Properties and Bounds on P/T Nets
Tutorial of PNPM'99 – PAPM'99 – NSMC'99

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Short description and motivation

A complementary approach to exact or approximation techniques for the analysis of *timed or stochastic Petri net models* is the computation of *bounds* for their *performance measures*. Performance bounds are useful in the preliminary phases of the design of a system, in which many parameters are not known accurately. Several alternatives for those parameters should be quickly evaluated, and rejected those that are clearly bad. Exact (and even approximate) solutions would be computationally very expensive. Bounds become useful in these instances since they usually require much *less computation effort*.

In this tutorial, *net-driven* techniques for the computation of bounds for the main performance indices of timed Petri net models are considered. Special attention is given to the intimate relationship between qualitative and quantitative aspects of Petri nets. In particular, the intensive use of *structure theory* of net models allows to obtain very *efficient computation* techniques.

The contents of the tutorial are the following: (1) Preliminary comments; (2) Introducing the ideas: Marked Graphs case; (3) Generalization: use of visit ratios; (4) Improvements of the bounds; (5) A general linear programming statement.

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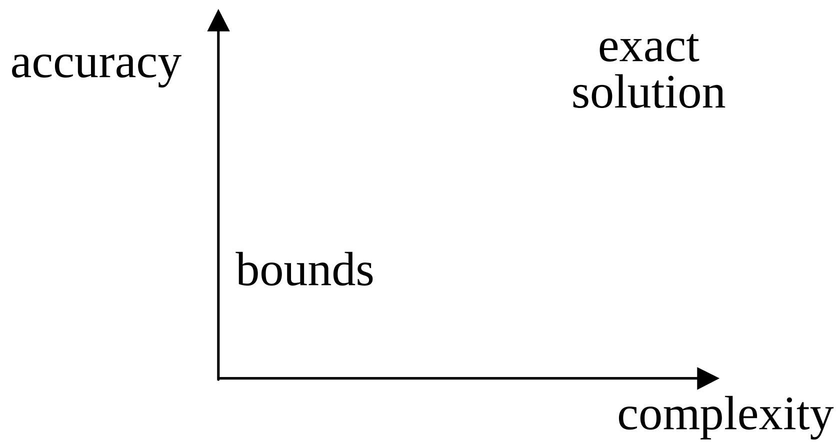
Preliminary comments (1)

- Interest of bounding techniques
  - preliminary phases of design
    - many parameters are not known accurately
    - quick evaluation and rejection of those clearly bad



Preliminary comments (2)

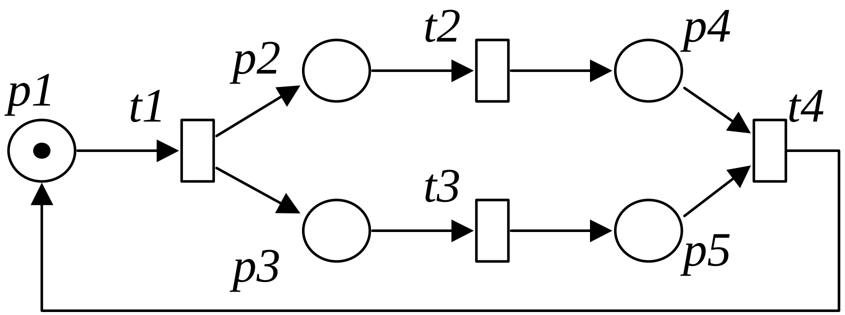
- Net-driven solution technique
  - stressing the intimate relationship between qualitative and quantitative aspects of PN’s
  - structure theory of net models

    → efficient computation techniques

Outline

- Introducing the ideas: Marked Graphs case
- Generalization: use of visit ratios
- Improvements of the bounds
- A general linear programming statement

Introducing ideas: MG's case (1)



generally distributed service times
(random variables X_i with mean $\bar{\mathbf{s}}[t_j]$)

we assume **infinite-server semantics**

exact cycle time (random variable): $X = X_1 + \max\{X_2, X_3\} + X_4$

average cycle time: $\Gamma = \bar{\mathbf{s}}[t_1] + \mathrm{E}[\max\{X_2, X_3\}] + \bar{\mathbf{s}}[t_4]$

but (non-negative variables): $X_2, X_3 \le \max\{X_2, X_3\} \le X_2 + X_3$

therefore: $\boxed{\bar{\mathbf{s}}[t_1] + \max\{\bar{\mathbf{s}}[t_2], \bar{\mathbf{s}}[t_3]\} + \bar{\mathbf{s}}[t_4] \le \Gamma \le \bar{\mathbf{s}}[t_1] + \bar{\mathbf{s}}[t_2] + \bar{\mathbf{s}}[t_3] + \bar{\mathbf{s}}[t_4]}$

Introducing ideas: MG's case (2)

Thus, the lower bound for the average cycle time is computed looking for the slowest circuit

$$\Gamma \geq \max_{\substack{C \in \{\text{circuits}\\ \text{of the net}\}}} \left(\sum_{t_i \in C} \bar{\mathbf{s}}[t_i] \Big/ \#\,\text{tokens in } C \right)$$

Interpretation:

an MG may be built synchronising circuits, so we look for the bottleneck

Introducing ideas: MG's case (3)

- Computation:

$$\begin{aligned} \Gamma \geq \quad & \text{maximum} \quad \mathbf{y} \cdot \mathbf{Pre} \cdot \bar{\mathbf{s}} \\ & \text{subject to} \quad \mathbf{y} \cdot \mathbf{C} = \mathbf{0} \\ & \qquad\qquad\quad \mathbf{y} \cdot \mathbf{m_0} = 1 \\ & \qquad\qquad\quad \mathbf{y} \geq \mathbf{0} \end{aligned}$$

($\bar{\mathbf{s}}$ is the vector of average service times)

(the proof of this comes later for a more general case)

solving a linear programming problem
(**polynomial complexity** on the net size)

Introducing ideas: MG's case (4)

- Even if naïf, the bounds are tight!
- Lower bound for the average cycle time

$$\max\{\bar{\mathbf{s}}[t_2], \bar{\mathbf{s}}[t_3]\} \leq \mathrm{E}[\max\{X_2, X_3\}]$$

  – it is exact for deterministic timing
  – it cannot be improved using only mean values of r.v. (it is reached in a limit case for a family of random variables with arbitrary means and variances)

Introducing ideas: MG's case (5)

$$X_{m,s}(a) = \begin{cases} ma & \text{with probability } 1-e \\ m\left(a + \dfrac{1-a}{e}\right) & \text{with probability } e \end{cases}$$

$$(0 \le a \le 1)$$

$$e = \frac{m^2(1-a)^2}{m^2(1-a)^2 + s^2}$$

$$\mathrm{E}\left[X_{m,s}(a)\right] = m \ ; \ \mathrm{Var}\left[X_{m,s}(a)\right] = s^2$$

$$\boxed{\begin{array}{l} \lim\limits_{a \to 1} \mathrm{E}\left[\max\left(X_{m,s}(a), X_{m',s'}(a)\right)\right] = \max(m, m') \\ \\ \mathrm{E}\left[X_{m,s}(a) + X_{m',s'}(a)\right] = m + m', \ \forall \ 0 \le a < 1 \end{array}}$$

they behave "as deterministic"
for the 'max' and '+' operators
in the limit (α→1)

Introducing ideas: MG's case (6)

- Upper bound for the average cycle time

$$\boxed{\Gamma \le \sum_{t \in T} \bar{\mathbf{s}}[t]}$$

  – it cannot be improved for 1–live MG's using only mean values of r.v. (it is reached in a limit case for a family of random variables with arbitrary means)

Introducing ideas: MG's case (7)

$$X^i_m(e) = \begin{cases} 0 & \text{with probability } 1 - e^i \\ \dfrac{m}{e^i} & \text{with probability } e^i \end{cases}$$

$(0 < e < 1)$

$$\mathrm{E}\left[X^i_m(e)\right] = m \ ; \ \mathrm{E}\left[X^i_m(e)^2\right] = \frac{m^2}{e^i}$$

If $X_j = X^{j-1}_{\bar{\mathbf{s}}[t_j]}(e), \ \forall t_j \in T,$ then for varying (decreasing) values of e:

$$\boxed{\mathrm{E}[\max(X_i, X_j)] = \bar{\mathbf{s}}[t_i] + \bar{\mathbf{s}}[t_j] + o(e)}$$

Generalization: visit ratios (1)

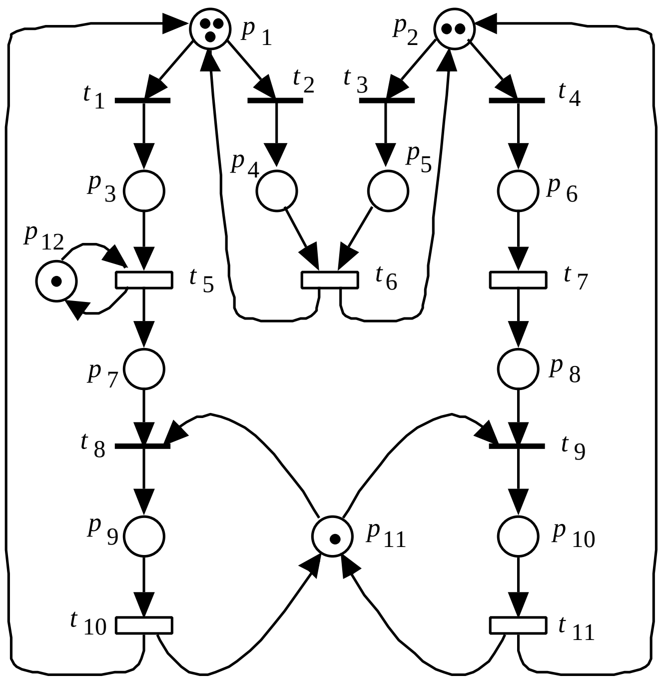
- Visit ratios = relative throughput

  (number of visits to t_i per each visit to t_1)

$$\mathbf{v}[t] = \frac{c[t]}{c[t_1]} = \underbrace{\Gamma[t_1]}_{\text{average interfiring time of } t_1} \; c[t]$$

Generalization: visit ratios (2)

- For some net classes **v** can be computed as:



$$\mathbf{C}\cdot\mathbf{v}=\mathbf{0};$$
$$r_1\mathbf{v}[t_2]=r_2\mathbf{v}[t_1];$$
$$r_3\mathbf{v}[t_4]=r_4\mathbf{v}[t_3];$$
$$\mathbf{v}[t_1]=1$$

Generalization: visit ratios (3)

- Little's law ($L = l W$) applied to a place p:

$$\overline{m}[p] = (\mathbf{Pre}[p,T] \cdot c) \;\; \overline{\mathbf{r}}[p]$$

Assume that timed transitions are never in conflict (**conflicts are modelled with immediate transitions**), then either all output transitions of p are immediate or p has a unique output transition, say t_1, and t_1 is timed, thus:

$$\overline{m}[p] = (\mathbf{Pre}[p,T] \cdot c) \;\; \overline{\mathbf{r}}[p] = \mathbf{Pre}[p,t_1] \;\; c[t_1] \;\; \overline{\mathbf{r}}[p]$$

$$\geq \mathbf{Pre}[p,t_1] \;\; c[t_1] \;\; \overline{\mathbf{s}}[t_1] = \sum_{j=1}^{m} \mathbf{Pre}[p,t_j] \;\; c[t_j] \;\; \overline{\mathbf{s}}[t_j]$$

Generalization: visit ratios (4)

Then: $$\Gamma[t_1]\ \bar{m}[p] \geq \sum_{j=1}^{m} \mathbf{Pre}[p,t_j]\ \Gamma[t_1]\ \ c[t_j]\ \ \bar{\mathbf{s}}[t_j] = \sum_{j=1}^{m} \mathbf{Pre}[p,t_j]\ \ \mathbf{v}[t_j]\ \ \bar{\mathbf{s}}[t_j]$$

Hence: $\Gamma[t_1]\ \ \bar{m} \geq \mathbf{Pre} \cdot \bar{\mathbf{D}}$ where $\bar{\mathbf{D}}[t] = \bar{\mathbf{s}}[t]\mathbf{v}[t]$ is the average service demand of t

Premultiplying by a P–semiflow $\mathbf{y}$ $(\mathbf{y} \cdot \mathbf{C} = \mathbf{0},\ \ \mathbf{y} \geq \mathbf{0},$ thus $\mathbf{y} \cdot \bar{m} = \mathbf{y} \cdot \mathbf{m_0})$,

$$\Gamma[t_1] \geq \ \text{maximum} \ \frac{\mathbf{y} \cdot \mathbf{Pre} \cdot \bar{\mathbf{D}}}{\mathbf{y} \cdot \mathbf{m_0}} \quad \text{subject to} \quad \begin{array}{l} \mathbf{y} \cdot \mathbf{C} = \mathbf{0} \\ \mathbf{1} \cdot \mathbf{y} > 0 \\ \mathbf{y} \geq \mathbf{0} \end{array}$$

$$\Longleftrightarrow$$

$$\Gamma[t_1] \geq \ \text{maximum} \ \frac{\mathbf{y} \cdot \mathbf{Pre} \cdot \bar{\mathbf{D}}}{q} \quad \text{subject to} \quad \begin{array}{l} \mathbf{y} \cdot \mathbf{C} = \mathbf{0} \\ \mathbf{1} \cdot \mathbf{y} > 0 \\ q = \mathbf{y} \cdot \mathbf{m_0} \\ \mathbf{y} \geq \mathbf{0} \end{array}$$

Generalization: visit ratios (5)

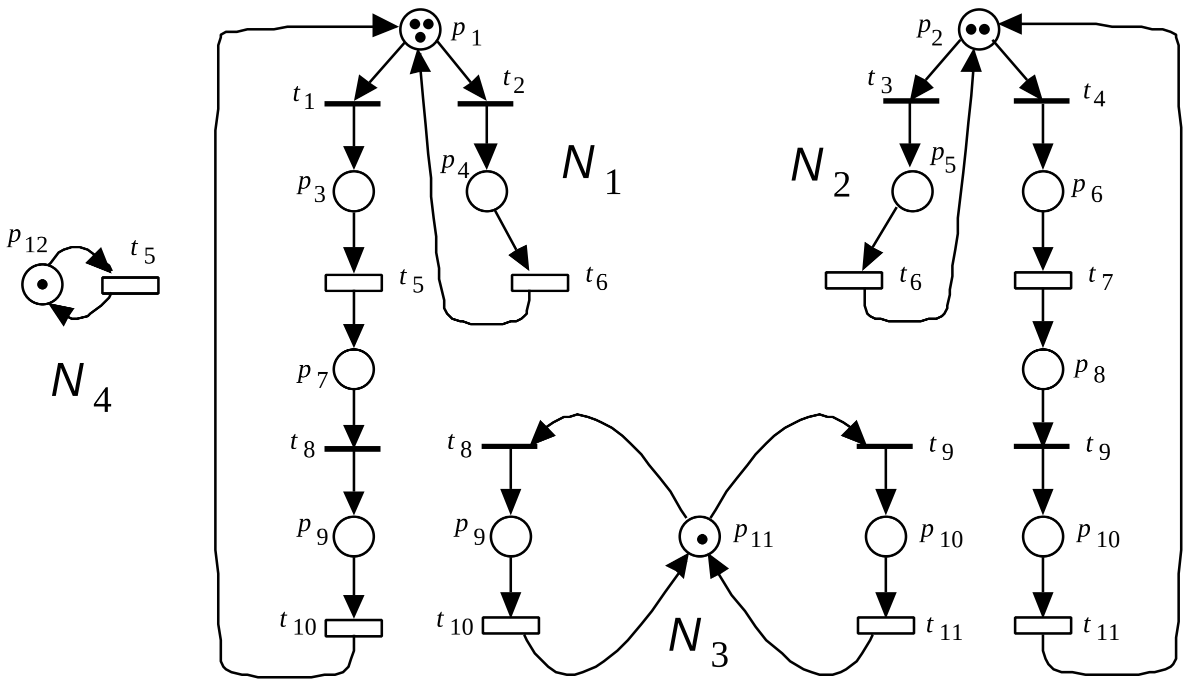
Since $\mathbf{y}\cdot\mathbf{m_0} > 0$ (live system), we change $\mathbf{y}/q$ to $\mathbf{y}$ and we obtain ($\mathbf{1}\cdot\mathbf{y} > 0$ is removed because $\mathbf{y}\cdot\mathbf{m_0} = 1$ implies $\mathbf{1}\cdot\mathbf{y} > 0$):

$$\boxed{\begin{array}{lll} \Gamma[t_1] \geq & \text{maximum} & \mathbf{y}\cdot\mathbf{Pre}\cdot\overline{\mathbf{D}} \\ & \text{subject to} & \mathbf{y}\cdot\mathbf{C} = \mathbf{0} \\ & & \mathbf{y}\cdot\mathbf{m_0} = 1 \\ & & \mathbf{y} \geq \mathbf{0} \end{array}}$$

again, a linear programming problem
(polynomial complexity on the net size)

Generalization: visit ratios (6)

Interpretation: **slowest subsystem generated by *P*–semiflows, in isolation**



minimal *P*–semiflows

$\mathbf{y_1}$ = (1,0,1,1,0,0,1,0,1,0,0,0)

$\mathbf{y_2}$ = (0,1,0,0,1,1,0,1,0.1,0,0)

$\mathbf{y_3}$ = (0,0,0,0,0,0,0,0,1,1,1,0)

$\mathbf{y_4}$ = (0,0,0,0,0,0,0,0,0,0,0,1)

$$\Gamma[t_1] \geq \max\{ (\bar{\mathbf{s}}[t_5]+\bar{\mathbf{s}}[t_6]+\bar{\mathbf{s}}[t_{10}])/3,\ (\bar{\mathbf{s}}[t_6]+\bar{\mathbf{s}}[t_7]+\bar{\mathbf{s}}[t_{11}])/2,\ \bar{\mathbf{s}}[t_{10}]+\bar{\mathbf{s}}[t_{11}],\ \bar{\mathbf{s}}[t_5] \}$$

Generalization: visit ratios (7)

- Upper bound for the average interfiring time

$$\boxed{\Gamma[t_1] \leq \sum_{t \in T} \mathbf{v}[t]\ \bar{\mathbf{s}}[t] = \sum_{t \in T} \overline{\mathbf{D}}[t]}$$

remember the marked graphs case (**v** = **1**): $\Gamma \leq \sum_{t \in T} \bar{\mathbf{s}}[t]$

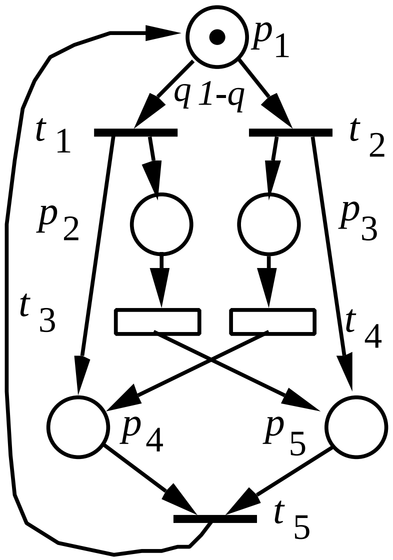
Improvements of the bounds

- Structural improvements

  bounds still based only on the mean values (not on higher moments of r.v., **insensitive** bounds)

  - lower bound for the average interfiring time:
    use of **implicit places** to increase the number of minimal P–semiflows
  - upper bound for the average interfiring time:
    use of **liveness bound of transitions** to improve the bound for some net subclasses

Use of implicit places (1)



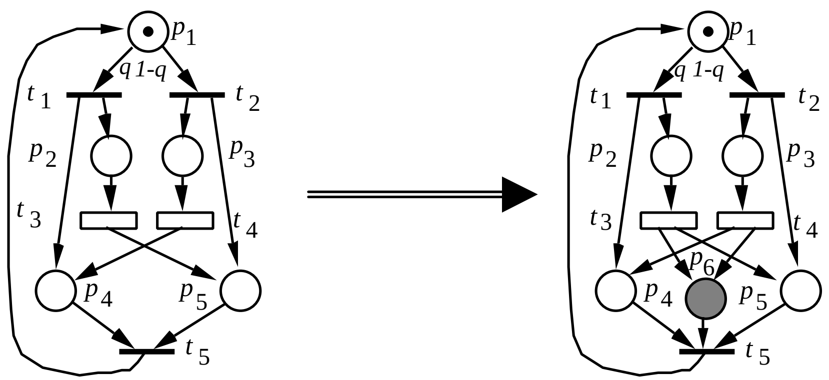
$$\Gamma[t_5] = q\bar{\mathbf{s}}[t_3] + (1-q)\bar{\mathbf{s}}[t_4]$$

$$\Gamma[t_1] \geq \begin{array}{ll} \text{maximum} & \mathbf{y} \cdot \mathbf{Pre} \cdot \overline{\mathbf{D}} \\ \text{subject to} & \mathbf{y} \cdot \mathbf{C} = \mathbf{0} \\ & \mathbf{y} \cdot \mathbf{m_0} = 1 \\ & \mathbf{y} \geq \mathbf{0} \end{array}$$

$$\Downarrow$$

$$\Gamma[t_5] \geq \max\left\{q\bar{\mathbf{s}}[t_3], (1-q)\bar{\mathbf{s}}[t_4]\right\}$$

Use of implicit places (2)



$$\Gamma[t_1] \geq \quad \text{maximum} \quad \mathbf{y} \cdot \mathbf{Pre} \cdot \overline{\mathbf{D}}$$
$$\text{subject to} \quad \mathbf{y} \cdot \mathbf{C} = \mathbf{0}, \quad \mathbf{y} \cdot \mathbf{m_0} = 1, \quad \mathbf{y} \geq \mathbf{0}$$

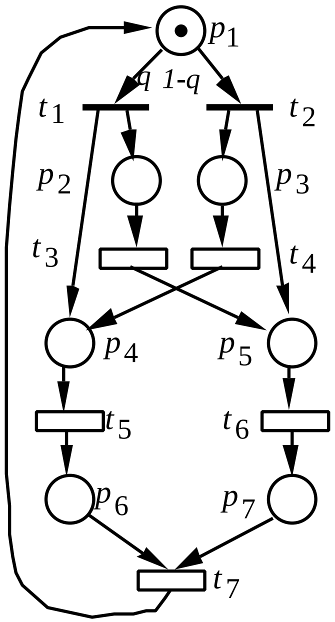
$$\Gamma[t_5] = q\bar{\mathbf{s}}[t_3] + (1-q)\bar{\mathbf{s}}[t_4]$$

$$\Gamma[t_5] \geq \max\left\{\, q\bar{\mathbf{s}}[t_3],\ (1-q)\bar{\mathbf{s}}[t_4],\ \underline{q\bar{\mathbf{s}}[t_3] + (1-q)\bar{\mathbf{s}}[t_4]} \,\right\}$$

in this case, we get the exact value!

Use of implicit places (3)

in general…



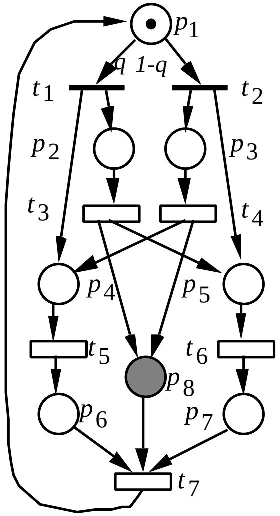
$$\Gamma[t_1] \geq \quad \text{maximum} \quad \mathbf{y} \cdot \mathbf{Pre} \cdot \overline{\mathbf{D}}$$
$$\text{subject to} \quad \mathbf{y} \cdot \mathbf{C} = \mathbf{0}$$
$$\mathbf{y} \cdot \mathbf{m_0} = 1$$
$$\mathbf{y} \geq \mathbf{0}$$

$$\Downarrow$$

$$\Gamma[t_7] \geq \max \{ \ q\bar{\mathbf{s}}[t_3] + \bar{\mathbf{s}}[t_6] + \bar{\mathbf{s}}[t_7], \quad (1-q)\bar{\mathbf{s}}[t_4] + \bar{\mathbf{s}}[t_5] + \bar{\mathbf{s}}[t_7] \ \}$$

Use of implicit places (4)

in general, the bound is non-reachable



$$\Gamma[t_7] \geq \max \{ \; q\bar{\mathbf{s}}[t_3]+\bar{\mathbf{s}}[t_6]+\bar{\mathbf{s}}[t_7], \; (1-q)\bar{\mathbf{s}}[t_4]+\bar{\mathbf{s}}[t_5]+\bar{\mathbf{s}}[t_7], \; q\bar{\mathbf{s}}[t_3]+(1-q)\bar{\mathbf{s}}[t_4]+\bar{\mathbf{s}}[t_7] \; \}$$

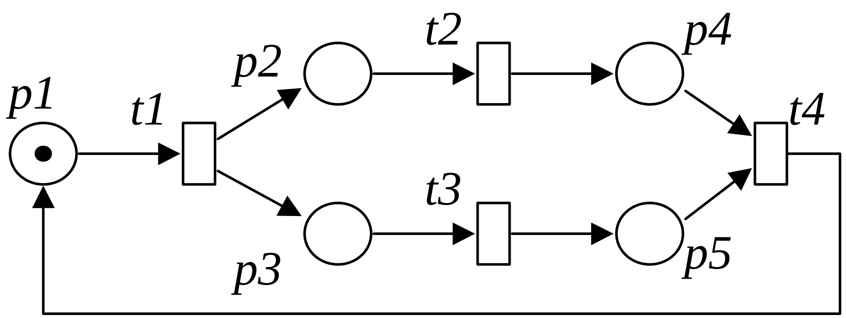
$$\Gamma[t_7] = q\max\{\bar{\mathbf{s}}[t_5],\bar{\mathbf{s}}[t_3]+\bar{\mathbf{s}}[t_6]\}+(1-q)\max\{\bar{\mathbf{s}}[t_4]+\bar{\mathbf{s}}[t_5],\bar{\mathbf{s}}[t_6]\}+\bar{\mathbf{s}}[t_7]$$

$$= \max \{ \; q\bar{\mathbf{s}}[t_3]+\bar{\mathbf{s}}[t_6]+\bar{\mathbf{s}}[t_7], \; (1-q)\bar{\mathbf{s}}[t_4]+\bar{\mathbf{s}}[t_5]+\bar{\mathbf{s}}[t_7], \; q\bar{\mathbf{s}}[t_3]+(1-q)\bar{\mathbf{s}}[t_4]+(1-q)\bar{\mathbf{s}}[t_5]+q\bar{\mathbf{s}}[t_6]+\bar{\mathbf{s}}[t_7], \; q\bar{\mathbf{s}}[t_5]+(1-q)\bar{\mathbf{s}}[t_6]+\bar{\mathbf{s}}[t_7] \; \}$$

(deterministic timing)

Use of liveness bounds (1)

- upper bound for the average interfiring time:

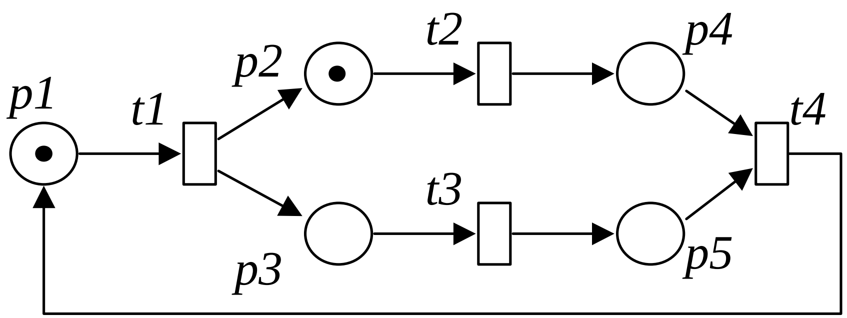


$$\Gamma \leq \sum_{t \in T} \bar{\mathbf{s}}[t]$$

reachable for 1-live marked graphs, but…

Use of liveness bounds (2)

it can be improved for *k*–live marked graphs



$$\Gamma \leq \bar{\mathbf{s}}[t_1] + \frac{\bar{\mathbf{s}}[t_2]}{2} + \bar{\mathbf{s}}[t_3] + \bar{\mathbf{s}}[t_4]$$

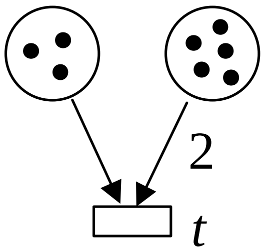
liveness bound of t_2

Use of liveness bounds (3)

- Definitions of enabling degree, enabling bound, structural enabling bound, and liveness bound

  – instantaneous enabling degree of a transition at a given marking

$$\mathbf{e}[t](\mathbf{m}) = \sup\left\{k \in \mathbb{N} : \ \forall p \in {}^{\bullet}t, \ \mathbf{m}[p] \geq k \ \mathbf{Pre}[p,t]\right\}$$

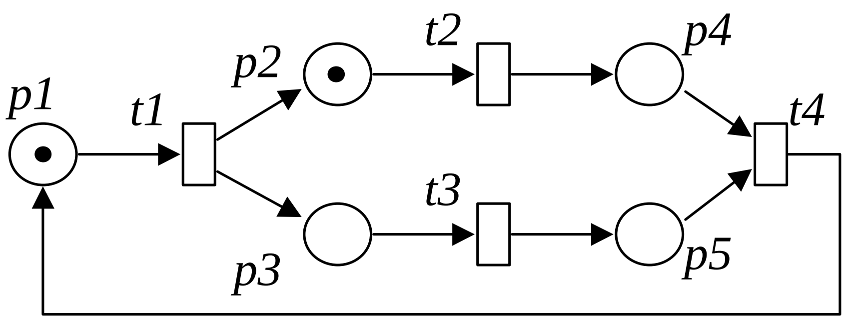


$\mathbf{e}[t](\mathbf{m}) = 2$

Use of liveness bounds (4)

– enabling bound of a transition in a given system:
maximum among the instantaneous enabling degree at all reachable markings

$$\mathbf{eb}[t] = \sup\left\{ k \in \mathrm{N} : \ \exists \mathbf{m}_0 \xrightarrow{s} \mathbf{m}, \ \forall p \in {}^{\bullet}t, \ \mathbf{m}[p] \geq k \ \mathbf{Pre}[p,t] \right\}$$

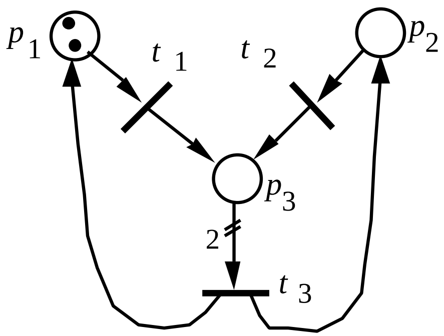


$\mathbf{eb}[t_2] = 2$

Use of liveness bounds (5)

– liveness bound of a transition in a given system:
  number of servers available in t in steady state

$$\mathbf{lb}[t] = \sup\left\{ k \in \mathrm{N} :\ \forall \mathbf{m}', \mathbf{m_0} \xrightarrow{s} \mathbf{m}', \exists \mathbf{m}, \mathbf{m}' \xrightarrow{s'} \mathbf{m} \wedge \forall p \in {}^{\bullet}t, \mathbf{m}[p] \geq k\ \mathbf{Pre}[p, t] \right\}$$



$\mathbf{lb}[t_1] = 1 < 2 = \mathbf{eb}[t_1]$

Use of liveness bounds (6)

– structural enabling bound of a transition in a given system: structural counterpart of the enabling bound (substitute reachability condition by $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot s$; $\mathbf{m}, s \bullet \mathbf{0}$)

$$
\begin{aligned}
\mathbf{seb}[t] = \quad & \text{maximum} \quad k \\
& \text{subject to} \quad \mathbf{m_0}[p] + \mathbf{C}[p,T] \cdot s \geq k \; \mathbf{Pre}[p,t], \quad \forall p \in P \\
& \qquad\qquad\quad s \geq 0
\end{aligned}
$$

Property: For any net system $\mathbf{seb}[t] \geq \mathbf{eb}[t] \geq \mathbf{lb}[t]$, for all t.

Property: For live and bounded free choice systems,

$$\mathbf{seb}[t] = \mathbf{eb}[t] = \mathbf{lb}[t], \text{ for all } t.$$

Use of liveness bounds (7)

improvement of the bound for live and bounded free choice systems:

$$\Gamma[t_1] \leq \sum_{t \in T} \frac{\mathbf{v}[t]\ \bar{\mathbf{s}}[t]}{\mathbf{seb}[t]} = \sum_{t \in T} \frac{\overline{\mathbf{D}}[t]}{\mathbf{seb}[t]}$$

this bound cannot be improved for marked graphs (using only the mean values of service times)

A general LP statement (1)

- The idea

a linear function

maximize [or minimize]	$f(\overline{m}, c)$
subject to	any linear constraint that we are able to state for $\overline{m}$, c, and other needed additional variables

linear operational laws

A general LP statement (2)

- A set of linear constraints:

$$\overline{m} = \mathbf{m_0} + \mathbf{C} \cdot s \qquad \textit{(state equation)}$$

$$\sum_{t \in {}^{\bullet}p} c[t]\ \mathbf{Post}[p,t] \geq \sum_{t \in p^{\bullet}} c[t]\ \mathbf{Pre}[p,t], \quad \forall p \in P$$

$$\sum_{t \in {}^{\bullet}p} c[t]\ \mathbf{Post}[p,t] = \sum_{t \in p^{\bullet}} c[t]\ \mathbf{Pre}[p,t], \quad \forall p \in P \text{ bounded } \textit{(flow balance equation)}$$

$$\frac{c[t_i]}{r_i} = \frac{c[t_j]}{r_j}, \quad \forall t_i, t_j \in T: \text{ behavioural free choice}$$

$$\text{(e.g. } \mathbf{Pre}[P, t_i] = \mathbf{Pre}[P, t_j])$$

$$\ldots \qquad \ldots$$

A general LP statement (3)

$$c[t] \;\; \bar{\mathbf{s}}[t] \leq \frac{\overline{m}[p]}{\mathbf{Pre}[p,t]}, \qquad \forall t \in T,\; \forall p \in {}^\bullet t \qquad \textit{(maximum throughput law)}$$

$$c[t] \;\; \bar{\mathbf{s}}[t] \geq \frac{\overline{m}[p] - \mathbf{Pre}[p,t] + 1}{\mathbf{Pre}[p,t]}, \qquad \forall t \in T \text{ persistent, age memory or immediate}: {}^\bullet t = \{p\} \qquad \textit{(minimum throughput law)}$$

$$\ldots \qquad\qquad \ldots$$

$$\overline{m}, \;\; c, \;\; s \geq \mathbf{0}$$

A general LP statement (4)

- It can be improved using second order moments
- It can be extended to well-formed coloured nets

A general LP statement (5)

- It is implemented in *GreatSPN*
  - select place (transition) object ( )
  - click right mouse button and select “show”
  - click again right mouse button and select “Average M.B.” (“LP Throughput Bounds”)
  - click left mouse button for upper bound
  - click middle mouse button for lower bound

Properties and Bounds on P/T Nets

More technical details

look at the bibliography

Properties and Bounds on P/T Nets: An Example of Application*

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Let us present an example of application for the computation of bounds in the case of the Timed Well-Formed Coloured Net (TWN) model of a shared-memory multiprocessor depicted in Figure 1. The architecture comprises a set of processing modules interconnected by a common

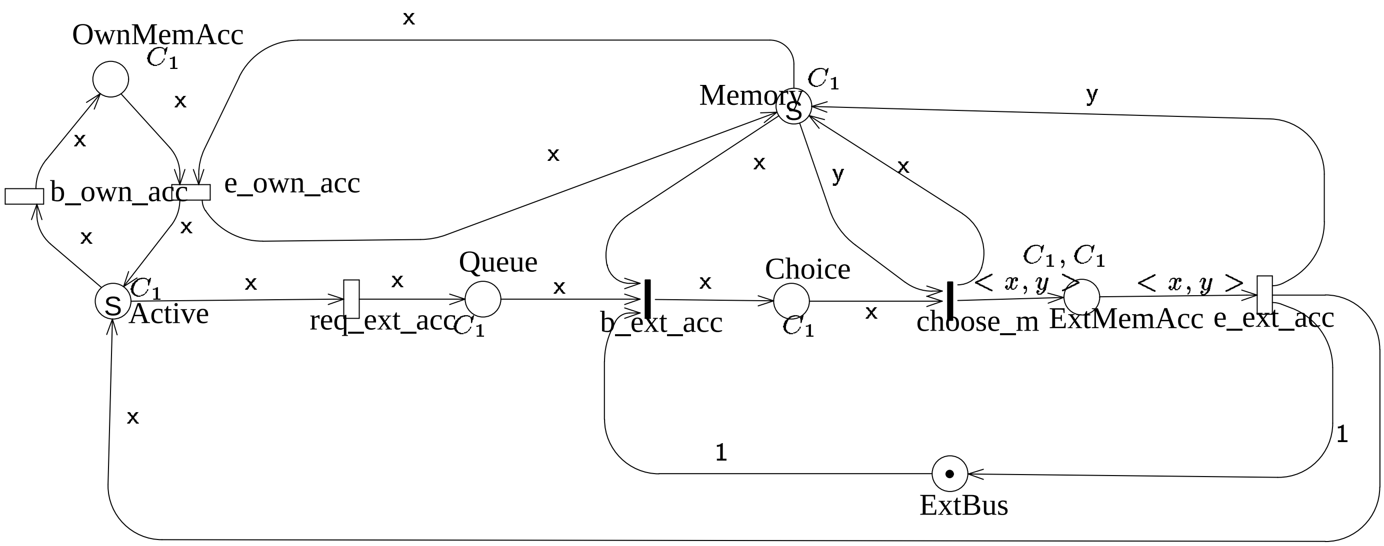

Figure 1: TWN model of a shared-memory multiprocessor.

bus called the "external bus". A processor can access its own memory module directly from its private bus through one port, or it can access non-local shared-memory modules by means of the external bus. In case of contention for the access to one shared-memory module, preemptive priority is given to external access through the external bus with respect to the accesses from the local processor. The experiments on the shared-memory model have been carried out assuming to have 4 processors and that the average service time of all the transitions are equal to 0.5.

According to the arguments presented in the Tutorial, bounds can be computed solving linear programming problems with constraints included in Table 1, where the first letters of each

*Tutorial of PNPM'99–PAPM'99–NSMC'99, Zaragoza (Spain), September 6-10, 1999. This text has been extracted from the paper "Operational analysis of timed Petri nets and application to the computation of performance bounds", by G. Chiola, C. Anglano, J. Campos, J.M. Colom, and M. Silva, in Proceedings of the *5th International Workshop on Petri Nets and Performance Models*, pp. 128-137, Toulouse, France, October 1993, IEEE Computer Society Press.

$$
\begin{array}{ll}
(c_1) & \overline{\boldsymbol{\mu}}[Active] = 4 + \boldsymbol{\sigma}[e\_e\_a] + \boldsymbol{\sigma}[e\_o\_a] - \boldsymbol{\sigma}[r\_e\_a] - \boldsymbol{\sigma}[b\_o\_a]; \\
& \overline{\boldsymbol{\mu}}[Memory] = 4 + \boldsymbol{\sigma}[e\_e\_a] - \boldsymbol{\sigma}[b\_e\_a]; \\
& \overline{\boldsymbol{\mu}}[OwnMemAcc] = \boldsymbol{\sigma}[b\_o\_a] - \boldsymbol{\sigma}[e\_o\_a]; \\
& \overline{\boldsymbol{\mu}}[Queue] = \boldsymbol{\sigma}[r\_e\_a] - \boldsymbol{\sigma}[b\_e\_a]; \\
& \overline{\boldsymbol{\mu}}[Choice] = \boldsymbol{\sigma}[b\_e\_a] - \boldsymbol{\sigma}[c\_m]; \\
& \overline{\boldsymbol{\mu}}[ExtMemAcc] = \boldsymbol{\sigma}[c\_m] - \boldsymbol{\sigma}[e\_e\_a]; \\
& \overline{\boldsymbol{\mu}}[ExtBus] = 1 + \boldsymbol{\sigma}[e\_e\_a] - \boldsymbol{\sigma}[b\_e\_a]; \\
(c_2') & \boldsymbol{\chi}[e\_e\_a] + \boldsymbol{\chi}[e\_o\_a] = \boldsymbol{\chi}[r\_e\_a] + \boldsymbol{\chi}[b\_o\_a]; \\
& \boldsymbol{\chi}[b\_e\_a] = \boldsymbol{\chi}[c\_m] = \boldsymbol{\chi}[e\_e\_a] = \boldsymbol{\chi}[r\_e\_a]; \\
(c_3) & \boldsymbol{\chi}[b\_o\_a] = \boldsymbol{\chi}[r\_e\_a]; \\
(c_4\&c_5) & \boldsymbol{\chi}[b\_o\_a]\ \overline{\mathbf{s}}[b\_o\_a] = \overline{\boldsymbol{\mu}}[Active]/2; \\
& \boldsymbol{\chi}[r\_e\_a]\ \overline{\mathbf{s}}[r\_e\_a] = \overline{\boldsymbol{\mu}}[Active]/2; \\
& \boldsymbol{\chi}[e\_e\_a]\ \overline{\mathbf{s}}[e\_e\_a] = \overline{\boldsymbol{\mu}}[ExtMemAcc]; \\
(c_4) & \boldsymbol{\chi}[e\_o\_a]\ \overline{\mathbf{s}}[e\_o\_a] \le \overline{\boldsymbol{\mu}}[OwnMemAcc]; \\
& \boldsymbol{\chi}[e\_o\_a]\ \overline{\mathbf{s}}[e\_o\_a] \le \overline{\boldsymbol{\mu}}[Memory]; \\
(c_6) & \boldsymbol{\chi}[e\_o\_a]\ \overline{\mathbf{s}}[e\_o\_a] \ge \overline{\boldsymbol{\mu}}[OwnMemAcc] + \frac{\mathbf{b}[OwnMemAcc]}{\mathbf{b}[Memory]}\overline{\boldsymbol{\mu}}[Memory] \\
& \qquad -\mathbf{b}[Memory]; \\
(c_7) & 4\left(\overline{\boldsymbol{\mu}}[ExtBus] - \mathbf{b}[ExtBus]\left(1 - \frac{\overline{\boldsymbol{\mu}}[Memory]}{\mathbf{b}[Memory]}\right)\right) \le 0; \\
& 4\left(\overline{\boldsymbol{\mu}}[ExtBus] - \mathbf{b}[ExtBus]\left(1 - \frac{\overline{\boldsymbol{\mu}}[Queue]}{\mathbf{b}[Queue]}\right)\right) \le 0;
\end{array}
$$

Table 1: Constraints for the model in Figure 1.

transition name have been used for reasons of space. The solution for the linear programming problems leads to upper and lower bounds, for the throughput of transitions, given by

$$\frac{8}{11} \le \boldsymbol{\chi}[e\_e\_a] \le 2$$

while the "exact" solution with exponential distribution is

$$\boldsymbol{\chi}[e\_e\_a] = 1.71999$$

An improvement in the lower bound can be obtained observing that when a token arrives in place Choice transition choose_m is enabled at least for one transition instance. This implies that the average marking of place Choice is equal to 0 (transition choose_m is immediate), so

$$\overline{\boldsymbol{\mu}}[Choice] = 0; \quad \mathbf{b}[Choice] = 0$$

(only tangible markings are considered) can be added to the set of constraints. Moreover place Memory is implicit with respect to the enabling of transition b_ext_acc, so we can consider this transition as having only two input places, so constraint (c_6) can be applied instead of constraint (c_7). Finally,

$$\mathbf{b}[Queue] = 3$$

can be added since the output transition of place Queue is immediate, and from the behaviour of the model it is clear that at most 3 processors can be waiting in the queue. The relations (c_7) in the above linear programming problem can thus be replaced with the new constraint:

$$4\left(\overline{\boldsymbol{\mu}}[ExtBus] + \frac{\mathbf{b}[ExtBus]}{\mathbf{b}[Queue]}\overline{\boldsymbol{\mu}}[Queue] - \mathbf{b}[ExtBus]\right) \leq 0$$

where $\mathbf{b}[Queue] = 3$. Solving this reduced linear programming problem the values obtained for the upper and lower bounds are:

$$1 \leq \boldsymbol{\chi}[e\_e\_a] \leq 2$$