Applying Sparse $\ell_1$-optimization to problems in robotics

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Abstract—Sparse $\ell_1$-optimization techniques have received a lot of attention in the signal processing and computer vision communities, where they have been applied to problems such as denoising [1], deblurring [2], and face recognition [3]. Using $\ell_1$-objective to solve an optimization problem has been shown to induce sparsity. Moreover, the problem is convex allowing a global minimum solution. Well studied techniques and solvers exist that allow efficient solutions for the optimization problem by posing it as either a Linear Problem (LP) or taking advantage of the sparse nature of the problem, i.e., homotopy based methods. In this work, we provide an overview of this sparse $\ell_1$-formulation and apply it to various problems in robotics including loop closure detection, place categorization and topological SLAM.

I. INTRODUCTION

In many robotics applications, we are interested in solving a one-to-many data-association problem, that is, given the current observation, we are interested in finding one or a few among all the previous observations that are in some sense similar to the present observation. For example, given the current image, is there a matching image that the robot has seen before? Or, given an image, can we find which class (kitchen, hallway, etc) does the image come from? In these problems, we expect the solution to be sparse, that is, only a few (if any) of the previous images match the current image and only a small number of different instances of the observed classes explain the current observation. With this intuition in mind, we want to represent these problems in a more general formulation which can give us “sparse” solutions, i.e. solutions in which only a small number of previous observations interact to explain the current observation.

We begin by formulating this as: $Ax = b$ where $A \in \mathbb{R}^{n \times m}$ is a function of all the previous observations, $b \in \mathbb{R}^n$ is the current observation and we want to find $x \in \mathbb{R}^m$, which would indicate how the previous observations in $A$ interact to generate $b$. We call $A$ the dictionary. Since there can be many more observations compared to the dimension of each observation ($m > n$), the problem is under-determined and infinitely many solutions exist (if $A$ has full row-rank, that is, rank($A$) = $m$). We need to “regularize” i.e select a desirable solution based on our prior knowledge about it.

A first and commonly used approach is to look for the least squares solution, i.e. a solution that minimizes

$$\arg\min_x \|Ax - b\|_2^2 \Rightarrow x^* = (AA^T)^{-1}b \quad (1)$$

This allows a closed-form and unique solution but the solution is not-sparse in general. All the elements of $x^*$ are non-zero indicating that all the columns of $A$ are utilized in explaining $b$. Instead, we want a solution that contains very few non-zero elements.

By quantifying the sparsity of a vector using the $\ell_0$-norm, which is defined as the number of non-zero elements in a vector $|\{x_i | x_i \neq 0\}|$, we look for a sparse solution under the constraint that this solution should explain our observation i.e.

$$\arg\min_x \|x\|_0 \text{ subject to } Ax = b \quad (2)$$

Solving for the $\ell_0$-norm is in general NP-hard [4] as all the possible solution have to be enumerated and the constrained checked for fulfillment. Instead, we can relax the problem by solving for $\ell_1$-norm. For a vector $x$, the $\ell_1$-norm is defined as the sum of absolute values of all the elements in the vector i.e $||x||_1 = \sum_{i=1}^{n} |x_i|$. (2) can now be relaxed to,

$$\arg\min_x \|x\|_1 \text{ subject to } Ax = b \quad (3)$$

Note that the above formulation (3) assumes perfect reconstruction and does not cater for any noise. In order to deal with real-world problems, we also introduce a sparse noise term to explain our observations with both the original dictionary and the added noise, i.e.,

$$\arg\min_{\alpha} \|\alpha\|_1 \text{ subject to } D\alpha = b \quad (4)$$

where $D = \begin{bmatrix} A & I \end{bmatrix}$ and $\alpha = \begin{bmatrix} x \\ e \end{bmatrix}$. Note that this still has the general form as (3).

The formulation in (4) can be posed as a linear program (LP), a class of convex optimization problems and solved using interior point methods [5], whose complexity is $O(m^3)$. This makes such methods computationally infeasible for long-term operation. Alternatively, homotopy methods [6], [7] are specifically designed to take advantage of the properties of $\ell_1$-minimization. Relaxing the equality constraint in (4), we have the following constrained problem:

$$\arg\min_{\alpha} \|\alpha\|_1 \text{ subject to } \|D\alpha - b\|_2 \leq \epsilon \quad (5)$$

where $\epsilon > 0$ is a pre-determined noise level. This is termed the Basis Pursuit Denoising (BPDN) problem in compressive sensing. A variant of (5) is the unconstrained minimization:

$$\arg\min_{\alpha} \lambda \|\alpha\|_1 + \frac{1}{2} \|D\alpha - b\|_2^2 \quad (6)$$

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where $\lambda$ is a scalar weight parameter. The complexity of the homotopy method is $O(dn^2 + dn m)$ for recovering a $d$-sparse signal in $d$ steps, but knowing that the solution is sparse, makes these methods more efficient compared to primal-dual methods.

Solving (6) gives us information about which basis (columns) of the dictionary are involved in explaining our current observation and well as their contribution towards making it. The noise component ($e$) can be thought of as “innovation”, it is information that can not be explained by any of the observations.

II. APPLICATIONS

In this sections, we show how the presented formulation can be applied to various problems in robotics.

(a) Representation: Vectorized raw image down-sampled (size $8 \times 6$)

(b) Representation: GIST descriptor calculated over full-sized image

Fig. 1: Loop closures detected with the proposed approach using two different bases for the New College data set. In these plots, visual odometry (as provided with the dataset) is shown in gray and the loop closures in blue. The vertical axis represents time (in seconds) and the $x$ and $y$ axes represent horizontal position (in meters).

Fig. 2: Sparsity pattern induced by solving (4) for all the images. Each column $i$ of this matrix corresponds to the solution for the $i$-th image, and the non-zeros are the values in each column that are greater than $\tau = 0.99$. Note that the main diagonal occurs due to the current image being best explained by its neighbouring image, while the off-diagonal non-zero elements indicate the loop closures.

A. Loop Closure Detection

Appearance based loop closing is the problem of finding images in the past that are similar to the image being observed, where similarity can be quantified as similarity in the image space or some descriptor space where each image is represented using a descriptor such as HOG [8] or GIST [9].

Loop closings occur sparsely and therefore it is natural to look for a sparse contribution from all the previous images. More detail can be found in [10] and a short summary is presented here. Until now, we have not placed any restriction on what the columns of $D$ in (6) represent. For the task of appearance based loop closing, each column in the dictionary represents one of the previously observed images. In the simplest case, this can be the scaled-down image, reshaped into a vector in $R^n$ where $n = r \times c$, a product of the scaled rows and columns. An alternate approach would be to extract a descriptor such a HOG representing each image. The important aspect of this approach is that there is no prior learning involved, as is the case for state-of-the-art methods. Also, the formulation does not pose any restrictions on the type of representation, allowing a flexible way of defining “similarity”.

Solving (6) then gives us the contribution of each previous image towards constructing the current image, most of which are zero. The relative contribution can then be obtained by normalizing the solution $\alpha$ to obtain the unit vector $\alpha$ and applying a threshold ($\tau$) to find the most dominant contribution, hence detecting a loop closure.

Sample results are presented in Fig. 1 for the New College dataset [11]. Flexibility of representation allows us to use a scaled down image (size : $8 \times 6$) and well as a GIST descriptor for each image and be able to detect similar loops. The induced sparsity pattern for a threshold ($\tau = 0.99$) is given in Fig. 2 which shows that the solution is sparse when a loop can be detected because the image is mostly explained by a single image in the past, however in absence
of any loops, the previous image is the best explanation for
the current image, still keeping the solution sparse. The first
three runs in the starting circular region can be observed in
the top-left corner of Fig. 2. The problem can be solved
efficiently both because of the inherent sparsity as well as
the availability of solvers due to the maturity of convex-
optimization techniques.

B. Visual Place Categorization

A closely related problem to loop closing is that of place
categorization in which we are interested in finding the class
to which the current image belongs rather than a single
image. Assuming we have annotated data that can provide
us image-class correspondence, where the classes may be
corridor, kitchen etc., we can collect images for each class
in a contiguous submatrix \( A_i \) for \( i = 1 \ldots n_c \) where \( n_c \) is
the number of classes. The dictionary in (6) will now have
the form \( D = [A_1 A_2 \ldots A_n I] \).

We can then solve (6) and normalize the solution to have
a unit norm as before. In this case, we are interested in
finding the class with the maximum contribution, which can
be obtained by summing up the contributions within each
submatrix of the dictionary. \( \alpha_i = \arg \max \sum_{j \in A_i} \alpha_j \) where
\( |A_i| \) are the indices of the columns in the dictionary \( (D) \)
corresponding to class \( i \). Additionally, the columns corre-
spending to the Identity matrix form a no-decision class.

Using this approach along with a GIST based image
representation, Carillo et. al [12] have demonstrated results
that are comparable to state-of-the-art methods and perform
well under different lighting conditions as well as minor
changes in the environment over time.

C. Surprising images / Key-frame discovery

In many applications, we are interested in finding images
that represent our lack of knowledge i.e. images that can con-
tribute to the expansion of current model or understanding of
the world. In these scenarios, given a image, we need to find
a way of quantifying how well our current understanding of
the world represents this image. That is exactly the question
that \( \ell_1 \)-optimization answers. More specifically, the solution
\( \alpha \) is composed of two parts, one coming from all the previous
observation \( x \) and one that comes from the noise (things that
cannot be explained by the dictionary yet) part \( e \). In order
to quantify how well our current image can be represented
by all (or a subset of) the previous images, we can solve (6)
given all the previous images and the current observation.

In order to find out the innovation (or noise), we find the
unit vector \( \alpha \) and calculate the sum of the component
corresponding the noise-term. This gives us an estimate of
how well the current image is represented by the dictionary.
If the contribution of the noise-term is above a certain
threshold, the current image can be added to the dictionary
as it represents information that we are currently lacking.

Some results obtained using different image sizes and error
threshold are give in Fig. 3, where the threshold \( \tau \) is
the maximum allowed contribution of error. If the error is greater
than this threshold, the image in added to the dictionary.

Fig. 3: Percentage of the total (2653) images retained using
different thresholds (\( \tau \)) for the allowed error. Each line
represents the size of the scaled image. All plots are for
the New College dataset.

III. Conclusion

We have presented a new formulation as an insight into
mapping related problems. The formulation takes advantage
of sparsity that is common and inherent to such problems.
We have presented three problems to which the formulation
can be applied.

IV. Questions

1. What other problems can be represented by this
formulation? 2. For loop closure detection, specifically in
topological SLAM, what strategies can be applied to ensure
robustness against perceptual aliasing?

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