A Reachable Throughput Upper Bound for Live and Safe Free Choice Nets *

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Abstract

This paper addresses the computation of upper bounds for the throughput of transitions of live and safe deterministically or stochastically timed free choice nets. The obtained results are extensions of the marked graph case, presented by the authors in previous works. Polynomial complexity algorithms are derived using linear programming techniques. The obtained values are tight in the sense that, with the only knowledge of the net topology, the mean service times of transitions, and the routing rates at conflicts, is not possible to improve the bounds.

Topics: Timed and stochastic nets. Analysis and synthesis, structure, and behaviour of nets.

1 Introduction

One of the main problems in the actual use of timed and stochastic Petri net models for the performance evaluation of large systems is the explosion of the computational complexity of the analysis algorithms. In general, *exact performance results* are obtained from the numerical solution of a continuous time Markov chain [BT81, Mol81, FN85]. This exact computation is only possible for bounded nets (finite state space) and some very restricted cases of unbounded nets, and most of them under exponential assumption for the service time of transitions. And the worst of it is that the dimension of the state space of the embedded Markov chain grows exponentially with the net size.

Two complementary approaches to the derivation of exact measures for the analysis of distributed systems are the utilization of approximation techniques and the computation of bounds. Performance bounds are useful in the preliminary phases of the design of a system, in which many parameters are not known accurately. Several alternatives for those parameters should be quickly evaluated, and rejected those that are clearly bad. Exact (and even approximate) solutions would be computationally very expensive. Bounds become useful in these instances since they usually require much less computation effort.

Inside the domain of Petri nets, many works exist related to the performance evaluation in the case of deterministically timed models, mainly for strongly connected marked graphs [Ram74, Sif78, RH80, Mag84, Mur85]. Extensions to non-ordinary nets have been presented in the case of deterministic timing [Hil88].

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Concerning stochastic nets, only a few works exist related with the computation of performance bounds. In [Mol85], M. Molloy noted that the average token flows in an ordinary Markovian network at steady-state are conserved. Therefore, a series of *flow balance equations* can be written. Token flows are conserved in places so the sum of all flows into a place equals the sum of all flows out of the place. On the other hand, all token flows on the input and output arcs of a transition are equal. These equations determine the average token flows in the cycles of the net to within a constant. This constant cannot be determined without Markovian analysis at the reachability graph level. However, limit flows when the number of tokens tends to infinity can be computed. In order to do that, bottleneck transitions must be first located. Then, the actual flow through a bottleneck transition is (under saturation conditions) equal to its potential firing rate.

It is well-known that the conservation of flows presented by M. Molloy is not only valid for Markovian nets. In fact, some of most important laws of queueing theory hold under very general assumptions. These general situations are considered in our work, and some fundamental laws taken from queueing theory (such as Little's formula) are applied to stochastic Petri net models.

S. Bruell and S. Ghanta [BG85] developed algorithms for computing upper and lower bounds for the throughput of a restricted subclass of generalized stochastic Petri nets (with immediate and exponentially timed transitions, [AMBC84]). The considered nets include *control tokens* to model a physical restriction, such as semaphores, which is not a design parameter. The rest of tokens of such nets, grouped in *classes*, correspond to the notion of a job or customer in a monoclass queueing network, and its number is treated as a parameter of the net. The upper and lower bounds on throughput are computed hierarchically estimating maximum and minimum time of the path followed by each class of jobs.

In the paper of S. Islam and H. Ammar [IA89], methods to compute upper and lower bounds for the steady-state token probabilities of a subclass of generalized stochastic Petri nets are presented. The considered nets are obliged to admit a *time scale decomposition*. This means that the transitions of the net are supposed to be divided into two classes: slow and fast transitions, with several orders of magnitude of difference in the duration of activities. Moreover, the subnets obtained after removing all slow transitions with their input and output arcs must be conservative and admit a reversible initial marking. The computation is based on *near-completely decomposability* of Markov chains.

The particular case of strongly connected marked graphs has been studied in [CCCS89]. The bounds obtained for this subclass of Petri nets are computable in polynomial time on the size of the net model. Moreover, both upper and lower bounds on throughput are tight, in the sense that for any marked graph model it is possible to define families of stochastic timings such that the steady-state performances of the timed Petri net models are arbitrarily close to either bound.

An extension to classes of nets behaviourally "similar" to strongly connected marked graphs is studied in [CCS91]. A characteristic of these nets is the existence of a unique consistent firing count vector. They can be obtained from two non-disjoint subclasses of live and bounded nets, named *persistent* and *mono-T-semiflow* nets. These nets are either decision-free or such that the decision policy at effective conflicts does not change the *vector of visit ratios for transitions*, which is needed for the computation of performance bounds (mono-T-semiflow nets allow concurrency and decision, in a particular way). Both the upper and lower bounds on throughput are independent of any assumption on the probability distribution of the delay associated with transitions, and their values can be computed based on the knowledge of the averages. Even more, the upper bound in case of persistent nets has been shown to be reachable.

The computation of a reachable lower bound for the steady-state performance of live and safe free choice nets with deterministic or stochastic timing of transitions is considered in this paper. The results presented here are extensions to live and safe free choice nets of the performance bounds for strongly connected marked graphs developed in [CCCS89], and constitute an improvement of those in [CCS90, Cam90], for the case of safe free choice nets.

The paper is organized as follows. In section 2, the basic notation of Petri net models as well as some preliminary considerations related to the introduction of a timing interpretation in the model are presented. In section 3, a result taken from [CCS90] is recalled in which Little's law [Lit61] and P-semiflows are applied for the derivation of a *linear programming problem* [Mur83]. Its optimum solution gives a lower bound for the *mean interfiring time of a transition*, defined as the average time between two consecutive firings (inverse of the throughput), of timed or stochastic live and bounded free choice nets. This problem includes structural information of the net by means of the global incidence matrix. All parameters defining stochastic interpretation are summarized in the vector of *average service demands for transitions* (products of visit ratios by mean service times), which can be efficiently computed for live and bounded free choice nets [CCS90].

The bound derived in section 3 is, in general, non-reachable. A reachable lower bound for the mean interfiring time of transitions is obtained in section 4 for the case of safe nets, by computing the bound for a behaviourally related strongly connected marked graph. This computation is very inefficient because of the size of the derived marked graph. A polynomial computation is presented in section 5. The idea is the following: a reachable bound for strongly connected marked graphs was derived in [CCCS89] using the circuits of the net. A natural extension of circuits of marked graphs for the case of free choice nets can be found in the framework of graph theory: *multisets of circuits*. From this approach, a lower bound for the mean interfiring time can be derived that is reachable for some distribution functions of service times of transitions with arbitrary mean values and for some deterministic conflict resolution policy, with arbitrary long run rates.

In section 6, we justify why the method developed for safe nets cannot be directly extended for bounded (non-safe) nets. Some concluding remarks are presented in section 7.

2 Preliminary concepts

We assume the reader is familiar with the structure, firing rules, and basic properties of net models (see [Mur89] for a recent survey). Let us recall some notation here: $\mathcal{N} = \langle P, T, Pre, Post \rangle$ is a net with $n = |P|$ *places* and $m = |T|$ *transitions*.

If Pre and $Post$ incidence functions take values in $\{0, 1\}$, $\mathcal{N}$ is said to be *ordinary*.

PRE, $POST$, and $C = POST - PRE$ are $n \times m$ matrices representing the Pre, $Post$, and global incidence functions.

The *pre-* and *post-sets* of a transition $t \in T$ are defined respectively as ${}^\bullet t = \{p | Pre(p,t) > 0\}$ and $t^\bullet = \{p | Post(p,t) > 0\}$. The *pre-* and *post-sets* of a place $p \in P$ are defined respectively as ${}^\bullet p = \{t | Post(p,t) > 0\}$ and $p^\bullet = \{t | Pre(p,t) > 0\}$.

Marked graphs are ordinary nets such that $\forall p \in P : |{}^\bullet p| = |p^\bullet| = 1$. *Free choice* nets are ordinary nets such that $\forall p \in P : |p^\bullet| > 1 \Rightarrow {}^\bullet(p^\bullet) = \{p\}$.

Vectors $X \geq 0$, $C \cdot X = 0$ ($Y \geq 0$, $Y^T \cdot C = 0$) represent *T-semiflows* (*P-semiflows*), also called consistent (conservative) components. The *support* of T-semiflows (P-semiflows) is defined by $||X|| = \{t \in T | X(t) > 0\}$ ($||Y|| = \{p \in P | Y(p) > 0\}$). A (T- or P-) semiflow I has *minimal support* iff there exist no other semiflow I' such that $||I'|| \subset ||I||$.

M (M_0) is a *marking* (*initial marking*). Finally, σ represents a *fireable sequence*, while $\vec{\sigma}$ is the *firing count vector* associated to σ. If M is *reachable* from M_0 (i.e. $\exists \sigma$ s.t. $M_0[\sigma\rangle M$), then $M = M_0 + C \cdot \vec{\sigma} \geq 0$ and $\vec{\sigma} \geq 0$.

The introduction of a timing specification is essential in order to use Petri net models for performance evaluation of distributed systems. We consider nets with *deterministically or stochastically* timed transitions with *one phase* firing rule, i.e., a timed enabling (called the *service time* of the transition) followed by an atomic firing. The service times of transitions are supposed to be mutually independent and time independent.

In order to avoid the coupling between resolution of conflicts and duration of activities, we suppose that transitions in conflict are *immediate* (they fire in zero time). Decisions at these conflicts are taken according to *routing rates* associated with immediate transitions (*generalized stochastic Petri nets* [AMBC84, AMBCC87]). In other words, each subset of transitions $\{t_1, \ldots, t_k\} \subset T$ that are in conflict in one or several reachable markings are considered immediate, and the constants $r_1, \ldots, r_k \in \mathbb{N}^+$ are explicitly defined in the net interpretation in such a way that when $t_1, \ldots, t_k$ are enabled, transition t_i $(i = 1, \ldots, k)$ fires with probability (or with long run rate, in the case of deterministic conflicts resolution policy) $r_i / (\sum_{j=1}^{k} r_j)$. Note that the routing rates (which are considered rational) are assumed to be strictly positive, i.e., all possible outcomes of any conflict have a non-null probability of firing. This fact guarantees a *fair* behaviour for the non-autonomous Petri nets that we consider.

3 A first lower bound for the mean interfiring time of live and bounded nets and its limitations

In this section, we recall the lower bound for the mean interfiring time of transitions, defined as the average time between two consecutive firings, of live and bounded free choice nets, presented in [CCS90]. It is computed by solving a linear programming problem which includes structural information by means of the incidence matrix of the net. On the other hand, the stochastic interpretation is summarized in the vector of average service demands for transitions, computed from mean service times and visit ratios for transitions. Mean service times are supposed to be given with the model, while visit ratios can be efficiently computed for live and bounded free choice nets.

3.1 Computation of the visit ratios for transitions

Let us denote by s_i, $i = 1, \ldots, m$, the arbitrary mean service times of transitions, and by $\overline{M}$ and $\vec{\sigma}^*$ the *limit average marking* and the *limit vector of transition throughputs* defined as [FN85]:

$$\overline{M} \stackrel{\text{def}}{=} \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} M_u \, du < \vec{\infty}, \text{ a.s.} \tag{1}$$

and

$$\vec{\sigma}^* \stackrel{\text{def}}{=} \lim_{\tau \to \infty} \frac{\vec{\sigma}_\tau}{\tau} < \vec{\infty}, \text{ a.s.} \tag{2}$$

where M_τ and $\vec{\sigma}_\tau$ represent the marking and the firing processes of the net, respectively.

The existency of the limits $\overline{M}$ and $\vec{\sigma}^*$, which is called *weak ergodicity* of the marking and firing processes (see [CCCS89]), is assured for live and bounded free choice nets:

Theorem 3.1 [CCS90] *Let $\langle \mathcal{N}, M_0 \rangle$ be a live and bounded free choice net with deterministic or stochastic service times of transitions. Then, both the marking and the firing processes of $\langle \mathcal{N}, M_0 \rangle$ are weakly ergodic.*

In order to compute the lower bounds for the mean interfiring time of transitions, defined as the inverse of the throughput, presented in [CCS90], it is necessary to obtain before the *relative throughputs* of transitions that, in order to approach with queueing theory terminology, we call *visit ratios*. The vector of visit ratios, normalized for instance for transition t_j, is:

$$\vec{v}^{(j)} = \frac{1}{\vec{\sigma}^*(t_j)} \, \vec{\sigma}^* = \Gamma_{(j)} \, \vec{\sigma}^* \tag{3}$$

where $\Gamma_{(j)}$ is called the mean interfiring time of t_j, i.e., the inverse of its throughput.

For live and bounded free choice nets, the vector of visit ratios for transitions can be computed in polynomial time, from the net structure and the routing rates at conflicts. This computation takes into account that the vector of visit ratios must be a T-semiflow, i.e., $C \cdot \vec{v}^{(j)} = 0$. Additionally, the components of $\vec{v}^{(j)}$ must verify the following relations with respect to the routing rates for each subset of transitions $T' = \{t_1, \dots, t_k\} \subset T$ in structural conflict (called structural conflict set):

$$\begin{aligned} r_2 \vec{v}^{(j)}(t_1) - r_1 \vec{v}^{(j)}(t_2) &= 0 \\ r_3 \vec{v}^{(j)}(t_2) - r_2 \vec{v}^{(j)}(t_3) &= 0 \\ &\dots \\ r_k \vec{v}^{(j)}(t_{k-1}) - r_{k-1} \vec{v}^{(j)}(t_k) &= 0 \end{aligned} \tag{4}$$

Expressing the former homogeneous system of equations in matrix form: $R_{T'} \cdot \vec{v}^{(j)} = 0$, where $R_{T'}$ is an $(k-1) \times m$ matrix. Now, by considering all structural conflict sets $T_1, \dots, T_r$: $R \cdot \vec{v}^{(j)} = 0$, where R is a matrix:

$$R = \begin{pmatrix} R_{T_1} \\ \vdots \\ R_{T_r} \end{pmatrix} \tag{5}$$

Therefore, the following theorem can be stated.

Theorem 3.2 [CCS90] *Let $\langle \mathcal{N}, M_0 \rangle$ be a live and bounded free choice net. Let C be the incidence matrix of $\mathcal{N}$, and R the matrix previously defined. Then, the vector of visit ratios $\vec{v}^{(j)}$ normalized, for instance, for transition t_j can be computed from C and R solving the following linear system of equations:*

$$\begin{pmatrix} C \\ R \end{pmatrix} \cdot \vec{v}^{(j)} = 0, \quad \vec{v}^{(j)}(t_j) = 1 \tag{6}$$

Corollary 3.1 *The computation of the vector of visit ratios for transitions for live and bounded free choice nets is polynomial on the net size.*

The following section presents the computation of a lower bound for the mean interfiring time using the vector of visit ratios.

3.2 A lower bound for the mean interfiring time

Little's theorem [Lit61] can be applied to each place of a weakly ergodic net. Denoting as $\overline{M}(p_i)$ the limit average number of tokens at place p_i, $\vec{\sigma}^*$ the limit vector of transition throughputs, and $\overline{R}(p_i)$ the average time spent by a token within the place p_i (average response time at place p_i), the above mentioned relationship is stated as follows (see [FN85]):

$$\overline{M}(p_i) = (PRE[p_i] \cdot \vec{\sigma}^*)\ \overline{R}(p_i) \tag{7}$$

where $PRE[p_i]$ is the i^{th} row of the pre-incidence matrix of the underlying Petri net, thus $PRE[p_i] \cdot \vec{\sigma}^*$ is the output rate of place p_i.

In the above equation $\vec{\sigma}^*$ is known except for a scaling factor (see equation (3) and theorem 3.2). The average response time $\overline{R}(p_i)$ at places with more than one output transition is null because such transitions are considered immediate. For the places p_i with only one output transition, the average response time can be expressed as sum of the *average waiting time due* *to a possible synchronization* in the output transition and the mean service time associated with that transition. Thus the average response times can be *lowerly bounded* from the knowledge of the mean service times of transitions, s_i, $i = 1, \dots, m$, and the following system of inequalities can be derived from (7):

$$\Gamma_{(j)}\overline{M} \geq PRE \cdot \vec{D}^{(j)} \tag{8}$$

which relates the mean interfiring time $\Gamma_{(j)}$ of transition t_j, the vector $\overline{M}$ of limit average markings, and the vector $\vec{D}^{(j)}$ of *average service demands* for transitions, with components $\vec{D}^{(j)}(t_i) \stackrel{\text{def}}{=} s_i \vec{v}^{(j)}(t_i)$, $i = 1, \dots, m$.

We remark that vector $\vec{D}^{(j)}$ can be efficiently computed for live and bounded free choice nets, if mean service times s_i are given, because the vector of visit ratios $\vec{v}^{(j)}$ can be derived for such nets by solving a linear system of equations (cfr. theorem 3.2).

The limit average marking $\overline{M}$ is unknown. However, taking the product with a P-semiflow Y (i.e., $Y^T \cdot C = 0$, thus $Y^T \cdot M_0 = Y^T \cdot M = Y^T \cdot \overline{M}$ for all reachable marking M), the following inequality can be derived:

$$\Gamma_{(j)} \geq \max_{Y \in \{P-semiflow\}} \frac{Y^T \cdot PRE \cdot \vec{D}^{(j)}}{Y^T \cdot M_0} \tag{9}$$

The previous lower bound can be formulated in terms of a *fractional programming problem* [Mur83] and later, after some considerations, transformed into a linear programming problem:

Theorem 3.3 [CCS90] *For live and bounded free choice nets, a lower bound for the mean interfiring time of transition t_j can be computed by the following linear programming problem:*

$$\begin{array}{rll} \Gamma_{(j)} \geq \Gamma_{(j)}^{PS} = & \text{maximum} & Y^T \cdot PRE \cdot \vec{D}^{(j)} \\ & \text{subject to} & Y^T \cdot C = 0 \\ & & Y^T \cdot M_0 = 1 \\ & & Y \geq 0 \end{array} \tag{LPP1}$$

We remark that the computation of the above bound for live and bounded free choice nets has polynomial complexity on the net size. This is because the computation of vector $\vec{D}^{(j)}$ is polynomial (by corollary 3.1) and because linear programming problems can be also solved in polynomial time [Kar84].

For live and bounded marked graphs (a subclass of free choice nets), the bound derived from theorem 3.3 has been shown to be reachable in the following sense:

Theorem 3.4 [CCCS89] *For any live and bounded marked graph with arbitrary values of mean and variance for transition service times, the lower bound for the mean interfiring time obtained from (LPP1) cannot be improved.*

Unfortunately, this is not the case for live and safe free choice net systems. Let us consider, for instance, the system depicted in figure 1. Let s_3 and s_4 be the mean service times associated with t_3 and t_4, respectively. Let t_1, t_2, and t_5 be immediate transitions (i.e., they fire in zero time). Let $q, 1-q \in (0,1)$ be the routing probabilities defining the resolution of conflict at place p_1. The vector of visit ratios normalized for t_5 is

$$\vec{v}^{(5)} = (q, 1-q, q, 1-q, 1)^T \tag{10}$$

All P-semiflows can be generated by non-negative linear combinations of:

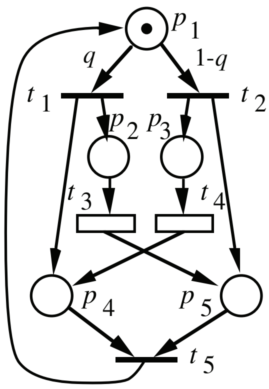

Figure 1: The lower bound for the mean interfiring time given by (LPP1) is non-reachable.

$$\begin{aligned} Y_1 &= (1,1,0,0,1)^T \\ Y_2 &= (1,0,1,1,0)^T \end{aligned} \tag{11}$$

Then, applying the problem (LPP1) to this net, the following lower bound for the mean interfiring time of transition t_5 is obtained:

$$\Gamma_{(5)} \geq \max\{qs_3, (1-q)s_4\} \tag{12}$$

while the actual mean interfiring time for this transition is

$$\Gamma_{(5)} = qs_3 + (1-q)s_4 \tag{13}$$

independently of the higher moments of the probability distribution functions associated with transitions t_3 and t_4. Therefore, the bound given by theorem 3.3 is non-reachable for the net in figure 1.

In the next section, a new method for the computation of a reachable lower bound for the mean interfiring time of transitions for live and safe free choice nets is derived. It is presented through the derivation of a marked graph which is behaviourally equivalent for a given deterministic conflict resolution policy.

4 A reachable bound for live and safe nets

In this section, let us consider live and safe free choice nets with arbitrary service times associated with transitions. The conflicts resolution policy is also arbitrary, but with some given routing rates. In fact, without loss of generality, we can restrict to deterministic resolution policies, which for safe free choice nets give the same performance than any probabilistic routing, in steady-state.

First, we give an algorithm to derive a live and safe marked graph which is behaviourally equivalent to the live and safe free choice net with deterministic routing. For this marked graph, well-known results [CCCS89] can be applied for the computation of bounds. After that, we interpret the computation of bounds for the behaviourally equivalent marked graph considering some collections of circuits, or *multisets* of the original net.

4.1 Derivation of a behaviourally equivalent marked graph

A deterministic resolution of the conflict between two transitions t_1 and t_2 is a rule that fixes which transition of them will be authorized to fire at the successive markings enabling both.

Thus, in some sense, the resulting *interpreted net* can be considered as a *conflict-free net*.

In the next paragraph, we present an algorithm for the computation of bounds for a live and safe free choice net $\langle \mathcal{N}, M_0 \rangle$ with deterministic routing, based on the fact that the behaviour of a safe conflict-free net can be represented by means of an equivalent marked graph [Ram74], for which the results of [CCCS89] can be applied.

Step 0. From the given deterministic resolution policy, compute the vector of visit ratios $\vec{v}^{(j)}$ in the net system $\langle \mathcal{N}, M_0 \rangle$, using the theorem 3.2.

Step 1. Steady-state markings must be *home states*. Let M_h be one of the home states (there always exist some for live and safe free choice nets [BV84]), and select it as the initial marking (i.e., $\langle \mathcal{N}, M_h \rangle$ is *reversible*).

Step 2. Apply the algorithm presented in [Ram74], with the initial marking of Step 1, in order to compute a behaviourally equivalent marked graph of safe conflict-free nets, with the following modifications: (1) each time one place enables more than one transition, select the transition authorized by the deterministic resolution policy; (2) select one *slice* [Ram74] of the behaviour graph (among those that occur repeatedly, according to Lemma 3.4.2 in [Ram74]) including the same places marked at the initial home state and such that the number of instances of each transition in the *frustum* [Ram74] is the same multiple of its corresponding entry in the vector of visit ratios.

Step 3. Compute the lower bound for the mean interfiring time of the marked graph obtained in Step 2, in which all instances of a same transition have a service time equal to that of the original one. The computation is made by solving the linear programming problem (LPP1). (Observe that the vector of visit ratios for the marked graph is $\vec{v} = \mathbb{1}$, where $\mathbb{1}$ denotes a vector with all entries equal to 1, but the number of instances of each transition is equal to the same multiple of its corresponding entry in the vector of visit ratios computed in Step 0.)

Step 4. A lower bound for the mean interfiring time of a given transition in the original net is computed by dividing the value obtained in Step 3 by the number of instances of this transition in the derived equivalent marked graph.

Observe that the smallest behaviourally equivalent marked graph that can be derived with previous algorithm is obtained by firing a sequence of transitions of the original net whose firing count vector is a multiple of the vector of visit ratios (let us denote as $\vec{v}$) such that: all its components are integer and their greatest common divisor es equal to 1. On the other hand, from the deterministic routing assumption follows that the only repetitive sequences of the interpreted net are such having a multiple of the vector of visit ratios as firing count vector. Therefore, for a given transition, the number of instances of it in the behaviourally equivalent marked graph is equal to its corresponding entry in vector $\vec{v}$. This is the reason why in Step 4 of above algorithm the value obtained from (LPP1) in Step 3 is divided by the number of instances of the considered transition.

It must be pointed out that since the marked graph derived in Step 2 is behaviourally equivalent to the original free choice net with deterministic conflict resolution policy then, in particular, their exact mean interfiring times are equal. Therefore, the lower bound for the mean interfiring time of transitions of the original free choice net (with the given deterministic conflicts resolution policy) can be derived from the mean interfiring time of the marked graph, after a normalization operation (dividing by the number of instances of the selected transition).

The bound computed for the marked graph by means of (LPP1) given by theorem 3.3 is reachable (see theorem 3.4). This provides a method for the computation of a reachable lower bound for the mean interfiring time of transitions for live and safe free choice nets.

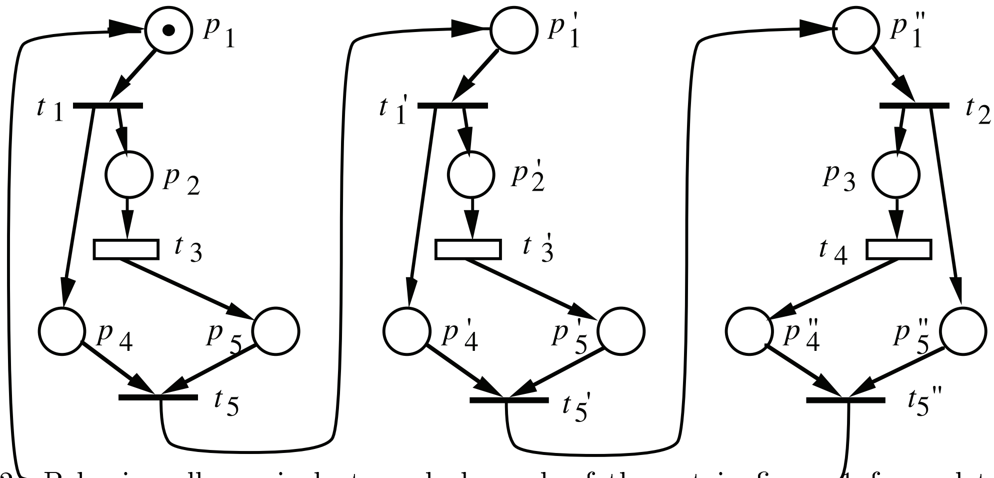

Figure 2: Behaviourally equivalent marked graph of the net in figure 1 for a deterministic resolution of conflict (the routing associated with t_1 is equal to twice the routing associated with t_2).

As an example, let us consider again the live and safe free choice net depicted in figure 1. Suppose that the deterministic conflict resolution policy at place p_1 is: "authorize twice transition t_1, then once transition t_2, and repeat it" (i.e., the routing rate for transition t_1 is twice the routing rate of t_2). The application of Step 0 gives the vector $\vec{v}^{(5)} = (2/3, 1/3, 2/3, 1/3, 1)^T$. The initial marking is already a home state and therefore the output of Step 1 is verified. The application of Step 2 gives the behaviourally equivalent marked graph depicted in figure 2. The lower bound for the mean interfiring time of the derived marked graph (which is equal for all its transitions) is $2s_3 + s_4$, according to Step 3. Finally, the application of Step 4 leads to a lower bound for the mean interfiring time of transition t_5 in the free choice net of figure 1 equal to $(2s_3 + s_4)/3$ (the one of the marked graph divided by the number of instances of transition t_5). Observe that this value coincides with that given by (13) with $q = 2/3$.

As can be seen in the example, the proposed method can become very expensive in amount of memory and computational time. Just consider, as an example, the net in figure 1 but with routing rates of transitions t_1 and t_2 being 11 and 21, only a bit different from the considered before (1 and 2, respectively). In this case the equivalent marked graph would have 96 transitions and 128 places !

4.2 Interpretation of previous result

This section is devoted to analyze the method to compute the lower bound for the mean interfiring time of the equivalent marked graph derived from the original safe free choice net. This is done in order to translate the underlying structural property to an equivalent structural property on the free choice net. This property is used in the next section to derive a polynomial method to compute the lower bound of the mean interfiring time for the original net without the generation of the marked graph.

It is known [Ram74, RH80, Mur85, CCCS89] that the problem of finding a lower bound for the mean interfiring time of transitions in a strongly connected stochastic marked graph can be solved looking at the cycle times associated with each minimal P-semiflow (circuits for marked graphs) of the net, considered in isolation (in fact, the simplex method used to solve the problem of Step 3 proceeds in this way). These cycle times can be computed making the summation of the mean service times of all the transitions involved in the P-semiflow, and dividing by the number of tokens present in it. Therefore, the performance computation of Step 3 looks at the circuits of the behaviourally equivalent marked graph.

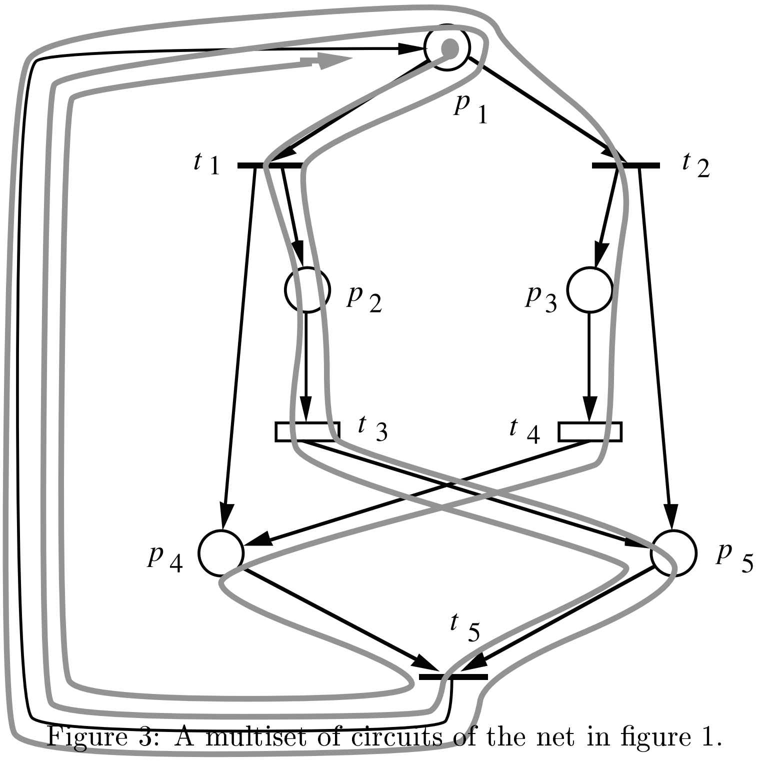

Figure 3: A multiset of circuits of the net in figure 1.

On the other hand, a circuit of the marked graph is composed by one or several instances of circuits of the original free choice net. This collection of circuits, including one or several instances of each circuit, is called *multiset* of circuits, and will be formally defined in the next section.

For the previous example, these multisets are

$$\begin{array}{rcl} \mathcal{M}_1 & = & \{\langle p_1 t_1 p_4 t_5\rangle, \langle p_1 t_1 p_4 t_5\rangle, \langle p_1 t_2 p_3 t_4 p_4 t_5\rangle\} \\ \mathcal{M}_2 & = & \{\langle p_1 t_1 p_4 t_5\rangle, \langle p_1 t_1 p_4 t_5\rangle, \langle p_1 t_2 p_5 t_5\rangle\} \\ \mathcal{M}_3 & = & \{\langle p_1 t_1 p_2 t_3 p_5 t_5\rangle, \langle p_1 t_1 p_2 t_3 p_5 t_5\rangle, \langle p_1 t_2 p_3 t_4 p_4 t_5\rangle\} \\ \mathcal{M}_4 & = & \{\langle p_1 t_1 p_2 t_3 p_5 t_5\rangle, \langle p_1 t_1 p_2 t_3 p_5 t_5\rangle, \langle p_1 t_2 p_5 t_5\rangle\} \\ \mathcal{M}_5 & = & \{\langle p_1 t_1 p_2 t_3 p_5 t_5\rangle, \langle p_1 t_1 p_4 t_5\rangle, \langle p_1 t_2 p_3 t_4 p_4 t_5\rangle\} \\ \mathcal{M}_6 & = & \{\langle p_1 t_1 p_2 t_3 p_5 t_5\rangle, \langle p_1 t_1 p_4 t_5\rangle, \langle p_1 t_2 p_5 t_5\rangle\} \end{array} \quad (14)$$

The reader can notice that multisets $\mathcal{M}_5$ and $\mathcal{M}_6$ (that we will call *non-minimal*) need not be considered in order to obtain the slowest path because if $\langle p_1 t_1 p_2 t_3 p_5 t_5\rangle$ is selected for the first time, it will be selected again instead of $\langle p_1 t_1 p_4 t_5\rangle$. We remark also that circuits $\langle p_1 t_1 p_4 t_5\rangle$ and $\langle p_1 t_1 p_2 t_3 p_5 t_5\rangle$ appear twice in multisets $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$, and $\mathcal{M}_4$, while circuits $\langle p_1 t_2 p_3 t_4 p_4 t_5\rangle$ and $\langle p_1 t_2 p_5 t_5\rangle$ appear only once, according to the routing rates associated with transitions t_1 and t_2. As an example, multiset $\mathcal{M}_3$ is depicted in figure 3. It can be interpreted as a path as follows: (1) a token, initially placed at p_1 enables transitions t_1 and t_2; (2) transition t_1 is authorized for firing, according to the given conflict resolution policy; (3) after the firing of t_1, the token splits into two tokens; (4) we follow one of them; for instance, the one that places at p_2; (5) after the firing of t_3 and t_5, it returns to p_1; (6) according to the conflict resolution policy, t_1 is authorized once more; (7) we follow the same path than in steps 4 and 5, until the token returns to p_1 again; (8) now, transition t_2 is autorized; (9) the path $\langle t_2 p_3 t_4 p_4 t_5\rangle$ is followed; (10) the situation now is the same than in step 1, so the previous steps can be executed ad infinitum.

The mean interfirng time of execution of previous path if the multiset of circuits $\mathcal{M}_3$ is considered in isolation is equal to the execution time of the corresponding isolated circuit

$\langle p_1 t_1 p_2 t_3 p_5 t_5 p_1' t_1' p_2' t_3' p_5' t_5' p_1'' t_2 p_3 t_4 p_4'' t_5'' \rangle$ of the marked graph depicted in figure 2, and it is, in general, a lower bound for the exact mean interfiring time. Therefore, a lower bound for the mean interfiring time can be computed taking the maximum among the mean interfiring time of execution of those multisets of circuits satisfying the routing rates, considered in isolation. In the particular case of marked graphs, since no decision exists for such nets, multisets of circuits were reduce to circuits, and these could be algebraically characterized as P-semiflows of the net [CCCS89].

The next step is to construct another net (with only a linear size increase of the original size) for which the previously considered multisets of circuits of live and safe free choice nets can be algebraically characterized (in fact, computed as P-semiflows). This can be done in a similar way to that presented in [Lau87] for the polynomial computation of the *minimal traps* of a net. In the next section we formalize the concept of multiset of circuits and present a net transformation for the efficient computation of mean interfiring time on multisets.

5 Polynomial computation: multisets of circuits

A *multiset* is a collection of elements that may contain several copies of an element. More formally, if $\mathcal{S}$ is a set, a multiset $\mathcal{M}$ of elements of $\mathcal{S}$ is an application $\mathcal{M} : \mathcal{S} \to \{0, 1, \ldots\}$.

If $\mathcal{N} = \langle P, T, Pre, Post \rangle$ is a Petri net and $\mathcal{M}$ a multiset of circuits of $\mathcal{N}$ (in what follows we write circuit instead of minimal circuit):

- $\mathcal{M}(y)$ denotes the number of circuits of $\mathcal{M}$ which pass through the node $y \in P \cup T$.
- $\mathcal{M}(p,t)$ (respectively $\mathcal{M}(t,p)$) denotes the number of circuits of $\mathcal{M}$ which pass through the arc (p,t) (respectively (t,p)), if $Pre(p,t) > 0$ (respectively $Post(p,t) > 0$).

In the next definitions, we limit the class of multisets of circuits to those that will correspond exactly with the circuits of the behaviourally equivalent marked graph that can give the optimum of the problem (LPP1).

Definition 5.1 ($\mathcal{R}$–multiset of circuits) *Let $\mathcal{N} = \langle P, T, Pre, Post \rangle$ be a net, $\mathcal{R}$ the definition of routing rates at conflicts, and $\mathcal{M}$ a non empty multiset of circuits of $\mathcal{N}$. $\mathcal{M}$ is called an $\mathcal{R}$–multiset of circuits iff for all $p \in P$ such that $|p^\bullet| > 1$ and $\mathcal{M}(p) > 0$: $r_j \mathcal{M}(p,t_i) = r_i \mathcal{M}(p,t_j)$, for all $t_i, t_j \in p^\bullet$, where r_i and r_j are the routing rates of transitions t_i and t_j in the conflict at place p.*

$\mathcal{R}$–multiset of circuits will be abreviated to $\mathcal{R}$–mc. The above definition constraints the multiset to contain the different circuits of the net according to the visit ratios for transitions derived from the routing rates at conflicts.

We define now the concept of set of places covered by an $\mathcal{R}$–mc and introduce *minimal $\mathcal{R}$–mcs.*

Definition 5.2 (Support of $\mathcal{R}$–multisets of circuits) *The support of an $\mathcal{R}$–mc $\mathcal{M}$ is the set of nodes $||\mathcal{M}|| \subseteq P \cup T$ covered by $\mathcal{M}$.*

Definition 5.3 (Minimal $\mathcal{R}$–multisets of circuits) *An $\mathcal{R}$–mc is called minimal iff*

a) its support does not contain the support of an $\mathcal{R}$–mc as a proper subset and

b) if $\mathcal{M}'$ is an $\mathcal{R}$–mc with $||\mathcal{M}'|| = ||\mathcal{M}||$, then $\mathcal{M}(y) \leq \mathcal{M}'(y)$, $\forall y \in P \cup T$.

In the above definition we denote the support with the same symbol than the support of a vector, as they are closely related concepts. We consider that the context eliminates confusion.

The consideration of minimal $\mathcal{R}$–mc discards the possibility of including two circuits that contain different output places of a *fork* transition (a transition with more than one output place). This constraint is not a problem in order to find the slowest path (with deterministic service times) because if a given output place of the fork transition is selected for the first time, it must be selected also the rest of the times.

Lemma 5.1 *Let $\langle \mathcal{N}, M_0 \rangle$ be a live and safe free choice net with deterministic conflict resolution policy, $\langle \mathcal{N}^{MG}, M_0^{MG} \rangle$ its behaviourally equivalent marked graph derived in previous section, and $\mathcal{M}$ a multiset of circuits of $\mathcal{N}$. $\mathcal{M}$ is a minimal $\mathcal{R}$–mc of $\mathcal{N}$ iff*

1. *There exists a circuit (minimal P-semiflow) c^{MG} of $\mathcal{N}^{MG}$ such that for all $t_1, t_2 \in c^{MG}$ instances of the same $t \in T$, then $c^{MG} \cap t_1^\bullet$ and $c^{MG} \cap t_2^\bullet$ are instances of the same $p \in P$, and*
2. *There exists an integer $k \geq 1$ such that $\mathcal{M}(x) = k.i_x$ where i_x is the number of instances of x in c^{MG}, for all $x \in P \cup T$.*

Proof sketch. Let c^{MG} be a circuit of $\mathcal{N}^{MG}$. If c^{MG} contains several instances of a place of $\mathcal{N}$, find a subpath of c^{MG} that begins at an instance p_1 of a given place $p \in P$ and ends at another instance p_2 of the same place and such that it does not include more than one instance of any other place (its existency is obvious). This path correspond with a circuit of the net $\mathcal{N}$. Substitute the subpath of c^{MG} by the single place p_1. Repeat the procedure of finding subpaths which correspond with circuits of the original net until c^{MG} has been reduced to a circuit without more than one instance of any place. It corresponds with a circuit of $\mathcal{N}$. Therefore, to each circuit of $\mathcal{N}^{MG}$ corresponds a multiset of circuits of $\mathcal{N}$. Moreover, by the method of derivation of $\mathcal{N}^{MG}$ (taking into account the deterministic routing at conflicts) the multiset of circuits of $\mathcal{N}$ corresponding with a circuit of $\mathcal{N}^{MG}$ is an $\mathcal{R}$–mc. If the number of copies of all the circuits in the multiset is a multiple of a given integer k, we consider the multiset obtained after dividing all the number of copies by k, and this multiset (which is also an $\mathcal{R}$–mc) verifies condition (b) of definition of minimality. Now, if condition (1) of the lemma is assumed for c^{MG}, condition (a) of minimality of the derived multiset follows.

Conversely, let $\mathcal{M}$ be a minimal $\mathcal{R}$–mc of $\mathcal{N}$. Then there exists a *non-minimal* circuit c of $\mathcal{N}$ with the same support than $\mathcal{M}$ and such that $c(p) = \mathcal{M}(p)$ for all $p \in ||\mathcal{M}|| \cap P$. Moreover, by minimality of $\mathcal{M}$, the circuit c verifies a condition analogue to (1), that is: if $t \in c$ then there exists only one $p \in t^\bullet$ such that $p \in c$. By the method of derivation of $\mathcal{N}^{MG}$ from $\mathcal{N}$ and since the original multiset was an $\mathcal{R}$–mc, there exists a path c^{MG} of $\mathcal{N}^{MG}$ equal (except instances) to the circuit c. Finally, either the obtained path of $\mathcal{N}^{MG}$ is a circuit that verifies (1) and $\mathcal{M}(x) = i_x$ (where i_x is the number of instances of x in c^{MG}), for all $x \in P \cup T$, or there exists a circuit of $\mathcal{N}^{MG}$ that consists of k repeated instances of the path c^{MG}, and that circuit verifies also (1) and (2). ■

Now, we define an expansion of a given live and safe stochastic free choice net with deterministic resolution of conflicts which allows a polynomial computation of its minimal $\mathcal{R}$–mcs.

Definition 5.4 (Expansion of a stochastic net) *Let $\mathcal{N} = \langle P, T, Pre, Post \rangle$ be a free choice net. The expanded net of $\mathcal{N}$, denoted as $\widehat{\mathcal{N}}$, is obtained from $\mathcal{N}$ after the following steps:*

Step 1. (Lautenbach expansion) Let $\overline{\mathcal{N}}$ be initially equal to $\mathcal{N}$. Replace each shared place $p_s \in P$ (i.e., such that $|^\bullet p_s| > 1 \vee |p_s^\bullet| > 1$) as follows (see figure 4):

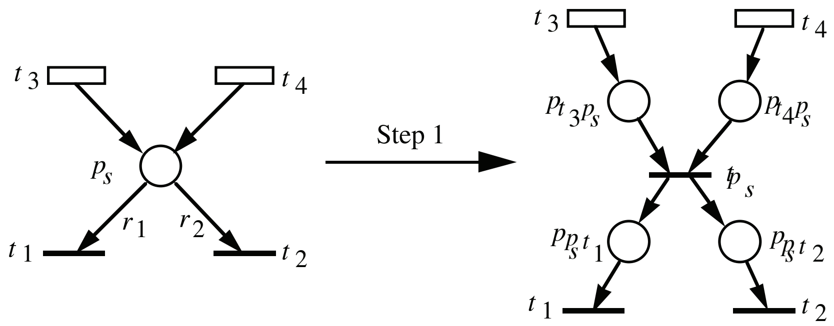

Figure 4: The replacement of a shared place: Step 1.

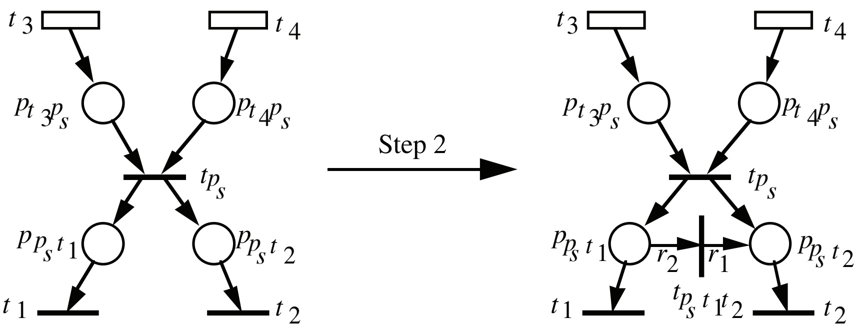

Figure 5: Additional constraint for an output-shared place: Step 2.

$$\begin{aligned}
&\overline{P} := (\overline{P} \setminus \{p_s\}) \cup \textstyle\bigcup_{t \in {}^\bullet p_s} p_{tp_s} \cup \bigcup_{t \in p_s^\bullet} p_{p_s t} \\
&\overline{T} := \overline{T} \cup t_{p_s} \\
&\overline{Pre}(p_{tp_s}, t_{p_s}) = 1,\ \forall t \in {}^\bullet p_s \\
&\overline{Post}(p_{p_s t}, t_{p_s}) = 1,\ \forall t \in p_s^\bullet \\
&\overline{Post}(p_{tp_s}, t) = Post(p_s, t),\ \forall t \in {}^\bullet p_s \\
&\overline{Pre}(p_{p_s t}, t) = Post(p_s, t),\ \forall t \in p_s^\bullet
\end{aligned}$$

Step 2. Derive a new net $\widehat{\mathcal{N}}$ from $\overline{\mathcal{N}}$ as follows: for each output-shared place $p_s \in P$ (i.e., such that $|p_s^\bullet| > 1$) and for each pair of output transitions t_1, t_2 of p_s add the transition $t_{p_s t_1 t_2}$ as follows (see figure 5):

$$\begin{aligned}
&\widehat{T} := \overline{T} \cup \{t_{p_s t_1 t_2}\} \\
&\widehat{Pre}(p_{p_s t_1}, t_{p_s t_1 t_2}) = r_2 \\
&\widehat{Post}(p_{p_s t_2}, t_{p_s t_1 t_2}) = r_1
\end{aligned}$$

where r_1, r_2 are positive integer numbers proportional to the routing rates associated with t_1, t_2 in the conflict at place p_s.

Step 3. Associate to each $t \in \widehat{T}$ the parameter $\widehat{s}(t)$ such that: $\widehat{s}(t_i) = s_i$ if $t_i \in \widehat{T} \cap T$ (where s_i is the mean service time of transition t_i in the original net), and $\widehat{s}(t) = 0$ if $t \in \widehat{T} \setminus T$.

For marked graphs, a graph theoretical concept (circuit) is related with another one of algebraic nature (minimal P-semiflow). Now, multisets of circuits are related with (non necessarily minimal) P-semiflows, for marked graphs.

Lemma 5.2 [Lau87] *Let $\mathcal{N} = \langle P, T, Pre, Post \rangle$ be a marked graph. Then Y is a P-semiflow of $\mathcal{N}$ iff there exists a multiset $\mathcal{M}$ of circuits of $\mathcal{N}$ such that $Y(p) = \mathcal{M}(p)$ for all $p \in P$.*

Now, the following result can be derived from lemma 5.2.

Theorem 5.1 *Let $\mathcal{M}$ be a multiset of circuits of a free choice net $\mathcal{N}$. $\mathcal{M}$ is a minimal $\mathcal{R}$–mc of $\mathcal{N}$ iff there exists a minimal P-semiflow $\widehat{Y}$ of the expanded net $\widehat{\mathcal{N}}$ such that:*

1. *For all $p \in P$ that is not a shared place then $\mathcal{M}(p) = \widehat{Y}(p)$.*
2. *For all $p \in P$ that is a shared place then $\mathcal{M}(p, t_i) = \widehat{Y}(p_{pt_i})$ and $\mathcal{M}(t_j, p) = \widehat{Y}(p_{t_jp})$ for all $t_i \in p^\bullet$ and for all $t_j \in {}^\bullet p$.*

The paragraphs below contain all technical details in order to prove this theorem. Previously, we introduce a simple definition and a technical lemma.

Definition 5.5 *Let $\mathcal{N} = \langle P, T, Pre, Post\rangle$ be a free choice net, $\widehat{\mathcal{N}}$ its corresponding expanded net, and $\widehat{Y}$ a P-semiflow of $\widehat{\mathcal{N}}$. The restricted support of $\widehat{Y}$ respect to places P of the original net $\mathcal{N}$, $||\widehat{Y}||_{\mathcal{N}}$, is given by:*

(a) If p is non-shared, then $p \in ||\widehat{Y}||_{\mathcal{N}} \Longleftrightarrow p \in ||\widehat{Y}||$.

(b) If p is shared, then $p \in ||\widehat{Y}||_{\mathcal{N}}$ if and only if all places p_{pt} resulting from the expansion of the output arcs of p belong to $||\widehat{Y}||$.

Now, as in [Lau87], the following result can be derived from lemma 5.2.

Lemma 5.3 *Let $\mathcal{M}$ be a multiset of circuits of a free choice net. $\mathcal{M}$ is an $\mathcal{R}$–mc of $\mathcal{N}$ iff there exists a P-semiflow $\widehat{Y}$ of $\widehat{\mathcal{N}}$ such that:*

1. *For all $p \in P$ that is non-shared $\mathcal{M}(p) = \widehat{Y}(p)$.*
2. *For all $p \in P$ that is a shared place $\mathcal{M}(p, t_i) = \widehat{Y}(p_{pt_i})$ and $\mathcal{M}(t_j, p) = \widehat{Y}(p_{t_jp})$ for all $t_i \in p^\bullet$ and for all $t_j \in {}^\bullet p$.*

Proof. Let $\widehat{Y}$ be a P-semiflow of $\widehat{\mathcal{N}}$. $\widehat{Y}$ is also a P-semiflow of $\overline{\mathcal{N}}$ because $\overline{\mathcal{N}}$ and $\widehat{\mathcal{N}}$ differ only in the transitions added in Step 2 of definition 5.4. By construction, the net $\overline{\mathcal{N}}$ is a marked graph and therefore (by lemma 5.2) there exists a multiset $\overline{\mathcal{M}}$ of circuits of $\overline{\mathcal{N}}$ such that for all $p \in \overline{P}$: $\widehat{Y}(p) = \overline{\mathcal{M}}(p)$.

Let us suppose that $\overline{\mathcal{M}}(t_p) > 0$, where t_p is generated in the expansion of an output shared place p and $t_p^\bullet = \{p_{pt_i} \mid t_i \in p^\bullet, 1 \leq i \leq v\}$ In $\widehat{\mathcal{N}}$, between two places $p_{pt_j}, p_{pt_{j+1}}$ $(1 \leq j \leq v-1)$ there exists a transition $t_{pt_jt_{j+1}}$ that verifies $Pre(p_{pt_j}, t_{pt_jt_{j+1}}) = r_j$ and $Post(p_{pt_{j+1}}, t_{pt_jt_{j+1}}) = r_{j+1}$ (see Step 2 in definition 5.4). Therefore, $\widehat{Y}$ verifies that $r_j\widehat{Y}(p_{pt_{j+1}}) = r_{j+1}\widehat{Y}(p_{pt_j})$. This implies that $r_j\overline{\mathcal{M}}(p_{pt_{j+1}}) = r_{j+1}\overline{\mathcal{M}}(p_{pt_j})$.

Finally, it is easy to see that, to $\overline{\mathcal{M}}$ corresponds a multiset of circuits $\mathcal{M}$ of $\mathcal{N}$ with $||\mathcal{M}|| \cap P = ||\widehat{Y}||_{\mathcal{N}}$, such that $\mathcal{M}(p, t_i) = \overline{\mathcal{M}}(p_{pt_i})$, $1 \leq i \leq v$ (being p a shared place), $\mathcal{M}(t_j, p) = \overline{\mathcal{M}}(p_{t_jp})$ and for all non-shared place $\mathcal{M}(p) = \overline{\mathcal{M}}(p)$. Therefore, $\mathcal{M}$ is an $\mathcal{R}$–mc.

The converse implication can be obtained by reversing the previous arguments. ∎

From the above lemma, we can deduce the following obvious result:

Corollary 5.1 *Let $\mathcal{M}$ be an $\mathcal{R}$–mc of a free choice net $\mathcal{N}$ and $\widehat{Y}$ the corresponding P-semiflow in the expanded net $\widehat{\mathcal{N}}$. Then $||\mathcal{M}|| \cap P = ||\widehat{Y}||_{\mathcal{N}}$.*

Now, we prove the theorem 5.1.

Proof of theorem 5.1. ($\Rightarrow$) By lemma 5.3, there exists a P-semiflow $\widehat{Y}_1$ of $\widehat{\mathcal{N}}$ such that $||\mathcal{M}|| = ||\widehat{Y}_1||_{\mathcal{N}}$ and the conditions (1) and (2) of the theorem hold. If $\widehat{Y}_1$ is minimal, we are done by taking $\widehat{Y} = \widehat{Y}_1$. Assume $\widehat{Y}_1$ is not minimal. Then, by definition of minimality, there exists a minimal P-semiflow $\widehat{Y}_2$ such that $||\widehat{Y}_2|| \subset ||\widehat{Y}_1||$. Moreover, by lemma 5.3 there exists an $\mathcal{R}$–mc $\mathcal{M}'$ that satisfies the conditions (1) and (2) of the theorem and $||\mathcal{M}'|| \cap P = ||\widehat{Y}_2||_{\mathcal{N}}$. Since $||\widehat{Y}_2|| \subset ||\widehat{Y}_1||$ implies that $||\widehat{Y}_2||_{\mathcal{N}} \subseteq ||\widehat{Y}_1||_{\mathcal{N}}$, it follows $||\mathcal{M}'|| \subseteq ||\mathcal{M}||$. Because the minimality of $\mathcal{M}$, the equality holds. Then we take $\widehat{Y} = \widehat{Y}_2$.

($\Leftarrow$) By lemma 5.3, there exists an $\mathcal{R}$–mc $\mathcal{M}_1$ of $\mathcal{N}$ such that $||Y||_{\mathcal{N}} = ||\mathcal{M}_1|| \cap P$ and the conditions (1) and (2) hold. If $\mathcal{M}_1$ is minimal, we are done by taking $\mathcal{M} = \mathcal{M}_1$. Assume that $\mathcal{M}_1$ is not minimal. Then, by definition of minimality, there exists a minimal $\mathcal{R}$–mc $\mathcal{M}_2$ such that the $\mathcal{R}$–mc $\mathcal{M}_2$ is enclosed into the $\mathcal{R}$–mc $\mathcal{M}_1$ (this inclusion is stronger than the one of the support of places). Moreover, by the previous implication (demostrated in this theorem) there exists a minimal P-semiflow $\widehat{Y}'$ that satisfies conditions (1) and (2) of the thorem. Since $\mathcal{M}_2$ is enclosed into $\mathcal{M}_1$ and this means inclusion at level of places, transitions, and arcs, $||\widehat{Y}'|| \subseteq ||\widehat{Y}||$ is verified. Because the minimality of $\widehat{Y}$, the equality holds. Then we take $\widehat{Y} = \widehat{Y}'$. ∎

The next theorem gives a lower bound for the mean interfiring time of a transition of a live and safe free choice net, using the P-semiflows of the expanded net defined above.

Theorem 5.2 *A lower bound for the mean interfiring time of transition t_j of a live and safe free choice net $\langle \mathcal{N}, M_0 \rangle$ is:*

$$\Gamma_{(j)}^{min} = \frac{k_{\widehat{Y}^*} \cdot \gamma(\widehat{Y}^*)}{k_{v(j)}}$$

where:

- $\gamma(\widehat{Y}^*)$ *can be obtained by solving the following linear programming problem:*

$$\begin{array}{rll} \gamma(\widehat{Y}^*) = & \text{maximum} & \widehat{Y}^T \cdot \widehat{PRE} \cdot \vec{\hat{s}} \\ & \text{subject to} & \widehat{Y}^T \cdot \widehat{C} = 0 \\ & & \mathbb{1}^T \cdot \widehat{Y} = 1 \\ & & \widehat{Y} \geq 0 \end{array} \qquad \text{(LPP2)}$$

  where $(\widehat{PRE})$ $\widehat{C}$ is the (pre-) incidence matrix of the expanded net $\widehat{\mathcal{N}}$ of $\mathcal{N}$ and $\vec{\hat{s}}$ is the vector defined in Step 3 of expansion of $\mathcal{N}$.

- $k_{\widehat{Y}^*}$ *is a non-negative number such that $k_{\widehat{Y}^*}\widehat{Y}^* = \widehat{Y}_Z^*$ where the vector $\widehat{Y}_Z^* \in \mathbb{Z}^{|\widehat{P}|}$ and the greatest common divisor of its components is equal to 1.*

- $k_{v(j)}$ *is a non-negative number such that $k_{v(j)}\vec{v}^{(j)} = \vec{v}_Z$ where the vector $\vec{v}_Z$ is such that $\vec{v}_Z \in \mathbb{Z}^{|T|}$ and the greatest common divisor of its components is equal to 1.*

Proof. The optimum solution of (LPP2) is always reached for a minimal P-semiflow because, taking into account Theorem 3.3 in [Mur83], if (LPP2) has an optimum feasible solution, then it has a basic feasible solution $\widehat{Y}$ that is optimum. Therefore, the set of rows that are used by $\widehat{Y}$ is linearly independent (i.e. full rank). Considering that $\widehat{Y}^T \cdot \widehat{C} = 0$, we obtain that the number of non-null entries of vector $\widehat{Y}$ (i.e. the number of rows used by $\widehat{Y}$) is equal to the rank of rows of $\widehat{C}$ used by $\widehat{Y}$ plus one. This last statement is precisely the characterization of a minimal P-semiflow, presented in [CS89].

Multiplying the optimum value of the above linear programming problem by the constant $k_{\widehat{Y}^*}$, we obtain the sum of mean service times of transitions covered by a minimal $\mathcal{R}$–mc. This is because the vector $\widehat{Y}^*_Z = k_{\widehat{Y}^*} \cdot \widehat{Y}^*$ is a minimal P-semiflow whose components are integer and minimal, thus there exists a minimal $\mathcal{R}$–mc, according to theorem 5.1.

Then the above computation can be rewritten in terms of minimal $\mathcal{R}$–mc in the following way:

$$k_{\widehat{Y}^*} \cdot \gamma(\widehat{Y}^*) = \begin{array}{ll} \text{maximum} & \mathcal{M}|_P \cdot PRE \cdot \vec{s} \\ \text{such that} & \mathcal{M} \text{ is a minimal } \mathcal{R}\text{–mc of } \mathcal{N} \end{array} \tag{15}$$

where $\mathcal{M}|_P$ denotes the row vector with components $\mathcal{M}|_P(p) = \mathcal{M}(p)$ for all place p of the original net.

Now, we consider the behaviourally equivalent (for a given deterministic conflicts resolution policy) marked graph derived in section 4. According to lemma 5.1, to each k multiple of a minimal $\mathcal{R}$–mc of the original net corresponds a circuit (i.e., minimal P-semiflow) of this marked graph, and each of them contains only one token. Then $k \cdot k_{\widehat{Y}^*} \cdot \gamma(\widehat{Y}^*)$ is the value computed in the Step 3 of the algorithm presented in section 4. Finally, since the number of instances of t_j in the behaviourally equivalent marked graph is $k \cdot \vec{v}_Z(t_j)$ (lemma 5.1) then, dividing $k \cdot k_{\widehat{Y}^*} \cdot \gamma(\widehat{Y}^*)$ by this constant, we obtain the mean interfiring time of transition t_j computed in Step 4 of the algorithm presented in section 4, hence being a lower bound for the mean interfiring time of t_j in the original net. ■

In fact, from the reachability of the bound for strongly connected marked graphs (see theorem 3.4) the reachability of the bound given by theorem 5.2 follows for live and safe free choice nets.

Theorem 5.3 *For live and safe free choice nets with arbitrary values of mean service times of transitions and arbitrary routing rates defining the resolution of conflicts, the lower bound for the mean interfiring time obtained from theorem 5.2 is reachable.*

Proof. The result follows from the following considerations: (1) deterministic service times and deterministic routing are particular cases of timing and conflict resolution policy, respectively; (2) for such policy, theorem 5.2 can be applied; and (3) for the case of marked graphs with deterministic timing, the derived bound is reached. ■

As in the case of strongly connected marked graphs (see [CCCS89]), a characterization of liveness and safeness for structurally live and structurally bounded free choice nets can be derived:

Theorem 5.4 *Liveness and safeness of structurally live and structurally bounded free choice nets can be characterized in polynomial time.*

Proof. For structurally live and structurally bounded free choice nets, marking bound of places equals structural marking bound [Esp90]. Structural marking bound can be computed by solving an LPP similar to that presented in [SC88], thus in polynomial time. Therefore, safeness can be characterized computing the structural marking bound of places. Liveness can be characterized checking the boundedness of the problem (LPP2): the value given by theorem 5.2 is a lower bound for the mean interfiring time; if this value is infinite the mean interfiring time is unbounded, and the net is non-live; if the value given by theorem 5.2 is finite, since it is reachable (cfr. theorem 5.3), the net must be *deadlock-free*. We know that for structurally live and structurally bounded free choice nets, liveness and deadlock-freeness are equivalent. Thus

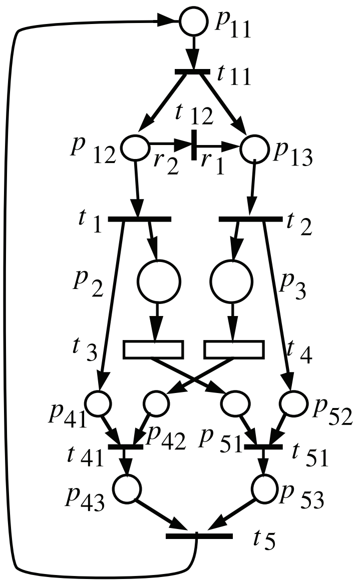

Figure 6: Expanded net of the one depicted in figure 1 (the weights r_1 and r_2 are such that $r_1/r_2 = q/(1-q)$).

the finiteness of the value given by theorem 5.2 is sufficient to establish the liveness of a safe structurally live and structurally bounded free choice net. ∎

As an example, let us consider once more the live and safe free choice net depicted in figure 1. Its expanded net according to definition 5.4 is depicted in figure 6. The application of theorem 5.2 for this net gives the value:

$$\Gamma_{(5)}^{min} = qs_3 + (1-q)s_4 \tag{16}$$

which is exactly the actual interfiring time of the net for deterministic service time of transitions.

6 The case of bounded (non-safe) nets

The natural extension of the results presented in previous section will consist on the computation of lower bounds for the mean interfiring time of transitions of live and bounded (non-safe) free choice nets. In this section we argue that a trivial extension cannot be obtained applying the techniques used for safe nets. However, a preliminary idea for the correct extension is outlined.

Let us consider once more the Petri net depicted in figure 1, but now with initial marking of place p_1 equal to 2 tokens. We assume *infinite-server semantics* for the timing of transitions, i.e., a transition enabled k times in a marking works at conditional speed k times that it would work in the case it was enabled only once. Suppose the following deterministic conflict resolution policy at place p_1: "select twice transition t_1, then once transition t_2, and repeat it" (i.e., the routing rate for transition t_1 is twice the routing rate of t_2). A trivial extension of the method presented in section 5 for the computation of a lower bound for the mean interfiring time of t_5 would be the following:

1. Derive the expanded net (figure 6).

2. Apply the theorem 5.2 with $j = 5$ and divide by 2 the obtained value (i.e., divide the interfiring time of the slowest circuit by the number of contained tokens).

But the obtained value is not a lower bound for the mean interfiring time of t_5 in the original net, in general. For instance, if transitions t_3 and t_4 are supposed to be exponentially timed with averages $s_3 = s_4 = 1$, the value that can be derived from theorem 5.2, dividing by 2, is $(2/3 + 1/3)/2 = 0.5$, while the actual mean interfiring time is $\Gamma_{(5)} = 0.387$.

The reason of this bad result is the following: when two tokens are initially put in p_1, three tokens in the subset $\{p_2, p_3\}$ will subsequently be possible (if t_1 and t_2 fire, transition t_5 becomes enabled and, being immediate, fires first so that a new token is put in p_1 and subsequently—always in zero time—delivered to place p_2). Then, the value obtained in step 2, dividing by two, is pessimistic since more than two tokens can be in the subset of places $\{p_2, p_3\}$. An optimistic assumption, thus leading to a throughput upper bound, will be to divide by three in step 2, which is the maximum number of tokens that can be in the subset $\{p_2, p_3\}$. Of course a general procedure must be found that deals with these more general cases.

7 Conclusions

Tight lower bounds for the steady-state mean interfiring time of transitions for live and safe free choice nets have been derived in this paper.

A direct application of the same result that gave a reachable bound for live and safe marked graphs (using P-semiflows) does not lead to a tight bound. A reachable lower bound for the mean interfiring time has been derived using another "natural" generalization of the marked graphs' result: multisets of circuits of free choice nets instead of circuits of marked graphs. A polynomial complexity algorithm for the computation of this bound has been obtained after the introduction of a particular net transformation, which is also of linear complexity.

A preliminary idea for the possible extension of the previous results for the case of bounded (non-safe) nets has been outlined.

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