum_{t \in p^\bullet} d(t) \cdot PRE(p, t)$. $M_{\mathcal{R}}^0$ is defined in such a way that any word $\sigma_R \in D^*$ such that $\#(t, \sigma_R) \leq d(t)$, for each $t \in D$, can be run in $\mathcal{R}_D$ from $M_{\mathcal{R}}^0$.

Let $M_{\mathcal{R}} \in \mathbb{N}^{P_R}$ be any reversible marking of $\mathcal{R}_D$ such that $M_{\mathcal{R}} \geq M_{\mathcal{R}}^0$. We'll prove that $M_0 \cup M_{\mathcal{R}}$ is live in $\mathcal{N} \cup \mathcal{R}_D$.

Take any $M \in [M_0] \cup M^{\mathcal{R}}$ and any $t \in T$. We'll prove the existence of such $\sigma \in T^*$ that $M[\sigma)$ activates t. First we show that there exists $M_1 \in \mathbb{N}^{P \cup P_R}$ such that $M_1|_{P_R} = M_{\mathcal{R}}$.

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From the reversibility of M_R we conclude that there exists a sequence $\sigma_R \in D^*$ such that $M'_R[\sigma_R\rangle M_R$. Let $\sigma_R = t_{D_1}, \ldots, t_{D_k}$. In the original net it was possible to fire a sequence of transitions of $T - D$ from $M|_P$ such that the resulting marking activated the set of transitions D (otherwise $\mathcal{N}$ wouldn't be live). In the scheduled net $\mathcal{N} \cup \mathcal{R}_D$ the same sequence activating D can be fired starting from M, since the local scheduler affects only firing the transitions of D. Having D activated we choose t_{D_1} to be fired, resulting some new marking M'. Now again we can activate D firing the transitions from $T - D$. We fire then t_{D_2}, and so on until we fire t_{D_k} coming to some marking M^k. We conclude that $M^k|_{P_R} = M_R$, because the marking of places from P_R can be affected only by firing transitions from the set D.

In the net $\mathcal{N}$ we could fire a sequence σt starting from $M^k|_P$, such that $\#(t_i, \sigma) \leq d(t_i)$ for each $t_i \in D$. The same σ can be fired from M^k in the scheduled net $\mathcal{N} \cup \mathcal{R}_D$, since the places from P_R hold enough tokens in order not to constraint firing the transitions of D during the execution of σ. ∎