Solution to Exercise 25.2-4

This solution is also posted publicly

With the superscripts, the computation is $d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$. If, having dropped the superscripts, we were to compute and store d_{ik} or d_{kj} before using these values to compute d_{ij}, we might be computing one of the following:

$$\begin{aligned} d_{ij}^{(k)} &= \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k)} + d_{kj}^{(k-1)}\right) , \\ d_{ij}^{(k)} &= \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k)}\right) , \\ d_{ij}^{(k)} &= \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k)} + d_{kj}^{(k)}\right) . \end{aligned}$$

In any of these scenarios, we're computing the weight of a shortest path from i to j with all intermediate vertices in $\{1, 2, \ldots, k\}$. If we use $d_{ik}^{(k)}$, rather than $d_{ik}^{(k-1)}$, in the computation, then we're using a subpath from i to k with all intermediate vertices in $\{1, 2, \ldots, k\}$. But k cannot be an *intermediate* vertex on a shortest path from i to k, since otherwise there would be a cycle on this shortest path. Thus, $d_{ik}^{(k)} = d_{ik}^{(k-1)}$. A similar argument applies to show that $d_{kj}^{(k)} = d_{kj}^{(k-1)}$. Hence, we can drop the superscripts in the computation.