

Points on Computable Curves

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Abstract

The “analyst’s traveling salesman theorem” of geometric measure theory characterizes those subsets of Euclidean space that are contained in curves of finite length. This result, proven for the plane by Jones (1990) and extended to higher-dimensional Euclidean spaces by Okikiolu (1992), says that a bounded set K is contained in some curve of finite length if and only if a certain “square beta sum”, involving the “width of K ” in each element of an infinite system of overlapping “tiles” of descending size, is finite.

In this paper we characterize those points of Euclidean space that lie on computable curves of finite length. We do this by formulating and proving a computable extension of the analyst’s traveling salesman theorem. Our extension, the computable analyst’s traveling salesman theorem, says that a point in Euclidean space lies on some computable curve of finite length if and only if it is “permitted” by some computable “Jones constriction”. A Jones constriction here is an explicit assignment of a rational cylinder to each of the above-mentioned tiles in such a way that, when the radius of the cylinder corresponding to a tile is used in place of the “width of K ” in each tile, the square beta sum is finite. A point is permitted by a Jones constriction if it is

contained in the cylinder assigned to each tile containing the point. The main part of our proof is the construction of a computable curve of finite length traversing all the points permitted by a given Jones constriction. Our construction uses the main ideas of Jones’s “farthest insertion” construction, but takes a very different form, because, having no direct access to the points permitted by the Jones constriction, our algorithm must work exclusively with the constriction itself.

1. Introduction

Where can an infinitely small robot go? This paper answers a precise form of this fanciful question by formulating and proving a computable extension of the celebrated “analyst’s traveling salesman theorem” of geometric measure theory.

The precise statement of our question is straightforward. Our robot is the size of a geometric point (the “ultimate nanobot”), and it moves in a Euclidean space \mathbb{R}^n , where $n \geq 2$. The robot’s motion is algorithmic, and there are no obstacles, thermal effects, or quantum effects, so its path is a computable curve, i.e., a curve traced by a computable function $f : [0, 1] \rightarrow \mathbb{R}^n$. The robot’s path has arbitrary but finite length. (The computable curve is *rectifiable*. Among other things, this implies that it is not a space-filling curve [19].) The robot’s motion is otherwise unrestricted. For example, it may cross or retrace its own path, so the function f

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is not required to be one-to-one. (In the terminology of some, f describes a tour, rather than a curve. In the terminology of others, f describes a curve that need not be simple.)

The collection of all possible paths of our robot forms a “computable transit network” $\mathcal{R} \subseteq \mathbb{R}^n$. This is the set of all *rectifiable points* in \mathbb{R}^n , i.e., all points $x \in \mathbb{R}^n$ lying on rectifiable computable curves. Our question is simple. Which points in \mathbb{R}^n lie in the set \mathcal{R} ?

A brief summary of some basic properties of \mathcal{R} (developed in detail in section 3) sets the stage for our main results. It is easy to see that \mathcal{R} has Hausdorff dimension 1, so most points in \mathbb{R}^n are not rectifiable. On the other hand, \mathcal{R} is a dense subset of \mathbb{R}^n , and \mathcal{R} is path-connected in the strong sense that any two points in \mathcal{R} lie on a *single* computable curve of finite length. Each point $x \in \mathcal{R}$ has dimension at most 1 (by which we mean that $\{x\}$ has constructive dimension at most 1 [12]), but the complement of \mathcal{R} contains points of arbitrarily small dimension, so this does not characterize membership in \mathcal{R} .

Our main theorem characterizes points in \mathcal{R} by extending the famous “analyst’s traveling salesman theorem” of geometric measure theory to a theorem in computable analysis. The analyst’s traveling salesman theorem, proven for \mathbb{R}^2 by Jones in 1990 [7] and extended to \mathbb{R}^n for $n \geq 2$ by Okikiolu in 1991 [17] (see also the monographs [14, 4]), gives a precise characterization of those subsets of \mathbb{R}^n that are contained in rectifiable curves.

For each $m \in \mathbb{Z}$, let \mathcal{Q}_m be the set of all *dyadic cubes of order m* , which are half-closed, half-open cubes

$$Q = [a_1, a_1 + 2^{-m}) \times \cdots \times [a_n, a_n + 2^{-m})$$

in \mathbb{R}^n with $a_1, \dots, a_n \in 2^{-m}\mathbb{Z}$. Note that such a cube Q has sidelength $\ell(Q) = 2^{-m}$ and all its vertices in $2^{-m}\mathbb{Z}^n$. Let $\mathcal{Q} = \bigcup_{m \in \mathbb{Z}} \mathcal{Q}_m$ be the set of all dyadic cubes of all orders. We regard each dyadic cube Q as an “address” of the larger cube $3Q$, which has the same center as Q and sidelength $\ell(3Q) = 3\ell(Q)$. The analyst’s traveling salesman theorem is stated in terms of the resulting system $\{3Q \mid Q \in \mathcal{Q}\}$ of overlapping cubes.

Let K be a bounded subset of \mathbb{R}^n . For each $Q \in \mathcal{Q}$, let $r(Q)$ be the least radius of any infinite closed cylinder in any direction in \mathbb{R}^n that contains all of $K \cap 3Q$. Then the *Jones beta-number* of K at Q is

$$\beta_Q(K) = \frac{r(Q)}{\ell(Q)},$$

and the *Jones square beta-number* of K is

$$\beta^2(K) = \sum_{Q \in \mathcal{Q}} \beta_Q(K)^2 \ell(Q)$$

(which may be infinite). Here is the analyst’s traveling salesman theorem.

Theorem 1.1 (Jones [7], Okikiolu [17]). *Let $K \subseteq \mathbb{R}^n$ be bounded. Then K is contained in some rectifiable curve if and only if $\beta^2(K) < \infty$.*

Jones’s proof of the “if” direction of Theorem 1.1 is an intricate “farthest insertion” construction of a curve containing K , together with an amortized analysis showing that the length of this curve is finite. This proof works in any Euclidean space \mathbb{R}^n . However, Jones’s proof of the “only if” direction of Theorem 1.1 uses non-trivial methods from complex analysis and only works in the Euclidean plane \mathbb{R}^2 (regarded as the complex plane \mathbb{C}). Okikiolu’s subsequent proof of the “only if” direction is a clever geometric argument that works in any Euclidean space \mathbb{R}^n . (It should also be noted that these papers establish a *quantitative* relationship between $\beta^2(K)$ and the infimum length of a curve containing K , and that the constants in this relationship have been improved in the recent thesis by Schul [20]. In contrast, in this paper, we are only concerned with the *qualitative* question of the *existence* of a rectifiable curve containing K .)

Theorem 1.1 is generally regarded as a solution of the “analyst’s traveling salesman problem” (analyst’s TSP), which is to characterize those sets $K \subseteq \mathbb{R}^n$ that can be traversed by curves of finite length. It is then natural to pose the *computable analyst’s TSP*, which is to characterize those sets $K \subseteq \mathbb{R}^n$ that can be traversed by *computable* curves of finite length. While the analyst’s TSP is only interesting for infinite sets K (because every finite set K is contained in a rectifiable curve), the computable analyst’s TSP is interesting for arbitrary sets K . In fact, the question posed at the beginning of this introduction is precisely the computable analyst’s TSP restricted to singleton sets $K = \{x\}$. (We repeat that we are focusing on the *qualitative* question here. The quantitative version of the analyst’s TSP is interesting for finite sets, though not for singletons.)

To solve the computable analyst’s TSP, we first replace the Jones square beta-number of the arbitrary set K with a data structure that can be required to be computable. To this end, we define a *cylinder assignment* to be a function γ assigning to each dyadic cube Q an (infinite) closed *rational cylinder* $\gamma(Q)$, by which we mean that $\gamma(Q)$ is a cylinder whose axis passes through two (hence infinitely many) points of \mathbb{Q}^n and whose radius $\rho(Q)$ is rational. (If $\rho(Q) = 0$, the cylinder is a line; if $\rho(Q) < 0$, the cylinder is empty.) The *set permitted* by a cylinder assignment γ is the (closed) set $\kappa(\gamma)$ consisting of all points $x \in \mathbb{R}^n$ such that, for all $Q \in \mathcal{Q}$,

$$x \in (3Q)^o \Rightarrow x \in \gamma(Q),$$

where $(3Q)^\circ$ is the interior of $3Q$.

There is one technical point that needs to be addressed here. If γ is a cylinder assignment that, at some $Q \in \mathcal{Q}$, prohibits a subcube $3Q'$ of $3Q$ (i.e., $\gamma(Q) \cap (3Q')^\circ = \emptyset$), then $\kappa(\gamma)$ contains no interior point of $3Q'$, so it is pointless and misleading for γ to assign Q' a cylinder $\gamma(Q')$ that meets $(3Q')^\circ$. We define a cylinder assignment γ to be *persistent* if it does not make such pointless assignments, i.e., if, for all $Q, Q' \in \mathcal{Q}$ with $Q' \subseteq Q$ and $\gamma(Q) \cap (3Q')^\circ = \emptyset$, we have $\gamma(Q') \cap (3Q')^\circ = \emptyset$. It is easy to transform a cylinder assignment γ into a persistent cylinder assignment γ' that is equivalent to γ in the sense that $\kappa(\gamma) = \kappa(\gamma')$, with γ' computable if γ is.

Definition. Let γ be a cylinder assignment.

1. The *Jones beta-number* of γ at a cube $Q \in \mathcal{Q}$ is

$$\beta_Q(\gamma) = \frac{\rho(Q)}{\ell(Q)}.$$

2. The *Jones square beta-number* of γ is

$$\beta^2(\gamma) = \sum_{Q \in \mathcal{Q}} \beta_Q(\gamma)^2 \ell(Q).$$

Note that $\beta^2(\gamma)$ may be infinite.

Definition. A *Jones constriction* is a persistent cylinder assignment γ for which $\beta^2(\gamma) < \infty$.

We can now state our main result, the computable analyst's traveling salesman theorem.

Theorem 1.2 *Let $K \subseteq \mathbb{R}^n$ be bounded. Then K is contained in some rectifiable computable curve if and only if there is a computable Jones constriction γ such that $K \subseteq \kappa(\gamma)$.*

Theorem 1.2 solves the computable analyst's TSP, and thus immediately solves our question about where an infinitely small robot can go:

Corollary 1.3 *A point $x \in \mathbb{R}^n$ is rectifiable if and only if x is permitted by some computable Jones constriction. That is,*

$$\mathcal{R} = \bigcup_{\text{computable } \gamma} \kappa(\gamma),$$

where the union is taken over all computable Jones constrictions.

It should be noted that (the proof of) Theorem 1.2 relativizes to arbitrary oracles, so it implies Theorem 1.1. This is the sense in which our computable analyst's traveling salesman theorem is an extension of the analyst's traveling salesman theorem.

Our proof of the “only if” direction of Theorem 1.2 is easy, because we are able to use the corresponding part of Theorem 1.1 as a “black box”. However, our proof of the “if” direction is somewhat involved. Given an arbitrary computable Jones constriction γ , we construct a rectifiable computable curve containing $\kappa(\gamma)$. In this construction, we are able to follow the broad outlines of Jones's “farthest insertion” construction and to use its key ideas, but we have an additional obstacle to overcome. The analyst's TSP does not require an algorithm, so Jones's proof can simply “choose” elements of the given set K according to various criteria at each stage of the construction (often moving these points later as needed). However, even if γ is computable, neither the set $\kappa(\gamma)$ nor its elements need be computable. Hence the algorithm for our computable curve cannot directly choose points in (or even reliably near) $\kappa(\gamma)$. Our construction succeeds by carefully separating the algorithm from the amortized analysis of the length of the curve that it computes. The proof is discussed in some detail in section 4 and at greater length in the full version of this paper.

2. Curves and Computability

We fix an integer $n \geq 2$ and work in the Euclidean space \mathbb{R}^n . A *curve* is a continuous function $f : [0, 1] \rightarrow \mathbb{R}^n$. The *length* of a curve f is

$$\text{length}(f) = \sup_{\vec{a}} \sum_{i=0}^{k-1} |f(a_{i+1}) - f(a_i)|,$$

where $|x|$ is the Euclidean norm of a point $x \in \mathbb{R}^n$ and the supremum is taken over all *dissections* \vec{a} of $[0, 1]$, i.e., all $\vec{a} = (a_0, \dots, a_k)$ with $0 = a_0 < a_1 < \dots < a_k = 1$. Note that $\text{length}(f)$ is the length of the actual path traced by f . If f is one-to-one (i.e., the curve is *simple*), then $\text{length}(f)$ coincides with $\mathcal{H}^1(f([0, 1]))$, which is the length (i.e., the one-dimensional Hausdorff measure [3]) of the range of f , but, in general, f may “retrace” parts of its range, so $\text{length}(f)$ may exceed $\mathcal{H}^1(f([0, 1]))$. A curve f is *rectifiable* if $\text{length}(f) < \infty$.

A *tour* of a set $K \subseteq \mathbb{R}^n$ is a curve $f : [0, 1] \rightarrow \mathbb{R}^n$ such that $K \subseteq f([0, 1])$.

Since curves are continuous, the extended computability notion introduced by Braverman [1] coincides with the computability notion formulated in the 1950s by Grzegorzczuk [5] and Lacombe [9] and expounded in the recent paper by Braverman and Cook [?] and in the monographs [18, 8, 22]. Specifically, a curve $f : [0, 1] \rightarrow \mathbb{R}^n$ is *computable* if there is an oracle

Turing machine M with the following property. For all $t \in [0, 1]$ and $r \in \mathbb{N}$, if M is given a function oracle $\varphi_t : \mathbb{N} \rightarrow \mathbb{Q}$ such that, for all $k \in \mathbb{N}$, $|\varphi_t(k) - t| \leq 2^{-k}$, then M , with oracle φ_t and input r , outputs a rational point $M^{\varphi_t}(r) \in \mathbb{Q}^n$ such that $|M^{\varphi_t}(r) - f(t)| \leq 2^{-r}$.

A point $x \in \mathbb{R}^n$ is *computable* if there is a computable function $\psi_x : \mathbb{N} \rightarrow \mathbb{Q}^n$ such that, for all $r \in \mathbb{N}$, $|\psi_x(r) - x| \leq 2^{-r}$. It is well known and easy to see that, if $f : [0, 1] \rightarrow \mathbb{R}^n$ and $t \in [0, 1]$ are computable, then $f(t)$ is computable.

3. The Set \mathcal{R}

As in the introduction, we let \mathcal{R} denote the set of all rectifiable points in \mathbb{R}^n , i.e., points that lie on rectifiable computable curves. We briefly discuss the structure of \mathcal{R} , referring freely to existing literature on fractal geometry [3] and effective dimension [11, 12, 2].

For each rectifiable curve f , we have $\mathcal{H}^1(f([0, 1])) \leq \text{length}(f) < \infty$, so the Hausdorff dimension of $f([0, 1])$ is 1, unless $f([0, 1])$ is a single point (in which case the Hausdorff dimension is 0). Since \mathcal{R} is the union of countably many such sets $f([0, 1])$, it follows by countable stability [3] that \mathcal{R} has Hausdorff dimension 1. This implies that \mathcal{R} is a Lebesgue measure 0 subset of \mathbb{R}^n , i.e., that almost every point in \mathbb{R}^n lies in the complement of \mathcal{R} .

Since \mathcal{R} contains every computable point in \mathbb{R}^n , \mathcal{R} is dense in \mathbb{R}^n . Also, if $x \in f([0, 1])$ and $y \in g([0, 1])$, where f and g are rectifiable computable curves, then we can use f , g , and the segment from $f(1)$ to $g(0)$ to assemble a rectifiable computable curve h such that $x, y \in h([0, 1])$. Hence, \mathcal{R} is path-connected in the strong sense that any two points in \mathcal{R} lie in a *single* rectifiable computable curve.

For each rectifiable computable curve f , the set $f([0, 1])$ is a computably closed (i.e., Π_1^0) subset of \mathbb{R}^n [16]. Since \mathcal{R} is the union of all such $f([0, 1])$, it follows by Hitchcock’s correspondence principle [6] that the constructive dimension of \mathcal{R} coincides with its Hausdorff dimension, which we have observed to be 1. (It is worth mention here that \mathcal{R} can easily be shown *not* to have computable measure 0, whence \mathcal{R} has computable dimension n [11]. By Staiger’s correspondence principle [21, 6], this implies that \mathcal{R} is not a Σ_2^0 set.) It follows that each point $x \in \mathcal{R}$ has dimension at most 1 (in the sense that $\{x\}$ has constructive dimension 1 [12]). It might be reasonable to conjecture that this actually characterizes points in \mathcal{R} , but the following example shows that this is not the case.

Construction 3.1 Given an infinite binary sequence R , define a sequence A_0, A_1, A_2, \dots of closed squares in \mathbb{R}^2 by the following recursion. First, $A_0 = [0, 1]^2$. Next, assuming that A_n has been defined, let a and b be the $2n$ th and $(2n + 1)$ st bits, respectively of R . Then A_{n+1} is the ab -most closed subsquare of A_n with $\text{area}(A_{n+1}) = \frac{1}{16} \text{area}(A_n)$, where 00 = “lower left”, 01 = “lower right”, 10 = “upper left”, and 11 = “upper right”. Let x_R be the unique point in \mathbb{R}^2 such that $x_R \in A_n$ for all $n \in \mathbb{N}$.

It is well known [15, 4] that the set K consisting of all such points x_R is a bounded set with positive, finite one-dimensional Hausdorff measure (and hence with Hausdorff dimension 1), but that K is not contained in any rectifiable curve. The next lemma is a constructive extension of this fact.

Lemma 3.2 For any sequence R that is random (in the sense of Martin-Löf [13]; see also [10, 2]), the point x_R of Construction 3.1 has dimension 1 and does not lie on any computable curve of finite length.

The following theorem shows that more is true, although the proof, a Baire category argument, does not yield such a concrete example.

Theorem 3.3 The complement of \mathcal{R} contains points of arbitrarily small dimension, including 0.

4. The Computable Analyst’s Traveling Salesman Theorem

This section presents the main ideas of the proof of Theorem 1.2. The detailed proof appears in preliminary form in the full version of this paper.

We first dispose of the “only if” direction. If we are given a rectifiable computable curve f and a rational $\epsilon > 0$, it is routine to construct a computable Jones constriction γ such that $f([0, 1]) \subseteq \kappa(\gamma)$ and $\beta^2(\gamma) \leq \beta^2(f([0, 1])) + \epsilon$. The “only if” direction of Theorem 1.2 hence follows easily from the “only if” direction of Theorem 1.1. We thus focus our attention on proving the “if” direction of Theorem 1.2.

As pointed out by Jones [7], the analyst’s TSP is significantly different from the classical TSP in that it typically involves uncountably many points at locations that are not explicitly specified. In his construction, he has the privilege to “know” whether a point is in the set K or not, since he is concerned only with the existence of a tour and not with the computability of the tour. This is no longer true in our situation, since we work with only

a computable constriction, from which we may not computably determine whether a point is in the set. Although the situations differ by so much, ideas with a flavor of the “farthest insertion” and “nearest insertion” heuristics that are used in Jones’s argument and the classical TSP are essential parts of our solution.

Given a computable Jones constriction γ , we construct computably a tour $f : [0, 1] \rightarrow \mathbb{R}^n$ of the set $K = \kappa(\gamma)$ permitted by γ such that $\kappa(\gamma) \subseteq f([0, 1])$ and the length of the tour is finite.

Our construction proceeds in stages. In each stage $m \in \mathbb{N}$, a set of points with regulated density is chosen according to the constriction and a tour f_m of these points is constructed so that every point in K is at most roughly 2^{-m} from the tour. Every tour is constructed by patching the previous tour locally so that the sequence of tours $\{f_m\}$ converges computably.

During the tour patching at each stage, the insertion ideas mentioned earlier are applied at different parts of the set K according to the local topology given by the constriction. Note that it is not completely clear that the use of “farthest insertion” is absolutely necessary. However, it greatly facilitates the associated amortized analysis of length, which is as crucial in our proof as it is in Jones’s. In the following, we describe in more detail how and when these ideas are applied in the algorithmic construction of the tour.

In each stage $m \in \mathbb{N}$, we look at cubes Q of side-length $A2^{-m}$, where $A = 2^{k_0}$ is a sufficiently large universal constant. We pick points so that they are at least 2^{-m} from each other and every point in K is at most 2^{-m} from some of those chosen points. Based on the value of $\beta_Q(\gamma)$, which measures the relative width of $3Q \cap K$, we divide cubes into “narrow” ones ($\beta_Q(\gamma) < \epsilon_0$) and “fat” ones ($\beta_Q(\gamma) \geq \epsilon_0$), where ϵ_0 is a small universal constant.

The fat cubes are easy to process, since the associated square beta-number is large. We connect the points in those cubes to nearby surrounding points, some of which are guaranteed to be in the previous tour due to the density of the points in the tour. Since the points are chosen with regulated density, the number of connections we make here is bounded by a universal constant. The length of each connection is proportional to the sidelength of the cube, which is proportional to 2^{-m} . Thus the total length we add to the tour is bounded by $c_0 \cdot \epsilon_0^2 \ell(Q)$, which is then bounded by $c_0 \cdot \beta_Q^2(\gamma) \ell(Q)$, where c_0 is a sufficiently large universal constant.

For the narrow cubes, we carry out either “farthest insertion” or “nearest insertion” depending on the local topology around each insertion point.

Suppose that we are about to patch the existing tour

to include a point x . Since from stage to stage, the points are picked with increasing density, there is always a point z_1 already in the tour inside the cube that contains x . However, there are two possibilities for the neighborhood of x . One is that there is another point z_2 already in the tour and z_2 is inside the cube that contains x . The other possibility is that z_1 is the only such point.

In the first case, point x lies in a narrow cube and there are points z_1 and z_2 in the narrow cube such that x is between z_1 and z_2 . Points z_1 and z_2 are in the existing tour and are connected directly with a line segment in the tour. In this case, we apply “nearest insertion” by letting z_1 and z_2 be the closest two neighbors of x in the existing tour, breaking the line segment between z_1 , z_2 , and connecting z_1 to x and x to z_2 . The increment of the length of the tour is $\ell([z_1, x]) + \ell([x, z_2]) - \ell([z_1, z_2])$, which is bounded by $c_1 \cdot \beta_Q^2(\gamma) \ell(Q)$ by an application of the Pythagorean theorem, since the cube is very narrow.

In the second case, point z_1 is the only point in the existing tour that is in the same cube as x . It is not guaranteed that x can be inserted between two points in the existing tour. Even when it is possible, the other point in the existing tour would be outside the cube that we are looking at and thus it might require backtracking an unbounded number of stages to bound the increment of length, which would make the proof extremely complicated (if even possible). Therefore, we keep the patching for every point local and, in this case, we make sure x is locally the “farthest” point from z_1 and connect x directly to z_1 . (Note that the actual situation is slightly more involved and is addressed in the full proof.) In this case, the Pythagorean theorem cannot be used and thus we cannot use the Jones square beta-number to directly bound the increment of length. To remedy this, we employ amortized analysis and save spare square beta-numbers in a savings account over the stages and use the saved values to bound the length increment. In order for this to work, we choose ϵ_0 so small that at a particular neighborhood, “farthest insertion” does not happen very frequently and we always have the time to save up enough of the square beta-number before we need to use it.

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