ON REACHABILITY IN AUTONOMOUS CONTINUOUS PETRI NET SYSTEMS

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Abstract. Fluidification is a common relaxation technique used to deal in a more friendly way with large discrete event dynamic systems. In Petri nets, fluidification leads to continuous Petri nets systems in which the firing amounts are not restricted to be integers. For these systems reachability can be interpreted in several ways. The concepts of *reachability* and lim-*reachability* were considered in [7]. They stand for those markings that can be reached with a finite and an infinite firing sequence respectively. This paper introduces a third concept, the δ*-reachability*. A marking is δ-reachable if the system can get arbitrarily close to it with a finite firing sequence. A full characterization, mainly based on the state equation, is provided for all three concepts for general nets. Under the condition that every transition is fireable at least once, it holds that the state equation does not have spurious solutions if δ-reachability is considered. Furthermore, the differences among the three concepts are in the border points of the spaces they define. For mutual lim-reachability and δ-reachability among markings, i.e., reversibility, a necessary and sufficient condition is provided in terms of liveness.

1 Introduction

Discrete systems with large populations or heavy traffic appear frequently in many fields: manufacturing processes, logistics, telecommunication systems, traffic systems,... It becomes, therefore, interesting to develop adequate formalisms and tools for the analysis and verification of such systems. The "natural" approach to study the above mentioned kind of systems consists in using discrete models. The main drawback is that often an exploration of the state space is needed for the verification of properties. Unfortunately, the size of the state space can grow exponentially with respect to the size of the population of the system, and so many properties are computationally too heavy to be verified.

An interesting approach to study discrete systems with large populations is based on the fluidification of the model. Thus, it is not discrete any more but continuous. This is a classical relaxation technique that can also be applied in

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the context of Petri nets. Usually, but not always [9], the greater the population of the discrete system the better the continuous approximation.

In PNs, fluidification has been introduced independently from three different perspectives:

- At the net level fluidification was introduced and developed by R. David and coauthors since 1987 [3, 1]. In this case, the fluidification of timed discrete systems generates deterministic continuous models, and also hybrid models if there is a partial fluidification.
- Analogously, fluidifying the firing count vector (thus also the marking) in the state equation allows the use of convex geometry and linear programming instead of integer programming, making possible the verification of some properties in polynomial time. The systematic use of linear programming on autonomous and timed system was proposed also in 1987 [8, 10].
- K. Trivedi and his group introduced [11, 2] a partial fluidification on some stochastic models. The fluidification only affects one or a limited number of places originating stochastic hybrid systems.

Like in [7], in this paper autonomous Petri net models will be considered. In particular, this means that no time interpretation will be applied on the firing of the transitions. A total nondeterminism on the evolution of the system exists. Notice, however, that if the transitions are timed, the evolution/behaviour of the system will always be constrained to some of the possible evolutions/behaviours of the autonomous system.

The paper is organized as follows: in Section 2 reachability in continuous systems is introduced formally and by means of examples. Three different ways of understanding (interpreting) reachability will be considered: reachability in a finite number of steps or simply reachability, reachability in an infinite number of steps or lim-reachability, and δ-reachability that has to do with the capacity of the system to get arbitrarily close to a given "continuous" marking. In order to make the paper more readable, a preview of the main results will be given in that section. Sections 3, 4 and 5 are devoted to the characterization of the sets of reachable markings according to the different concepts: reachability, lim-reachability and δ-reachability respectively. Moreover, it will be seen that it is decidable whether a given "continuous" marking belongs to any of those three concepts. Finally, Section 6 studies reversibility in continuous systems.

2 Definitions and Preview

In the following it is assumed that the reader is familiar with Petri nets (PNs) (see [6, 4] for example). The usual PN system will be denoted as $\langle \mathcal{N}, \mathbf{m_0} \rangle$, where $\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post} \rangle$. If not explicitly said, all the Petri nets systems considered here are *continuous*. A continuous system is understood as a relaxation of a *discrete* system. The main difference between continuous and discrete PNs is in the firing count vector and consequently in the marking, which in discrete PNs are restricted to be in the naturals, while in continuous PNs are relaxed into the non-negative real numbers. The marking of a place can be seen as an amount of fluid being stored, and the firing of a transition can be considered as a flow of this fluid going from a set of places (input places) to another set of places (output places). Thus, instead of tokens and discrete firings, it is more convenient to talk of levels in the places (deposits/reservoirs) and flows through transitions (valves).

The firing of a transition is also modified and brought to the non-negative real domain. A transition t is *enabled* at $\mathbf{m}$ iff for every $p \in {}^\bullet t$, $\mathbf{m}[p] > 0$. In other words, the enabling condition of continuous systems and that of discrete ordinary systems can be expressed in an "analogous" way: every input place should be marked. Notice that to decide whether a transition in a continuous system is enabled or not it is not necessary to consider weights of the arcs going from the input places to the transition. However, the arc weights are important to compute the enabling degree of a transition and to obtain the new marking after a firing. As in discrete systems, the *enabling degree* at $\mathbf{m}$ of a transition measures the maximal amount in which the transition can be fired in a single occurrence, i.e., $\text{enab}(t, \mathbf{m}) = \min_{p \in {}^\bullet t}\{\mathbf{m}[p]/\mathbf{Pre}[p, t]\}$. The firing of t in a certain amount $\alpha \leq \text{enab}(t, \mathbf{m})$ leads to a new marking $\mathbf{m'}$, and it is denoted as $\mathbf{m} \xrightarrow{\alpha t} \mathbf{m'}$. It holds $\mathbf{m'} = \mathbf{m} + \alpha \cdot \mathbf{C}[P, t]$, where $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$ is the token flow matrix (incidence matrix if $\mathcal{N}$ is self-loop free). Hence, as in discrete systems, $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}$, the state (or fundamental) equation summarizes the way the marking evolves. Right and left natural annullers of the token flow matrix are called T- and P-semiflows, respectively. As in discrete systems, when $\mathbf{y} \cdot \mathbf{C} = \mathbf{0}$, $\mathbf{y} > \mathbf{0}$ the net is said to be *conservative*, and when $\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ the net is said to be *consistent*. A set of places Θ is a *trap* iff $\Theta^\bullet \subseteq {}^\bullet\Theta$. Similarly, a set of places Σ is a *siphon* iff ${}^\bullet\Sigma \subseteq \Sigma^\bullet$. The support of a vector $\mathbf{x} \geq \mathbf{0}$ will be denoted as $\|\mathbf{x}\|$ and represents the set of positive elements of $\mathbf{x}$.

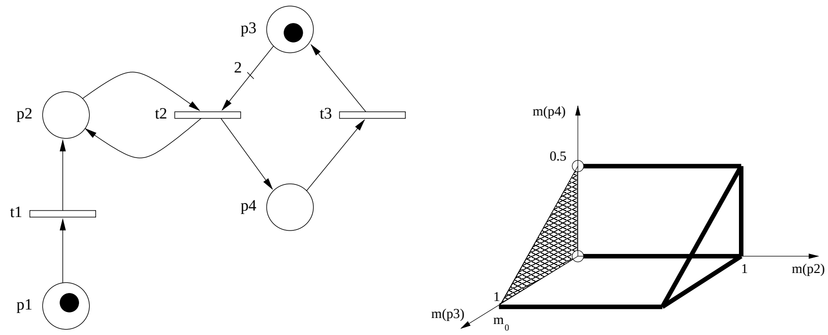
In order to illustrate the firing rule in a continuous system, let us consider the system in Figure 1(a). The only enabled transition at the initial marking is t_1 whose enabling degree is 1. Hence, it can be fired in any real quantity going from 0 to 1. For example, firing by 0.5 would yield marking $\mathbf{m}_1 = (0.5,\ 0.5,\ 1,\ 0)$. At $\mathbf{m}_1$ transition t_2 has enabling degree equal to 0.5; if it is fired in this amount the resulting marking is $\mathbf{m}_2 = (0.5,\ 0.5,\ 0,\ 0.5)$. Both $\mathbf{m}_1$ and $\mathbf{m}_2$ are reachable markings with finite firing sequences, or simply reachable markings.

The set of all reachable markings for a given system $\langle \mathcal{N}, \mathbf{m_0} \rangle$ is denoted as $\text{RS}(\mathcal{N}, \mathbf{m_0})$:

Definition 1 $\text{RS}(\mathcal{N}, \mathbf{m_0}) = \{\ \mathbf{m}|$ *a finite fireable sequence* $\sigma = \alpha_1 t_{a_1} \ldots \alpha_k t_{a_k}$ *exists such that* $\mathbf{m_0} \xrightarrow{\alpha_1 t_{a_1}} \mathbf{m}_1 \xrightarrow{\alpha_2 t_{a_2}} \mathbf{m}_2 \ldots \xrightarrow{\alpha_k t_{a_k}} \mathbf{m}_k = \mathbf{m}$ *where* $t_{a_i} \in T$ *and* $\alpha_i \in \mathbb{R}^+\}$.

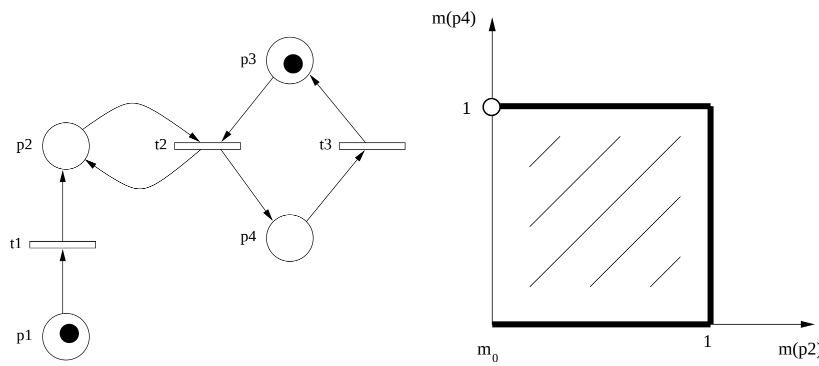
An interesting property of $\text{RS}(\mathcal{N}, \mathbf{m_0})$ is that it is a *convex* set (see [7]). That is, if two markings $\mathbf{m}_1$ and $\mathbf{m}_2$ are reachable, then for any $\alpha \in [0, 1]$ $\alpha\mathbf{m}_1 + (1 - \alpha)\mathbf{m}_2$ is also a reachable marking.

Let us consider again the system in Figure 1(a) with initial marking $\mathbf{m}_0 = (0.5,\ 0.5,\ 0,\ 0.5)$. At this marking either transition t_1 or transition t_3 can be



(a) (b)

Fig. 1. (a) Autonomous continuous system (b) Lim-Reachability space



(a) (b)

Fig. 2. (a) Autonomous continuous system (b) Reachability space and Lim-Reachability space coincide

fired. The firing of t_3 in an amount of 0.5 makes the system evolve to marking (0.5, 0.5, 0.5, 0) from which t_2 can be fired in an amount of 0.25 leading to marking (0.5, 0.5, 0, 0.25). Now, the markings of places p_1, p_2 and p_3 are the same that those of the system at $\mathbf{m}_0$, but the marking of p_4 is half of its marking at $\mathbf{m}_0$. The continuous firing of transitions t_2 and t_3 by its maximum enabling degree causes the elimination of half of the marking of p_4. Assume that we go on firing transitions t_2 and t_3. Then, as the number of firings increases the marking of p_4 approaches 0, value that will only be reached in the limit. Notice that the marking reached in the limit (0.5, 0.5, 0, 0) corresponds to the emptying of an initially marked trap ($\Theta = \{p_3, p_4\}, \Theta^{\bullet} = {}^{\bullet}\Theta = \{t_2, t_3\}$), fact that does not occur in discrete systems. From the point of view of the analysis of the behaviour of the system, it is interesting to consider this marking as limit-reachable, since it is the one to which the state of the system may converge. We will define the set of such markings that are reachable with a finite/infinite firing sequence:

Definition 2 *[7] Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a continuous system. A marking $\mathbf{m} \in (\mathbb{R}^+ \cup \{0\})^{|P|}$ is* lim-reachable*, iff a sequence of reachable markings $\{\mathbf{m}_i\}_{i \geq 1}$ exists such that*

$$\mathbf{m_0} \xrightarrow{\sigma_1} \mathbf{m}_1 \xrightarrow{\sigma_2} \mathbf{m_2} \cdots \mathbf{m}_{i-1} \xrightarrow{\sigma_i} \mathbf{m}_i \cdots$$

and $\lim\limits_{i \to \infty} \mathbf{m}_i = \mathbf{m}$. The lim*-reachable space is the set of* lim*-reachable markings, and will be denoted* lim-RS$(\mathcal{N}, \mathbf{m_0})$.

Figure 1(b) depicts the lim-reachability space of system in Figure 1(a). It is not necessary to represent the marking of place p_1 since $\mathbf{m}_1 = 1 - \mathbf{m}_2$. The set of lim-reachable markings is composed of the points inside the prism, the points in the non shadowed sides, the points in the thick edges and the points in the non circled vertices.

For some systems, the sets RS$(\mathcal{N}, \mathbf{m_0})$ and lim-RS$(\mathcal{N}, \mathbf{m_0})$ are identical. This means that in this case, with regard to the set of reachable markings, there is no difference between considering sequences of finite or infinite length. See Figure 2 for an example. Only $\mathbf{m}_2$ and $\mathbf{m}_4$ are represented since $\mathbf{m}_1 = 1 - \mathbf{m}_2$ and $\mathbf{m}_3 = 1 - \mathbf{m}_4$. The innner points of the square defined by the vertices $(0,\ 0)$, $(0,\ 1)$, $(1,\ 1)$ and $(1,\ 0)$, and the thick lines in Figure 2(b) are part of the reachability and the lim-reachability space, while the points going from $\mathbf{m_0}$ to $(0,\ 1)$ (including $(0,\ 1)$) do not belong to these sets.

However, in general, the set of reachable markings, RS$(\mathcal{N}, \mathbf{m_0})$ is a subset of the set of lim-reachable markings, lim-RS$(\mathcal{N}, \mathbf{m_0})$. For the system in Figure 3(b), neither p_1 nor p_2 can be emptied with a finite firing sequence because every time a transition is fired some marks are put in both places. For that system the set of reachable markings is $(\alpha,\ 2 - \alpha),\ 0 < \alpha < 2$. Nevertheless, considering the sequence $\frac{1}{2}t_1, \frac{1}{4}t_1, \frac{1}{8}t_1, \ldots$, in the k-th step, the system reaches the marking $(2^{-k},\ 2-2^{-k})$. When k tends to infinity the marking of the system tends to $(0,\ 2)$. Therefore the infinite firing of t_1 (t_2) will converge to a marking in which p_1 (p_2) is empty. Thus the set of markings reachable in the limit is $(\alpha,\ 2-\alpha),\ 0 \leq \alpha \leq 2$. Notice that the only difference between both sets lim-RS$(\mathcal{N}, \mathbf{m_0})$ and RS$(\mathcal{N}, \mathbf{m_0})$ is in the markings $(0,\ 2)$ and $(2,\ 0)$. Observe that even under consistency and conservativeness RS$(\mathcal{N}, \mathbf{m_0}) \neq$ lim-RS$(\mathcal{N}, \mathbf{m_0})$.

For the system in Figure 3(a), p_1 (p_2) can be emptied with the firing of t_1 (t_2) in an amount of 1. Hence, although the systems in Figure 3 have the same incidence matrix, their sets of finitely reachable markings are not the same.

Both RS$(\mathcal{N}, \mathbf{m_0})$ and lim-RS$(\mathcal{N}, \mathbf{m_0})$ are not in general closed sets. For example in Figure 2(b) the points on the segment going from $(0,\ 0)$ (initial marking) to $(0,\ 1)$ do neither belong to RS$(\mathcal{N}, \mathbf{m_0})$ nor to lim-RS$(\mathcal{N}, \mathbf{m_0})$. Nevertheless, any point on the right of this segment does belong to both sets RS$(\mathcal{N}, \mathbf{m_0})$ and lim-RS$(\mathcal{N}, \mathbf{m_0})$. For a given set A, the closure of A is equal to the points in A

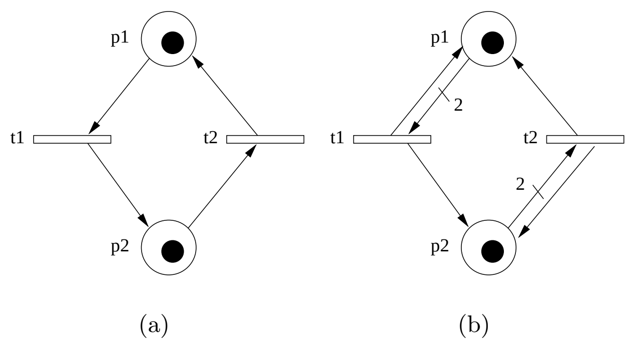


Fig. 3. Continuous systems that have the same incidence matrix and whose reachability spaces do not coincide.

plus those points which are infinitely close to points in A, but are not contained in A. In the case of the set depicted in Figure 2(b) its closure is equal to the inner and edge points of the square defined by the vertices $(0,\ 0)$, $(0,\ 1)$, $(1,\ 1)$ and $(1,\ 0)$, that is, it is obtained by adding the segment $[(0,\ 0),(0,\ 1)]$.

Focusing on the spaces defined by $\mathrm{RS}(\mathcal{N},\mathbf{m_0})$ and lim-$\mathrm{RS}(\mathcal{N},\mathbf{m_0})$ and closing them, it will be noticed that the points limiting both spaces are exactly the same. This is because if the system can get as close as desired to a given point with an infinite sequence, it can also get as close as desired with a finite sequence and vice versa. Hence, the following property can be stated:

Property 3 *The closure of* $\mathrm{RS}(\mathcal{N},\mathbf{m_0})$ *is equal to the closure of* lim-$\mathrm{RS}(\mathcal{N},\mathbf{m_0})$.

Assume that, given a system, $\mathrm{RS}(\mathcal{N},\mathbf{m_0})$ and lim-$\mathrm{RS}(\mathcal{N},\mathbf{m_0})$ are not identical sets, i.e., $\mathrm{RS}(\mathcal{N},\mathbf{m_0}) \subsetneq$ lim-$\mathrm{RS}(\mathcal{N},\mathbf{m_0})$. This means that for every $\mathbf{m}$ in lim-$\mathrm{RS}(\mathcal{N},\mathbf{m_0}) \setminus \mathrm{RS}(\mathcal{N},\mathbf{m_0})$, $\mathbf{m}$ is a *border* point of lim-$\mathrm{RS}(\mathcal{N},\mathbf{m_0})$; that is, there are markings in $\mathrm{RS}(\mathcal{N},\mathbf{m_0})$ infinitely close to $\mathbf{m}$ that do not belong to lim-$\mathrm{RS}(\mathcal{N},\mathbf{m_0})$. Let us make a final consideration on the system of Figure 1(a). It has been seen that the initial firing of t_1 enables t_2 and that an infinite sequence consisting on firing t_2 and t_3 will empty p_3 and p_4, reaching marking $(0.5,\ 0.5,\ 0,\ 0)$. In that example t_1 was fired in an amount of 0.5. Nevertheless, p_3 and p_4 can be emptied also if t_1 is fired in an amount α such that $0 < \alpha \leq 1$. For example, if we take $\alpha = 0.1$, we fire t_1 in an amount of 0.1 and then fire t_2 five times in an amount of 0.1. Now we can fire completely, in an amount of 0.5, transition t_3. Repeating this procedure, in the limit p_3 and p_4 become empty. Thus, it can be said that the marking $(1-\alpha,\ \alpha,\ 0,\ 0)$ is lim-reachable for any α such that $0 < \alpha \leq 1$. Hence, marking $(1,\ 0,\ 0,\ 0)$ is not lim-reachable but the system can get as close as desired to it by taking a small enough α. This marking can then be interpreted as the fact that a little leak of fluid from p_1 to p_2 can cause the emptying of p_3 and p_4. In some situations, it may be useful to consider those markings like $(1,\ 0,\ 0,\ 0)$, that are not reachable, but for which the system can get as close as desired.

Let us consider a norm in order to determine the proximity of two markings. Let $|\mathbf{x}|$ denote the norm of vector $\mathbf{x} = (x_1, \ldots, x_n)$ defined as: $|\mathbf{x}| = |x_1| + \ldots + |x_n|$. A new reachability concept for continuous systems will be introduced: the δ-reachability. The set of δ-reachable markings will be written as δ-RS$(\mathcal{N}, \mathbf{m_0})$ and accounts for those markings to which the system can get as close as desired firing a finite sequence. Formally:

Definition 4 *δ-RS$(\mathcal{N}, \mathbf{m_0})$ is the closure of* RS$(\mathcal{N}, \mathbf{m_0})$ *: δ-RS$(\mathcal{N}, \mathbf{m_0}) = \{\ \mathbf{m} \mid$ for every $\epsilon > 0$ a marking $\mathbf{m'} \in$ RS$(\mathcal{N}, \mathbf{m_0})$ exists such $|\mathbf{m'} - \mathbf{m}| < \epsilon\}$.*

Since the closure of RS$(\mathcal{N}, \mathbf{m_0})$ is equal to the closure of lim-RS$(\mathcal{N}, \mathbf{m_0})$, δ-RS$(\mathcal{N}, \mathbf{m_0})$ is also equal to the set of markings to which the system can get as close as desired firing an infinite sequence. RS$(\mathcal{N}, \mathbf{m_0})$ and lim-RS$(\mathcal{N}, \mathbf{m_0})$ are, therefore, subsets of δ-RS$(\mathcal{N}, \mathbf{m_0})$.

Therefore, till now three different kinds of reachability concepts have been defined:

- Markings that are reachable with a finite firing sequence, RS$(\mathcal{N}, \mathbf{m_0})$.
- Markings to which the system converges, eventually, with an infinitely long sequence, lim-RS$(\mathcal{N}, \mathbf{m_0})$.
- Markings to which the system can get as close as desired with a finite sequence, δ-RS$(\mathcal{N}, \mathbf{m_0})$.

Let us finish this section by defining the linearized reachability set with respect to the state equation:

Definition 5 LRS$(\mathcal{N}, \mathbf{m_0}) = \{\mathbf{m} | \mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma} \geq \mathbf{0}$ *with* $\boldsymbol{\sigma} \in (\mathbb{R}^+ \cup \{0\})^{|T|}\}$.

Notice that given a consistent system (i.e., $\exists\ \mathbf{x} > \mathbf{0} | \mathbf{C} \cdot \mathbf{x} = \mathbf{0}$) it holds: LRS$(\mathcal{N}, \mathbf{m_0}) = \{\mathbf{m} | \mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma} \geq \mathbf{0}$ with $\boldsymbol{\sigma} \in \mathbb{R}^{|T|}\}$. In [7] it was shown that for consistent systems in which every transition is fireable at least once, the sets LRS$(\mathcal{N}, \mathbf{m_0})$ and lim-RS$(\mathcal{N}, \mathbf{m_0})$ are the same. This result will be generalized by describing the set of lim-reachable markings of a general system.

By definition LRS$(\mathcal{N}, \mathbf{m_0})$ is a closed set. $\mathbf{m}$ is a *border* point of LRS$(\mathcal{N}, \mathbf{m_0})$ iff for every $\epsilon > 0$ there exists $\mathbf{m'}$, $|\mathbf{m'} - \mathbf{m}| < \epsilon$ such that $\mathbf{m'} \not\in$ LRS$(\mathcal{N}, \mathbf{m_0})$.

The *open* set of LRS$(\mathcal{N}, \mathbf{m_0})$ is the result of removing every border point from LRS$(\mathcal{N}, \mathbf{m_0})$ and will be denoted as]LRS$(\mathcal{N}, \mathbf{m_0})$[.

Notice that given a system $\langle \mathcal{N}, \mathbf{m_0} \rangle$ if there exists $\mathbf{y} \neq \mathbf{0}$ such that $\mathbf{y} \cdot \mathbf{C} = 0$ then every $\mathbf{m} \in$ LRS$(\mathcal{N}, \mathbf{m_0})$ is a border point of LRS$(\mathcal{N}, \mathbf{m_0})$, and so in this case]LRS$(\mathcal{N}, \mathbf{m_0})$[$= \emptyset$. If such $\mathbf{y}$ exists all the points in LRS$(\mathcal{N}, \mathbf{m_0})$ are contained in a hyperplane of smaller dimension than the number of places. In particular, if a system has a p-semiflow, every marking in LRS$(\mathcal{N}, \mathbf{m_0})$ is a border point. Those markings having null components are also border points of LRS$(\mathcal{N}, \mathbf{m_0})$.

Since all reachable, lim-reachable and δ-reachable markings are solution of the state equation, the following relation is satisfied:

RS($\mathcal{N},\mathbf{m_0}$) $\subseteq$ lim-RS($\mathcal{N},\mathbf{m_0}$) $\subseteq$ δ-RS($\mathcal{N},\mathbf{m_0}$) $\subseteq$ LRS($\mathcal{N},\mathbf{m_0}$).

Along the paper this relationship among the different sets will be completed showing that the open linearized set,]LRS($\mathcal{N},\mathbf{m_0}$)[, is contained in RS($\mathcal{N},\mathbf{m_0}$) and that δ-RS($\mathcal{N},\mathbf{m_0}$) = LRS($\mathcal{N},\mathbf{m_0}$) if every transition is fireable at least once.

3 RS($\boldsymbol{\mathcal{N}},\mathbf{m_0}$)

The goal of this section is first to provide a full characterization of the set of reachable markings (Subsection 3.1) and then to show a computation algorithm that decides the reachability of a given target marking (Subsection 3.2).

3.1 Reachability characterization

Before showing the main result (Theorem 12), some intermediate lemmas will be presented in order to ease the final characterization. First, let us introduce an algorithm to compute the sets of transitions fireable from the initial marking, and some interesting results dealing with continuous systems.

Let $FS(\mathcal{N},\mathbf{m_0})$ be the set of sets of transitions for which there exists a sequence fireable from $\mathbf{m_0}$ that contains those and only those transitions in the set. Formally,

Definition 6 $FS(\mathcal{N},\mathbf{m_0}) = \{\ \theta|$ *there exists a sequence fireable from* $\mathbf{m_0}$, σ, *such that* $\theta = \|\boldsymbol{\sigma}\|\}$.

Algorithm 7 (Computation of the set $FS(\mathcal{N},\mathbf{m_0})$)

1. *Let V be the set of transitions enabled at* $\mathbf{m_0}$
2. $FS := \{v|v \subseteq V\}$ *% all the subsets of V including the empty set*
3. *Repeat*
  3.1. *take $f \in FS$ such that it has not been taken before*
  3.2. *fire sequentially from $\mathbf{m_0}$ every transition in f without disabling any enabled transition. Let $\mathbf{m}$ be the reached marking.*
  3.3. $V := \{t|\ t$ *is enabled at* $\mathbf{m}$ *and* $t \notin f\}$
  3.4. $FS := FS \cup \{f \cup v|v \subseteq V\}$
4. *until FS does not increase*

Notice that step **3.2.** can always be achieved since for any element $f \in FS(\mathcal{N},\mathbf{m_0})$ there exists a fireable sequence that contains every transition in f. Algorithm 7 accounts for all possible subsets of transitions that can become enabled, and so its complexity is exponential on the number of transitions and so is the size of the set $FS(\mathcal{N},\mathbf{m_0})$. As an example, considering the net in Figure 4 with initial marking $\mathbf{m_0} = (1,\ 0,\ 1,\ 1,\ 0)$ the result of Algorithm 7 is $FS(\mathcal{N},\mathbf{m_0}) = \{\ \{\},\{t_2\},\{t_3\},\{t_4\},\{t_2,\ t_3\},\{t_2,\ t_4\},\{t_3,\ t_4\},\{t_2,\ t_3,\ t_4\},$ $\{t_1,\ t_2\},\{t_4,\ t_5\},\{t_1,\ t_2,\ t_3\},\{t_1,\ t_2,\ t_4\},\{t_2,\ t_4,\ t_5\},\{t_1,\ t_2,\ t_4,\ t_5\},\{t_3,\ t_4,\ t_5\},$ $\{t_1,\ t_2,\ t_3,\ t_4\},\ \{t_2,\ t_3,\ t_4,\ t_5\},\ \{t_1,\ t_2,\ t_3,\ t_4,\ t_5\}\}$.

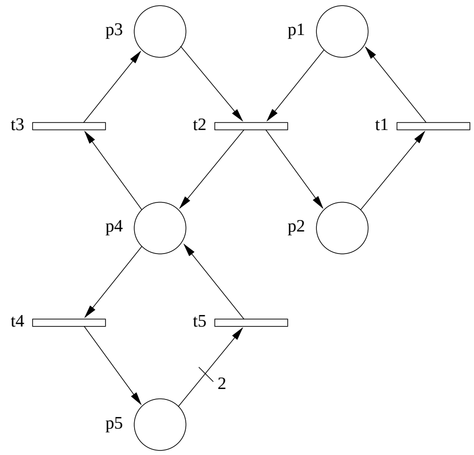


Fig. 4. Non-consistent continuous system

Now let us introduce four lemmas that will help to characterize the set of reachable markings. The first one simply states that continuous systems are homothetic w.r.t. the scaling of $\mathbf{m_0}$.

Lemma 8 *[7] Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a continuous system. If σ is a fireable sequence yielding marking $\mathbf{m}$, then for any $\alpha \geq 0$, $\alpha\sigma$ is fireable at $\alpha\mathbf{m_0}$ yielding marking $\alpha\mathbf{m}$, where $\alpha\sigma$ represents a sequence that is equal to σ except in the amount of each firing, that is multiplied by α.*

Although this section deals with those markings that are reachable with a finite firing sequence, a lemma that has to do with the markings that can be reached in the limit will be presented. Lemma 9 establishes that if all the transitions in the support of a given firing vector $\boldsymbol{\sigma}$ are enabled, then $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma} \geq 0$ is reachable in the limit, whatever the value of $\boldsymbol{\sigma}$ is. Furthermore, there exists a sequence of reachable markings that are "in the direction" of $\mathbf{m}$.

Lemma 9 *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a continuous system. Let $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma} \geq 0$, $\boldsymbol{\sigma} \geq \mathbf{0}$ and $\mathbf{m_0}$ such that for every $t \in \|\boldsymbol{\sigma}\|$ $\text{enab}(t, \mathbf{m_0}) > 0$. Then, there exists a succession of reachable markings $\mathbf{m}_1, \mathbf{m}_2, \ldots$ fulfilling $\mathbf{m}_1 = \mathbf{m}_0 + \beta_1 \mathbf{C} \cdot \boldsymbol{\sigma}$, $\mathbf{m}_2 = \mathbf{m}_0 + \beta_2 \mathbf{C} \cdot \boldsymbol{\sigma}, \ldots$ with $0 < \beta_1 < \beta_2 < \ldots$ that converges to $\mathbf{m}$.*

Proof. Since at $\mathbf{m_0}$ every transition of $\|\boldsymbol{\sigma}\|$ is enabled, α and $\boldsymbol{\sigma'}$ exist such that $\boldsymbol{\sigma'}$ is fireable from $\mathbf{m_0}$ and $\boldsymbol{\sigma'} = \alpha\boldsymbol{\sigma}$, i.e., a sequence proportional to the vector leading from $\mathbf{m_0}$ to $\mathbf{m}$ can be fired. If $\alpha \geq 1$, it is clear that $\mathbf{m}$ can be reached from $\mathbf{m_0}$. Otherwise, the firing of $\boldsymbol{\sigma'}$ leads to $\mathbf{m_0} + \mathbf{C} \cdot \alpha\boldsymbol{\sigma} = (1-\alpha)\mathbf{m_0} + \alpha\mathbf{m_0} + \mathbf{C} \cdot \alpha\boldsymbol{\sigma} = (1-\alpha)\mathbf{m_0} + \alpha\mathbf{m}$. By Lemma 8, if $\boldsymbol{\sigma'}$ was fireable from $\mathbf{m_0}$, then $(1-\alpha)\boldsymbol{\sigma'}$ is fireable from $(1-\alpha)\mathbf{m_0}$. In this way, we have

$$\alpha\mathbf{m} + (1-\alpha)\mathbf{m_0} \stackrel{(1-\alpha)}{\longrightarrow}\ '\alpha\mathbf{m} + \alpha(1-\alpha)\mathbf{m} + (1-\alpha)^2\mathbf{m_0}$$

Repeating this procedure, in the iteration n we reach the marking

$$\alpha\mathbf{m}(1+(1-\alpha)+(1-\alpha)^2+\ldots+(1-\alpha)^n)+(1-\alpha)^n\mathbf{m_0}$$

Thus, the marking of the system as n goes to infinity converges to $\mathbf{m}$. $\square$

Based on this result a part of the set of reachable markings can be described.

Lemma 10 *Let $\langle \mathcal{N}, \mathbf{m_0}\rangle$ be a continuous system. Let $\mathbf{m} = \mathbf{m_0} + \mathbf{C}\cdot\boldsymbol{\sigma} \geq 0$, $\boldsymbol{\sigma} \geq \mathbf{0}$ and for every $t \in \|\boldsymbol{\sigma}\|$ $\mathrm{enab}(t, \mathbf{m_0}) > 0$ and $\mathrm{enab}(t, \mathbf{m}) > 0$. Then $\mathbf{m} \in RS(\mathcal{N}, \mathbf{m_0})$.*

Proof. Every $t \in \|\boldsymbol{\sigma}\|$ is enabled at $\mathbf{m}$. This means that for every $t \in \|\boldsymbol{\sigma}\|$ all its input places are positively marked at $\mathbf{m}$. Then, we can define an $\mathbf{m}'$ such that $\mathbf{m}' = \mathbf{m_0} + \mathbf{C}\cdot(1+\alpha)\boldsymbol{\sigma} \geq 0$ with $\alpha > 0$. According to Lemma 9 there is a succession of markings that converges to $\mathbf{m}'$. Since $\mathbf{m}$ is in the line that goes from $\mathbf{m_0}$ to $\mathbf{m}'$ we can stop that sequence at a given step and reach exactly $\mathbf{m}$ in a finite number of firings. $\square$

The following last lemma imposes a necessary and sufficient condition for the fireability of a transition in terms of siphons.

Lemma 11 *Let $\mathbf{m} \in \mathrm{RS}(\mathcal{N}, \mathbf{m_0})$. Transition t is not fireable for any successor of $\mathbf{m}$ iff there exists an empty siphon at $\mathbf{m}$ containing a place p such that $p \in {}^\bullet t$.*

Proof.
($\Leftarrow$)
If there exists such an empty siphon Θ, no transition in $\Theta^\bullet$ is fireable.
($\Rightarrow$)
Assume t is not fireable for any successor of $\mathbf{m}$. Then there exists a place p such that $p \in {}^\bullet t$ and $\mathbf{m}(p) = 0$. Furthermore, no input transition of p, t', can ever be fired. Hence, for every t' there exists an empty input place p'. Repeating this reasoning we obtain a set of empty places Q. This set Q has the property that all its input transitions (${}^\bullet Q$) are output transitions ($Q^\bullet$). Hence Q is an empty siphon. $\square$

Before going on with the characterization of the set of reachable markings let us make some considerations on the conditions a given marking $\mathbf{m}$ should fulfill in order to be reachable. First of all, it is clear that a necessary condition for $\mathbf{m}$ to be reachable is that it has to be solution of the state equation, that is, there must exist $\boldsymbol{\sigma}$ such that $\mathbf{m} = \mathbf{m_0} + \mathbf{C}\cdot\boldsymbol{\sigma}$. Furthermore, $\|\boldsymbol{\sigma}\|$ must be in $FS(\mathcal{N}, \mathbf{m_0})$ in order to have a fireable sequence. In Section 2 it has been seen that some marked traps can be emptied in a continuous system with the firing of an infinite sequence. If only finite firing sequences are considered, no marked trap can be emptied. Since now the interest lies in finite firing sequences those $\boldsymbol{\sigma}$'s that correspond to a firing count vector that empties (or fills and then empties) a trap have to be explicitly forbidden. As it will be seen, these necessary conditions are also sufficient for a marking to be reachable.

Given a net $\mathcal{N}$ and a firing sequence $\boldsymbol{\sigma}$, let us denote as $\mathcal{N}$ the net obtained removing from $\mathcal{N}$ the transitions not in the support of $\boldsymbol{\sigma}$ and the resulting isolated places. In other words, $\mathcal{N}$ is the net composed of the transitions of $\mathcal{N}$ in the support of $\boldsymbol{\sigma}$ and their input and output places. Using the previous lemmas a full characterization of the set of reachable markings is obtained.

Theorem 12 *A marking* $\mathbf{m} \in \mathrm{RS}(\mathcal{N}, \mathbf{m_0})$ *iff*

1. $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma} \geq 0,\ \boldsymbol{\sigma} \geq 0$
2. $\|\boldsymbol{\sigma}\| \in FS(\mathcal{N}, \mathbf{m_0})$
3. *there is no empty trap in* $\mathcal{N}$ *at* $\mathbf{m}$

Proof.
$\subseteq$
Let $\mathbf{m} \in \mathrm{RS}(\mathcal{N}, \mathbf{m_0})$. Then, there exists $\boldsymbol{\sigma} \geq \mathbf{0}$ such that $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}$ and $\|\boldsymbol{\sigma}\| \in FS(\mathcal{N}, \mathbf{m_0})$. Furthermore, there cannot be an empty trap in $\mathcal{N}$ at $\mathbf{m}$ since it would mean that the trap was emptied with a finite firing sequence.
$\supseteq$
Let $\mathbf{m}$ be such that $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma} \geq \mathbf{0}$, $\boldsymbol{\sigma} \geq \mathbf{0}$, $\|\boldsymbol{\sigma}\| \in FS(\mathcal{N}, \mathbf{m_0})$ and there is no empty trap in $\mathcal{N}$ at $\mathbf{m}$. It will be shown that $\mathbf{m}$ can be reached from $\mathbf{m_0}$ by a finite firing sequence. This will be done in three steps: from $\mathbf{m_0}$ we will reach a marking $\mathbf{m}'$ at which every transition $t \in \|\boldsymbol{\sigma}\|$ is enabled. From $\mathbf{m}'$ we will make the system evolve to a marking $\mathbf{m}''$ at which also every transition $t \in \|\boldsymbol{\sigma}\|$ is enabled and is as closed as desired to $\mathbf{m}$. Finally, due to the way $\mathbf{m}''$ is defined it is shown that $\mathbf{m}$ is reachable from $\mathbf{m}''$. Although the order of the sequence of reachable markings is $\mathbf{m_0}$, $\mathbf{m}'$, $\mathbf{m}''$ and $\mathbf{m}$, we will start by defining $\mathbf{m}''$ and showing how it can be reached.

If every $t \in \|\boldsymbol{\sigma}\|$ is enabled at $\mathbf{m}$ then $\mathbf{m}'' = \mathbf{m}$. Otherwise, we will consider the system with marking $\mathbf{m}$ and we will fire backwards and sequentially transitions in $\|\boldsymbol{\sigma}\|$ until we reach a marking ($\mathbf{m}''$) at which every transition in $\|\boldsymbol{\sigma}\|$ is enabled. Notice that this backward firing is equivalent to a forward firing in the reverse net (changing directions of arcs). We will reason that such a firing from $\mathbf{m}$ to $\mathbf{m}''$ is always feasible. Notice that in the reverse net traps have become siphons (structural deadlocks) and the forward firing in the reverse net of transitions in $\|\boldsymbol{\sigma}\|$ never involves the filling of empty siphons of $\mathcal{N}$ at $\mathbf{m}$. This is because according to the initial condition 3, "there is no empty trap in $\mathcal{N}$ at $\mathbf{m}$". Therefore, by Lemma 11 we can assure that every transition $t \in \|\boldsymbol{\sigma}\|$ can be fired in the reverse net. Let us denote $\widehat{\boldsymbol{\sigma}}$ the firing count vector such that $\mathbf{m} = \mathbf{m}'' + \mathbf{C} \cdot \widehat{\boldsymbol{\sigma}}$ with $\|\widehat{\boldsymbol{\sigma}}\| = \|\boldsymbol{\sigma}\|$.

Now we will define $\mathbf{m}'$ as a marking reached from $\mathbf{m_0}$ at which every transition $t \in \|\boldsymbol{\sigma}\|$ is enabled, $\mathbf{m}' = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}' \leq \boldsymbol{\sigma}$. This is always possible since $\|\boldsymbol{\sigma}\|$ belongs to $FS(\mathcal{N}, \mathbf{m_0})$ and so we can fire small amounts of the transitions in $\|\boldsymbol{\sigma}\|$ until every transition in $\|\boldsymbol{\sigma}\|$ is enabled. We will define $\boldsymbol{\sigma}''$ as $\boldsymbol{\sigma}'' = \boldsymbol{\sigma} - \boldsymbol{\sigma}' - \widehat{\boldsymbol{\sigma}}$, then $\mathbf{m}'' = \mathbf{m}' + \mathbf{C} \cdot \boldsymbol{\sigma}''$. Notice that since $\boldsymbol{\sigma}'$ and $\widehat{\boldsymbol{\sigma}}$ can be taken as small as wanted and their supports are contained in the support of $\boldsymbol{\sigma}$, it can always be verified that $\boldsymbol{\sigma}'' \geq 0$ and $\|\boldsymbol{\sigma}\| = \|\boldsymbol{\sigma}''\|$. Moreover, Lemma 10 can be directly applied on $\mathbf{m}'$ and $\boldsymbol{\sigma}''$ obtaining that $\mathbf{m}''$ is reachable from $\mathbf{m}'$. And finally, we can conclude that $\mathbf{m}$ is reachable from $\mathbf{m_0}$. $\square$

Figure 5 sketches the trajectory built by the proof of Theorem 12 to reach $\mathbf{m}$.

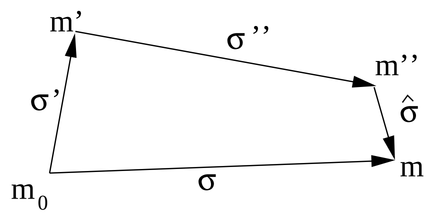


Fig. 5. Trajectory to reach $\mathbf{m}$ with a finite firing sequence

As an example, let us take the system in Figure 6. The marking $\mathbf{m} = (0,\ 0,\ 0,\ 0,\ 1)$ is solution of the state equation and can be obtained with vectors: $\boldsymbol{\sigma}_1 = (1,\ 0,\ 1,\ 1,\ 0,\ 0)$ and $\boldsymbol{\sigma}_2 = (0,\ 1,\ 0,\ 0,\ 1,\ 0)$. Obviously, $\boldsymbol{\sigma}_2$ fulfills the conditions of Theorem 12, and so it can be concluded that $\mathbf{m}$ can be reached. However, if we consider the system that results of removing transitions t_2, t_5 and the place p_4, then the only possibility to reach $\mathbf{m}$ is with $\boldsymbol{\sigma}_1$ or with $\boldsymbol{\sigma}_1 + \mathbf{x}$ where $\mathbf{x}$ is a T-semiflow. Notice that the nets $\mathcal{N}_{1}$ and $\mathcal{N}_{1+\mathbf{x}}$ have an empty trap at $\mathbf{m}$ composed of $\{p_2, p_3\}$. Hence, the third condition of Theorem 12 is violated and $\mathbf{m}$ cannot be reached with a finite sequence.

3.2 Deciding reachability

Based on Theorem 12, an algorithm that decides whether a given marking $\mathbf{m}$ is reachable or not is introduced. A necessary condition for $\mathbf{m}$ to be reached is that there must exist a $\boldsymbol{\sigma} \geq 0$ such that $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma} \geq 0$. Given a marking $\mathbf{m}$ the number of $\boldsymbol{\sigma} \geq 0$ fulfilling the state equation can be infinite. However, as stated in Theorem 12, it is only interesting to consider those $\boldsymbol{\sigma}$'s such that $\|\boldsymbol{\sigma}\| \in FS(\mathcal{N}, \mathbf{m_0})$. Furthermore, it is not necessary to consider two different $\boldsymbol{\sigma}$'s that are solution of the state equation and have the same support, since clearly one of those $\boldsymbol{\sigma}$'s fulfills the condition on the traps of Theorem 12 iff the other one also fulfills it, and the support of one belongs to $FS(\mathcal{N}, \mathbf{m_0})$ iff the other one also belongs to it. This reasoning reduces the number of $\boldsymbol{\sigma}$'s to be considered to a finite number.

Now let us take into account a set Σ of $\boldsymbol{\sigma}$'s that are solution of the state equation, have different supports and the support of all of them is in $FS(\mathcal{N}, \mathbf{m_0})$. To decide reachability it is only necessary to consider those $\boldsymbol{\sigma}$'s with minimal support. This is because if there is a non minimal $\boldsymbol{\sigma} \in \Sigma$ fulfilling the condition on the traps of Theorem 12, then its support contains the support of a $\boldsymbol{\sigma}' \in \Sigma$ that also fulfills this condition.

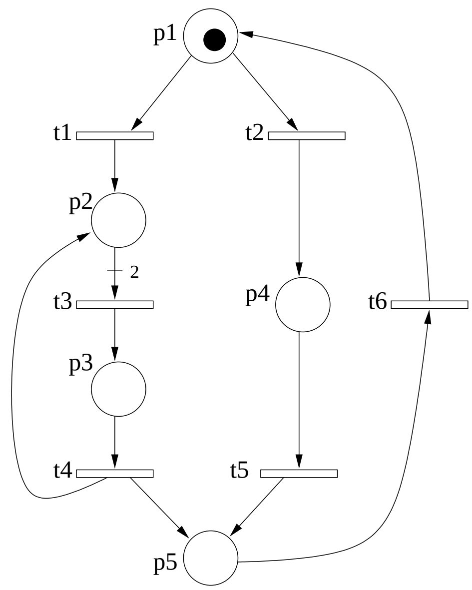


Fig. 6. Marking (0, 0, 0, 0, 1) can be reached with a finite and with an infinite firing sequence

Summing up, to decide reachability it is only necessary to consider the $\boldsymbol{\sigma}$'s in a set $\Sigma = \{\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k\}$ that fulfills the following conditions:

1. $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}_i \geq 0$ and $\|\boldsymbol{\sigma}_i\| \in FS(\mathcal{N}, \mathbf{m_0})$
2. $\|\boldsymbol{\sigma}_i\|$ is minimal, i.e., for every $j \neq i$ $\|\boldsymbol{\sigma}_j\| \not\subseteq \|\boldsymbol{\sigma}_i\|$
3. for every $\boldsymbol{\gamma}$ such that $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\gamma} \geq 0$ and $\|\boldsymbol{\gamma}\| \in FS(\mathcal{N}, \mathbf{m_0})$ there exists $i \in \{1, \ldots, k\}$ such that $\|\boldsymbol{\sigma}_i\| \subseteq \|\boldsymbol{\gamma}\|$

The third condition guarantees that every $\boldsymbol{\sigma}$ that verifies the first two conditions is included in Σ.

The following algorithm takes as inputs a continuous system, a target marking $\mathbf{m}$, and a set Σ verifying the above conditions for the target marking. The output of the algorithm is the boolean variable *answer* that takes the value *YES* iff $\mathbf{m}$ is reachable. The general idea of the algorithm is first checking whether all traps are marked at $\mathbf{m}$, step 2, and whether there is an empty trap at $\mathbf{m}$ that was marked at $\mathbf{m_0}$, step 3. In these both cases a quick answer to reachability can be given. Otherwise, it is required to iterate on the elements of Σ.

Algorithm 13

INPUT: $\langle\mathcal{N}, \mathbf{m_0}\rangle$, $\mathbf{m}$, $\Sigma = \{\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k\}$

OUTPUT: answer

1. *If* $\Sigma = \emptyset$ *then answer:=NO; exit; end if*

2. *If there is no empty trap in* $\mathcal{N}$ *at* $\mathbf{m}$ *then answer:=YES; exit; end if*

3. *If there is an empty trap in $\mathcal{N}$ at* $\mathbf{m}$ *that was not empty at* $\mathbf{m_0}$ *then answer:=NO; exit; end if*
4. *i:=0*
5. *loop*
  5.1. *i:=i+1*
  5.2. *If there is no empty trap in* $\mathcal{N}_i$ *at* $\mathbf{m}$ *then answer:=YES; exit; end if*
6. *until i=k*
7. *answer:=NO*

In [5] a method to compute traps based on the solution of a system of linear equations was proposed. According to this method the support of a solution of that system represents the places of a trap. In steps 2 and 5.2 of Algorithm 13, we are interested only in empty traps at $\mathbf{m}$, therefore only the subnet composed of empty places at $\mathbf{m}$ has to be considered. In step 3, the focus is on the empty traps at $\mathbf{m}$ that were marked at $\mathbf{m_0}$. The only thing that has to be included in the system of inequalities proposed in [5] is forcing that a solution of the system must have at least one non null component corresponding to a non empty place at $\mathbf{m_0}$.

4 lim-RS($\mathcal{N}, \mathbf{m_0}$)

As it has been shown, some traps (for example the one composed of p_3 and p_4 in the system of Figure 1(a)) can be emptied with an infinite firing sequence. Hence when facing the problem of describing the set of lim-reachable markings, it is not necessary to exclude those markings that are result of the state equation and that have empty traps that were previously filled. In this way, the characterization of the lim-RS($\mathcal{N}, \mathbf{m_0}$) is easier and it is only necessary to care about the fireability of the firing count vector $\boldsymbol{\sigma}$ (conditions 1. and 2. of Theorem 12).

Theorem 14 *A marking* $\mathbf{m} \in$ lim-RS($\mathcal{N}, \mathbf{m_0}$) *iff*

1. $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma} \geq 0, \ \boldsymbol{\sigma} \geq 0$
2. $\|\boldsymbol{\sigma}\| \in FS(\mathcal{N}, \mathbf{m_0})$

Proof.
$\subseteq$
Let $\mathbf{m} \in$ lim-RS($\mathcal{N}, \mathbf{m_0}$). Since $\mathbf{m}$ is reached by a finite or infinite firing sequence there must exist a firing count vector, $\boldsymbol{\sigma} \geq \mathbf{0}$, corresponding to this sequence such that $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}$. If the sequence was fireable then $\|\boldsymbol{\sigma}\| \in FS(\mathcal{N}, \mathbf{m_0})$.
$\supseteq$
Let $\mathbf{m}$ be such that $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma} \geq 0$, $\boldsymbol{\sigma} \geq 0$ and $\|\boldsymbol{\sigma}\| \in FS(\mathcal{N}, \mathbf{m_0})$. From $\mathbf{m_0}$ it is possible to fire sequentially a subset of transitions in $\|\boldsymbol{\sigma}\|$, since it belongs to $FS(\mathcal{N}, \mathbf{m_0})$, reaching marking $\mathbf{m'} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma'}$ at which every transition in $\|\boldsymbol{\sigma}\|$ is enabled. Since $\boldsymbol{\sigma'}$ can be taken arbitrarily small, it can always

fulfill $\boldsymbol{\sigma} - \boldsymbol{\sigma}' \geq 0$. Lemma 9 can be applied on the system $(\mathcal{N}, \mathbf{m}')$ and therefore marking $\mathbf{m}$ can be reached in the limit. $\square$

According to Theorem 14 checking whether a given marking is reachable in the limit is a *decidable* problem. For the system in Figure 6 without transitions t_2, t_5 and place p_4 it can can be assured that the marking $\mathbf{m} = (0,\ 0,\ 0,\ 0,\ 1)$ is lim-reachable (but not reachable) since it is solution of the state equation with $\boldsymbol{\sigma} = (1,\ 0,\ 1,\ 1,\ 0,\ 0)$ and $\|\boldsymbol{\sigma}\| \in FS(\mathcal{N}, \mathbf{m_0})$.

If the system fulfills some initial conditions, then the set lim-RS($\mathcal{N}$,$\mathbf{m_0}$) can be described without the use of $FS(\mathcal{N}, \mathbf{m_0})$. Furthermore, those conditions can be checked in *polynomial* time. For example, for a system, $\langle\mathcal{N}, \mathbf{m_0}\rangle$, in which every transition is enabled at $\mathbf{m_0}$, it holds $FS(\mathcal{N}, \mathbf{m_0}) = \{q | q \subseteq T\}$ and therefore every $\boldsymbol{\sigma} \geq \mathbf{0}$ belongs to $FS(\mathcal{N}, \mathbf{m_0})$.

Corollary 15 *If for every transition t* $\mathrm{enab}(t, \mathbf{m_0}) > 0$ *then* $\text{lim-RS}(\mathcal{N}, \mathbf{m_0}) = \mathrm{LRS}(\mathcal{N}, \mathbf{m_0})$.

Let $\langle\mathcal{N}, \mathbf{m_0}\rangle$ be a consistent system in which every transition is fireable at least once, i.e., for every transition t there exists $\mathbf{m}' \in \mathrm{RS}(\mathcal{N}, \mathbf{m_0})$ such that $\mathrm{enab}(t, \mathbf{m}') > 0$. Clearly $T \in\ FS(\mathcal{N}, \mathbf{m_0})$. Since the system is consistent it has a T-semiflow $\mathbf{x} > \mathbf{0}$ that can be added to a given $\boldsymbol{\sigma}$, $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma} \geq \mathbf{0}$, fulfilling $\boldsymbol{\sigma} + \mathbf{x} > \mathbf{0}$. It is obvious that $\mathbf{C} \cdot \boldsymbol{\sigma} = \mathbf{C} \cdot (\boldsymbol{\sigma} + \mathbf{x})$ and that $\|\boldsymbol{\sigma} + \mathbf{x}\| = T$. Therefore, $\mathbf{m}$ is lim-reachable.

Corollary 16 ([7]) *If* $(\mathcal{N}, \mathbf{m_0})$ *is consistent and every transition is fireable at least once, then* $\text{lim-RS}(\mathcal{N}, \mathbf{m_0}) = \mathrm{LRS}(\mathcal{N}, \mathbf{m_0})$.

5 $\boldsymbol{\delta}$-RS($\boldsymbol{\mathcal{N}}$, $\mathbf{m_0}$)

Let us now assume that given a system, $\langle\mathcal{N}, \mathbf{m_0}\rangle$, every transition is fireable at least once. That is for every transition t there exists $\mathbf{m} \in \mathrm{RS}(\mathcal{N}, \mathbf{m_0})$ such that $\mathrm{enab}(t, \mathbf{m}) > 0$. The existence of transitions that do not fulfill this condition can be easily detected (see [7]): it is sufficient to iterate on the enabled transitions firing them in half its enabling degree until no more transitions become enabled. Those transitions that are not enabled after the iteration can never be fired. Notice that this assumption does not imply a loss of generality in the following results, since if a transition can never be enabled it can be removed without affecting any possible evolution of the system or changing the set of reachable markings.

In this section the set of markings to which the system can get as close as desired is described. For example, in Figure 4 with $\mathbf{m_0} = (1,\ 0,\ 0,\ 0,\ 1)$, $\mathbf{m} = (0,\ 1,\ 0,\ 0,\ 1)$ does not belong neither to RS($\mathcal{N}$,$\mathbf{m_0}$) nor to lim-RS($\mathcal{N}$,$\mathbf{m_0}$), however $\mathbf{m} = (0,\ 1,\ 0,\ \alpha,\ 1 - 2 \cdot \alpha)$ belongs to RS($\mathcal{N}$,$\mathbf{m_0}$) (hence also to lim-RS($\mathcal{N}$,$\mathbf{m_0}$)) for every α fulfilling $0 < \alpha \leq 0.5$.

For this set of markings, that will be called δ-reachable, there are no spurious solutions of the state equation.

Theorem 17 *If every transition is fireable at least once from the initial marking, then a marking* $\mathbf{m} \in \delta\text{-RS}(\mathcal{N}, \mathbf{m_0})$ *iff*

1. $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma} \geq 0,\ \boldsymbol{\sigma} \geq 0$

i.e., $\delta\text{-RS}(\mathcal{N}, \mathbf{m_0}) = \text{LRS}(\mathcal{N}, \mathbf{m_0})$.

Proof.
$\subseteq$
$\delta\text{-RS}(\mathcal{N}, \mathbf{m_0}) \subseteq \text{LRS}(\mathcal{N}, \mathbf{m_0})$ since $\text{LRS}(\mathcal{N}, \mathbf{m_0})$ is a closed set that includes the $\text{RS}(\mathcal{N}, \mathbf{m_0})$.
$\supseteq$
Let $\mathbf{m}$ be a solution of the state equation, i.e., $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma} \geq 0$. Since every transition is fireable at least once, let us consider a sequence, $\boldsymbol{\sigma}'$, that reaches a marking, $\mathbf{m}'$, at which every transition in the support of $\boldsymbol{\sigma}$ is enabled. Let us consider the real quantity α determined by $\alpha = \min\{1, \max\{\beta | \mathbf{m}' + \mathbf{C} \cdot \beta \cdot \boldsymbol{\sigma} \geq 0\}\}$. Then, according to Theorem 14, the marking $\mathbf{m}'' = \mathbf{m}' + \mathbf{C} \cdot \alpha \cdot \boldsymbol{\sigma}$ is reachable in the limit from $\mathbf{m}'$. And clearly, it is also reachable in the limit from $\mathbf{m_0}$ ($\mathbf{m}'' \in \text{lim-RS}(\mathcal{N}, \mathbf{m_0})$). Notice that if $|\boldsymbol{\sigma}'|$ tends to zero, then the value of α goes to one and $\mathbf{m}''$ approaches $\mathbf{m}$. Thus, firing a finite sequence we can get as close to $\mathbf{m}$ as desired. □

Establishing a bridge to discrete systems, it can be said that if the system is highly populated and it is not necessary to exactly determine the marking at places, then the system can evolve to any marking that is solution of the state equation.

Summarizing on reachability, the following relationship among the different sets of reachable markings can be stated. It asserts that the only differences among the described sets of reachable markings are in the border points of the space defined by the state equation.

Corollary 18 *If every transition is fireable then:*

1. $]\text{LRS}(\mathcal{N}, \mathbf{m_0})[\subseteq \text{RS}(\mathcal{N}, \mathbf{m_0}) \subseteq \text{lim-RS}(\mathcal{N}, \mathbf{m_0}) \subseteq \delta\text{-RS}(\mathcal{N}, \mathbf{m_0}) = \text{LRS}(\mathcal{N}, \mathbf{m_0})$.
2. *Under consistency of* $\mathcal{N}$*:* $\text{lim-RS}(\mathcal{N}, \mathbf{m_0}) = \delta\text{-RS}(\mathcal{N}, \mathbf{m_0}) = \text{LRS}(\mathcal{N}, \mathbf{m_0})$.

Proof. 1. is a direct consequence of the fact that $\delta\text{-RS}(\mathcal{N}, \mathbf{m_0})$ is the closure of $\text{RS}(\mathcal{N}, \mathbf{m_0})$ and $\text{lim-RS}(\mathcal{N}, \mathbf{m_0})$, and $\delta\text{-RS}(\mathcal{N}, \mathbf{m_0}) = \text{LRS}(\mathcal{N}, \mathbf{m_0})$.

2. is immediate from Corollary 16 and Theorem 17. □

6 Reversibility and Liveness

Reversibility is a basic property that has to do with mutual reachability among all markings of the system, or equivalently with the ability to reach the initial marking from any reachable one. Liveness is the capacity of the system of potentially firing any transition from any reachable marking. In discrete systems *if every transition is fireable at least once, then reversibility implies liveness and* *consistency*: if a system is reversible, it can always get back to the initial marking, therefore it is live because from the initial marking every transition is fireable at least once. Moreover, if a system can always return to the initial marking after every transition has fired, it means that a T-semiflow covering every transition has been fired, that is, the system is consistent.

However, liveness and consistency are not sufficient conditions for reversibility in discrete systems. For example, the system in Figure 7 is consistent and live as discrete, however once t_1 has fired it is impossible to get back to the initial marking. Thus the system is not reversible as discrete.

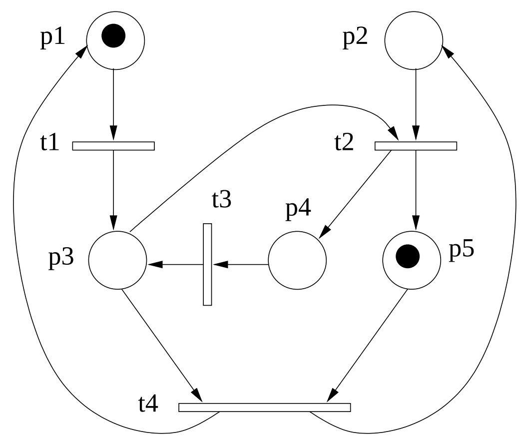


Fig. 7. Non reversible system as discrete or continuous with finite number of firings, but lim-reversible and δ-reversible

In continuous systems, assuming that every transition is fireable at least once, it can be observed that reversibility also implies consistency and liveness. As in discrete systems, if reachability with finite sequences is considered, liveness and consistency are not sufficient conditions for reversibility. The system in Figure 6 is consistent and live as continuous considering finite firing sequences. If transition t_1 is fired in any amount, the trap $\{p_2, p_3, p_4\}$ becomes marked, and cannot be emptied with a finite firing sequence. Hence, once t_1 has fired it is not possible to go back to the initial marking, and therefore it can be said that the system is not reversible. Nevertheless, as it will be seen, the system is reversible if lim-reachability and δ-reachability are considered.

In [7] lim-liveness was defined in order to extend the liveness concept to continuous systems regarding lim-reachability. Let us now define also δ-liveness and lim-(δ-)reversibility as the natural extensions of the classical definitions for the concepts of lim-reachability and δ-reachability respectively:

Definition 19 $\langle \mathcal{N}, \mathbf{m_0} \rangle$ *is* lim-*(δ-)live iff for every* $\mathbf{m} \in$ lim-(δ-)RS$(\mathcal{N}, \mathbf{m_0})$ *and for every* $t \in T$ *there exist* $\mathbf{m}' \in$ RS$(\mathcal{N}, \mathbf{m})$ *such that* enab$(t, \mathbf{m}') > 0$.

Definition 20 $\langle \mathcal{N}, \mathbf{m_0} \rangle$ *is* lim-*(δ-)reversible iff for every* $\mathbf{m} \in$ lim-(δ-)RS$(\mathcal{N}, \mathbf{m_0})$, $\mathbf{m_0} \in$ lim-(δ-)RS$(\mathcal{N}, \mathbf{m})$.

The following theorem states that under lim-reachability and δ-reachability, if every transition can be fired at least once, consistency and lim-(δ-)liveness are not only necessary conditions for lim-(δ-)reversibility but also sufficient.

Theorem 21 *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be such that every transition is fireable at least once. $\langle \mathcal{N}, \mathbf{m_0} \rangle$ is consistent and* lim-*(δ-)live iff $\langle \mathcal{N}, \mathbf{m_0} \rangle$ is* lim-*(δ-)reversible.*

Proof.
($\Rightarrow$)
Since the system is consistent and every transition is fireable at least once, it holds by Corollary 18 that lim-RS$(\mathcal{N}, \mathbf{m_0}) = \delta$-RS$(\mathcal{N}, \mathbf{m_0}) =$ LRS$(\mathcal{N}, \mathbf{m_0})$. Let us consider the lim-(δ-)reachable marking $\mathbf{m}$, $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}$. It will be seen that $\mathbf{m_0}$ is lim-(δ-)reachable from $\mathbf{m}$. Since the system is lim-(δ-)live, every transition is fireable from $\mathbf{m}$, and therefore a strictly positive marking, $\mathbf{m'} > \mathbf{0}$, can be reached, $\mathbf{m'} = \mathbf{m_0} + \mathbf{C} \cdot (\boldsymbol{\sigma} + \boldsymbol{\sigma'})$. The net is consistent, hence a T-semiflow, $\mathbf{x} > \mathbf{0}$, exists such that $\mathbf{x} - \boldsymbol{\sigma} - \boldsymbol{\sigma'} \geq \mathbf{0}$. By Corollary 15, $\mathbf{m_0} = \mathbf{m'} + \mathbf{C} \cdot (\mathbf{x} - \boldsymbol{\sigma} - \boldsymbol{\sigma'})$ is lim-(δ)reachable from $\mathbf{m'}$.
($\Leftarrow$)
If the system is reversible and every transition can be fired at least once, then it clearly cannot lim-(δ-)reach a marking in which one transition is not fireable any more. It would mean that it cannot get back to the initial marking. Moreover, if after the firing of every transition the system always can return to the initial marking, it means that it is consistent. □

For example, the system in Figure 7 is consistent and lim-(δ-)live, therefore according to Theorem 21 it is lim-(δ-)reversible. If from the initial marking t_1 is fired in an amount of 1, the marking (0, 0, 1, 0, 1) is reached. Applying the infinite firing sequence $\frac{1}{2}, t_4 \frac{1}{2} t_2, \frac{1}{2} t_3, \frac{1}{4}, t_4 \frac{1}{4} t_2, \frac{1}{4} t_3, \ldots$ from (0, 0, 1, 0, 1) the system converge to the initial marking.

From Theorem 21, the following Corollary is immediate:

Corollary 22 *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be* lim-*(δ-)live. $\langle \mathcal{N}, \mathbf{m_0} \rangle$ is* lim-*(δ-)reversible iff $\mathcal{N}$ is consistent.*

Notice that the lim-(δ-)liveness condition in Theorem 21 and Corollary 22 cannot be relaxed to lim-(δ-)deadlock-freeness, where lim-(δ-)deadlock-freeness means that the system cannot lim-(δ-)reach a marking in which no transition is fireable. In other words, as in discrete systems, lim-(δ-)deadlock-freeness does not imply lim-(δ-)liveness, even under consistency and conservativeness. For example, the consistent and conservative system in Figure 8 is lim-(δ-)deadlock-free but not lim-(δ-)live: transitions t_3 and t_4 are potentially fireable from any lim-(δ-)reachable marking, but once t_1 is fired in an amount of 1, neither t_1 nor t_2 will ever be fireable.

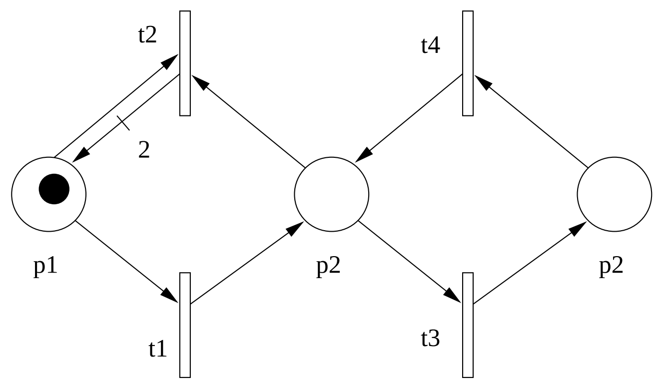


Fig. 8. A lim-(δ-)deadlock-free and not lim-(δ-)live system

7 Conclusions

In continuous nets the concept of “reachable marking” can be interpreted in three different ways:

a) *reachability*, a marking can be reached with a finite firing sequence.

b) lim-*reachability*, a marking can be reached with a finite or with an infinite firing sequence.

c) δ-*reachability*, the system can get as close as desired to a marking with a finite firing sequence.

Each of the three concepts has its own reachability space. These reachability spaces can be fully characterized using, among other elements, the state equation. Moreover, it is decidable whether a marking is reachable according to each concept. Furthermore, there is an inclusion relationship among the sets of markings associated to each concept: $a \subseteq b \subseteq c$. The only differences among these sets are in the border points of the spaces (i.e., the convex hull).

Moreover, as the level of “exigency” regarding reachability decreases (a is the “strongest” and c the “weakest”) the characterization of the reachability space becomes progressively easier. In particular, if every transition is fireable at least once, a very weak condition because otherwise unfireable transitions can be simply removed, the set of the markings in c is equal to the solutions of the state equation. In other words, for this last case there exists no spurious solution of the state equation.

Finally, a necessary and sufficient condition for reversibility with respect to lim-reachability and δ-reachability has been provided.

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