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The "bound reaching problem" on the fluidization of timed Petri nets ⋆

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Abstract: Fluidization is a classical technique to overcome the *state explosion problem*, which consists in relaxing its behaviour, dealing with hybrid or continuous systems. In the Petri nets framework, continuous net systems are the result of removing the integrality constraint in the firing of transitions. This relaxation may highly reduce the complexity of analysis techniques but may not approximate some properties of the original system, such as its throughput. This paper deals with the basic operation of fluidization of discrete timed Petri nets. More precisely, the "bound reaching problem" is identified, which points out the differences between discrete and continuous behaviour in a case in which the probability of a transition to be enabled is low in the discrete case. An approach denoted ρ*-semantics* is proposed to tackle this problem and compared with other methods.

Keywords: fluidization; continuous Petri nets; stochastic Petri nets; throughput;

1. INTRODUCTION

Petri nets (PN) are a well known formalism for the analysis of Discrete Event Systems (DES). Continuous Petri nets are the result of removing the integrality constraint in the firing of transitions. This process is known as *fluidization* being its result a continuous Petri net in which both the firing amounts of transitions and the marking of places are non-negative real quantities (David and Alla, 2010; Silva et al., 2011). In this work we will focus on timed systems.

At first glance, the simple way in which the basic definitions of discrete models are extended to continuous ones may make us naively think that their behaviour will be similar. However, the behaviour of the continuous model can be completely different just because the integrality constraint has been dropped. Some works have been proposed to improve the accuracy of the original fluid approximation, such as Vázquez and Silva (2012), Lefebvre and Leclercq (2012).

When the population of the system is high, the continuous PN system often approximates adequately the behaviour of the discrete one (see Section 2.3). Unfortunately, this is not the case in systems in which the population is "relatively small" in a part of the system (if a place has a big population, but also a big amount of tokens is required for its output transition, we can say that it is "relatively small"). This paper focuses on the approximation of timed discrete PN by means of continuous PN in the cases in which the maximum marking of a place is equal to the weight of one of its output arcs. It is what is denoted here as the "bound reaching problem" (BRP).

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The BRP is a challenging problem that appears in many practical cases. It can arise in systems in which very low and very high populations are combined. In particular, it also appears when inhibitor arcs of a bounded system are removed and simulated with regular arcs and places, because the complementary place that is added presents exactly this problem: its marking bound is equal to the weight of at least one of its output arcs.

Among the different concerns related to the BRP, we will concentrate on the approximation of the mean throughput of a stochastic PN system by its continuous counterpart.

The rest of the work is organized as follows. Section 2 recalls some definitions that will be used in the rest of the paper. In Section 3, we introduce the *bound reaching problem* and we describe two preliminary approaches to address it. Section 4 proposes a continuous approach to tackle the *bound reaching problem*, the ρ*-semantics*. Two case studies are discussed in Section 5. Section 6 concludes the paper.

2. PRELIMINARY CONCEPTS AND DEFINITIONS

This section defines some of the concepts used in the rest of the paper. First, discrete stochastic Petri nets and continuous Petri nets are introduced. Then, the relationship between them as the system size tends to infinity is established. In the following, it is assumed that the reader is familiar with Petri nets (see Murata (1989); DiCesare et al. (1993) for a gentle introduction).

2.1 Stochastic Petri nets

A Petri net (PN) is a tuple $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post} \rangle$ where $P = \{p_1, p_2, ..., p_n\}$ and $T = \{t_1, t_2, ..., t_m\}$ are disjoint and finite sets of places and transitions, and $\boldsymbol{Pre}$, $\boldsymbol{Post}$

are $|P| \times |T|$ sized, natural valued, incidence matrices. The *preset* and *postset* of a node $X \in P \cup T$ are denoted by ${}^{\bullet}X$ and $X^{\bullet}$, respectively. A discrete PN system is a tuple $\langle \mathcal{N}, \boldsymbol{M}_0 \rangle$ where $\mathcal{N}$ is the structure and $\boldsymbol{M}_0 \in \mathbb{N}_{\geq 0}^{|P|}$ is the initial marking (denoted in upper case $\boldsymbol{M}$ for the discrete).

The enabling degree of transition t_i at marking $\boldsymbol{M}$ is defined as $Enab(t_i, \boldsymbol{M}) = min_{p_j \in {}^{\bullet}t_i} \lfloor M[p_j]/Pre[p_j, t_i] \rfloor$. The firing of a transition t_i in a certain *natural* amount $\alpha \leq Enab(t_i, \boldsymbol{M})$ leads to a new marking $\boldsymbol{M'}$, which is denoted as $\boldsymbol{M} \xrightarrow{\alpha t_i} \boldsymbol{M'}$, and it that satisfies $\boldsymbol{M'} = \boldsymbol{M} + \alpha \cdot \boldsymbol{C}[P, t_i]$, where $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the token flow matrix (incidence matrix if $\mathcal{N}$ is self-loop free) and $C[P, t_i]$ denotes the i^{th} column $\boldsymbol{C}$. Hence, $\boldsymbol{M} = \boldsymbol{M}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}$, the state (or fundamental) equation summarizes the way the marking evolves; where $\boldsymbol{\sigma}$ is the firing count vector associated to the fired sequence.

The set of all the markings reachable in $\langle \mathcal{N}, \boldsymbol{M}_0 \rangle$ is defined as: $RS(\mathcal{N}, \boldsymbol{M}_0) = \{\boldsymbol{M} \mid \exists \sigma = \alpha_1 t_1 \ldots \alpha_k t_k$ s.t. $\boldsymbol{M}_0 \xrightarrow{\alpha_1 t_1} \boldsymbol{M}_1 \xrightarrow{\alpha_2 t_2} \boldsymbol{M}_2 \cdots \xrightarrow{\alpha_k t_k} \boldsymbol{M}_k = \boldsymbol{M}\}$.

A Markovian stochastic Petri net system (SPN) is a discrete PN system in which the transitions fire at independent exponentially distributed random time delays (see Molloy (1982)). Hence, the firing time of each transition is characterized by its firing rate. More formally, a SPN is a tuple $\langle \mathcal{N}, \mathbf{M_0}, \boldsymbol{\lambda} \rangle$, where $\boldsymbol{\lambda} \in \mathbb{R}_{>0}^{|T|}$ is the vector of rates associated to the transitions. In this paper, infinite-server semantics is assumed for all transitions, and therefore, the system evolves as a jump Markov process where the time to fire a transition t_i, at a given marking $\boldsymbol{M}$, follows an exponentially distributed function with parameter $\lambda_i \cdot Enab(t_i, \boldsymbol{M})$.

Assuming the SPN system is bounded and ergodic, the steady state throughput of a transition t_i, denoted as $\chi(t_i)$, is the limit average number of times t_i fires per time unit when the time tends to infinity.

2.2 Continuous Petri nets

The main difference between continuous and discrete PNs is in the firing amounts and consequently in the marking, which in discrete PNs are restricted to be in the naturals, while in continuous PNs are relaxed into the non-negative real numbers. Thus, a *continuous* PN system is understood as a relaxation of a *discrete* one.

A continuous PN system is a tuple $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ where $\mathcal{N}$ is the net structure (as defined for discrete PNs) and $\boldsymbol{m}_0 \in \mathbb{R}_{\geq 0}^{|P|}$ is the initial marking. In a continuous PN, the enabling degree of transition t_i at marking $\boldsymbol{m}$ is defined as $enab(t_i, \boldsymbol{m}) = \min_{p_j \in {}^{\bullet}t_i} \{ \frac{\boldsymbol{m}[p_j]}{\boldsymbol{Pre}[p_j, t_i]} \}$. The firing of a transition t_i in a certain *real* amount $\alpha \leq enab(t_i, \boldsymbol{m})$ leads to a new marking $\boldsymbol{m'}$ that satisfies $\boldsymbol{m'} = \boldsymbol{m} + \alpha \cdot \boldsymbol{C}[P, t_i]$. Notice that in contrast to discrete PNs, a continuous transition can fire if all its input places are positively marked, i.e., $enab(t_i, \boldsymbol{m}) > 0$, regardless of the arc weights.

As in discrete PNs the state equation $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}$ summarizes the system evolution. The derivative of this equation with respect to time is $\dot{\boldsymbol{m}} = \boldsymbol{C} \cdot \dot{\boldsymbol{\sigma}}$ where $\dot{\boldsymbol{\sigma}} = \boldsymbol{f}$ is the vector of instantaneous continuous flows of transitions. Different semantics exist to define the flow $\boldsymbol{f}$ of transitions, the two most important being infinite server and finite server semantics (David and Alla, 2010; Silva et al., 2011). Here, infinite server semantics (ISS) will be considered.

A Timed Continuous Petri Net (TCPN) is a continuous PN together with a vector $\boldsymbol{\lambda} \in \mathbb{R}_{>0}^{|T|}$ defining the speed associated to transitions. Similarly to purely markovian discrete net models, under ISS, the flow through a continuous timed transition t_i is the product of the speed, λ_i, and the instantaneous enabling of the transition, i.e., $f_i = \lambda_i \cdot enab(t_i, \boldsymbol{m})$.

For the flow to be well defined, every transition must have at least one input place, hence in the following we will assume that $|{}^{\bullet}t_i| \geq 1$ for each transition t_i. Thus, the time evolution of a TCPN can be expressed by the following system of differential equations:

$$\dot{\boldsymbol{m}} = \boldsymbol{C} \cdot \boldsymbol{f} \tag{1}$$

where the i^{th} component of $\boldsymbol{f}$ is $f_i = \lambda_i \cdot enab(t_i, \boldsymbol{m})$.

If there exists a steady state in a TCPN system, the throughput of t_i, denoted as $\chi(t_i)$, is equal to its flow f_i.

2.3 Deterministic limit of SPN

In population dynamics, the *deterministic limit* (Jacod and Shiryaev, 2002) describes the trajectory towards which the population densities of a discrete stochastic system converge as its size tends to infinity. Let us consider a SPN with initial marking $\boldsymbol{M}_0 = k \cdot \mathbf{x}_0 \in \mathbb{N}_{\geq 0}^{|P|}$ where $\mathbf{x}_0 \in \mathbb{R}_{\geq 0}^{|P|}$ represents the initial marking density of the system, and $k \in \mathbb{R}$ represents the system size (or volume).

Let us define the vector field for place p_j as $F_j(\mathbf{x}) = \sum_{t_i \in ({}^{\bullet}p_j \cup p_j{}^{\bullet})} C[p_j, t_i] \cdot f_i$, where $f_i = \lambda_i \cdot enab(t_i, \mathbf{x})$ (notice that F_j is a nonnegative function of real arguments on the system densities). Let $F(\mathbf{x})$ be a vector composed of the vector field functions $F_j(\mathbf{x})$ of every place p_j. The two following conditions can be easily checked: a) $F(\mathbf{x})$ is Lipschitz continuous, i.e., $\exists H \geq 0$ such that $|F(\mathbf{x}) - F(\mathbf{y})| \leq H \cdot |\mathbf{x} - \mathbf{y}|$; b) $\sum_{t_i \in ({}^{\bullet}p_j \cup p_j{}^{\bullet})} |C[p_j, t_i]| \cdot f_i(\mathbf{x}) < \infty$. Then, the deterministic limit behaviour of the marking densities $\mathbf{x}$ of the SPN when k tends to infinity is given by the following set of differential equations (Ethier and Kurtz, 1986; Jacod and Shiryaev, 2002): $\dot{\mathbf{x}} = F(\mathbf{x}) = \boldsymbol{C} \cdot \boldsymbol{f}$.

Thus, the deterministic limit of a SPN matches with the time evolution defined for TCPN, and therefore a TCPN captures faithfully the behaviour of a SPN with high markings. However, in order to obtain a suitable continuous approximation for SPN with low markings, further manipulations are required on the TCPN.

3. THE BOUND REACHING PROBLEM

The "bound reaching problem" (BRP) studies a particular situation in which the continuous approximation does not approximate correctly the behaviour of the discrete PN.

As previously pointed (Section 2.3), TCPN approximate reasonably well the behaviour of SPN when the populations are relatively high. However, when "relatively small" populations are also considered it is not the case.

Table 1. $\chi(t_1)$ in the PN in Fig. 1, with $\boldsymbol{\lambda} = (10, 1, 1)$; considered as SPN and as TCPN.

Method	k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8	k = 9	k = 10
SPN	0.833	0.417	0.242	0.144	0.085	0.049	0.028	0.016	0.008	0.005
TCPN	0.833	0.833	0.833	0.833	0.833	0.833	0.833	0.833	0.833	0.833

This lack of accuracy is related to the fact that synchronizations are strongly relaxed when the net is fluidified. Consider a transition t and a place p such that ${}^\bullet t = \{p\}, Pre[p,t] = k$ (for example, t_1 and p_1 in Fig. 2(a)). Considered as a discrete system, t is only enabled when $M[p] \geq k$, and the probability of that transition to be enabled can be low. However, as continuous, t is enabled for any positive amount of tokens $m[p] > 0$, regardless of the arc weight k. An extreme case occurs when the maximum possible amount of tokens in p is equal to k. Then, this place p needs to "reach its bound" in order to enable transition t. This lack of accuracy and the search of alternative fluid schemes to improve it the BRP.

We identify that a system $\langle \mathcal{N}, \boldsymbol{M}_0 \rangle$ suffers from the BRP when the maximum number of tokens in a place is k, and the arc from that place to a transition has weight k. More formally, there exists BRP if $\exists p, t$, s.t. $\boldsymbol{M}[p] \leq Pre[p,t]$, $\forall \boldsymbol{M} \in RS(\mathcal{N}, \boldsymbol{M}_0)$. If the inequality is strict, i.e., $M[p] < Pre[p,t]$, $\forall \boldsymbol{M} \in RS(\mathcal{N}, \boldsymbol{M}_0)$, then t is a transition which will never be enabled in the net system, and consequently the system is not live. We will not consider this case along this work, because it would be enough to remove t.

Let us define the set of places which suffer from the *bound reaching problem* as *bound reaching set* (BRS):

$$BRS = \{p | M[p] \leq Pre[p,t], \forall \boldsymbol{M} \in RS(\mathcal{N}, \boldsymbol{M}_0)\}$$

Equivalently, the set of transitions which suffer the BRP is defined as the *bound reaching transition set* ($BRTS$):

$$BRTS = \{t | \exists p \in {}^\bullet t, M[p] \leq Pre[p,t], \forall \boldsymbol{M} \in RS(\mathcal{N}, \boldsymbol{M}_0)\}$$

In this paper, we focus on the relatively frequent particular case of the BRP affecting only one transition t, and t has only one input place (i.e., it is not a *join* transition): $|BRTS| = 1$ and $|{}^\bullet BRTS| = 1$.

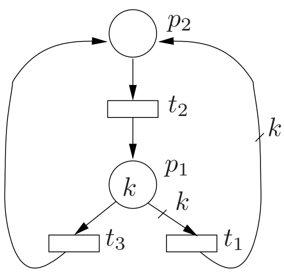

Fig. 1. PN system in which place p_1 and transition t_1 suffer from the BRP, $\boldsymbol{\lambda} = (10, 1, 1)$.

Consider the PN example in Fig. 1. Apparently, it has four parameters (λ_1, λ_2, λ_3 and k). However, one of the firing rates can be fixed (here, $\lambda_2 = 1$), which seen just as a time scale: we can consider three parameters, without lost of generality. Here, In which the BRP appears in p_1, t_1. Transitions t_1 and t_3 are enabled at $\boldsymbol{M}_0 = (k, 0)$. Suppose t_1 is fired first. Then, all the tokens are moved to p_2, and only t_2 is enabled. After one or more firings of t_2, transitions t_2, t_3 are enabled an can fired. Transition t_3 is enabled when $M[p_1] \geq 1$. However, t_1 can only fire when $M[p_1] = k$. Consequently, transition t_1 is not fired very often and its throughput is low.

Moreover, if k grows, then the probability of having k tokens in p_1 decreases, and hence the steady state throughput $\chi(t_1)$ of the SPN decreases. It can be seen in Table 1, in which the steady state throughput of t_1 for different values of k is shown in the first row for SPN. The steady state throughput also depends on the ratio among the firing rates λ_i. Assuming λ_2 a scale constant, $\chi(t_1)$ increases monotonically with λ_1, and $\chi(t_1)$ decreases when λ_3 increases.

However, in the case of the TCPN, $\chi(t_1)$ is independent of k, as it can be seen in the second row in Table 1.

The schema of this example is important in practice. It appears when an inhibitor arc is removed by the system with the addition of a *complementary* place which usually has this structure.

3.1 A basic schema in which BRP may appear

In order to tackle the BRP, we will start with the the simple but representative example in Fig. 2(a), in which p_1 has only one input transition, $|p_1{}^\bullet| = 1$. In other words, transition t_3 is dropped from Fig. 1.

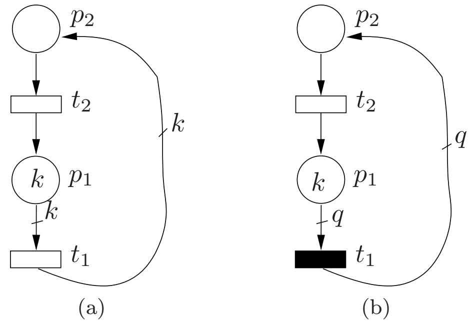

Fig. 2. (a) SPN system in which p_1 and t_1 suffer from the BRP, its firing rates are $\boldsymbol{\lambda} = (10, 1)$. (b) Hybrid PN system in which t_1 is discrete (black transition) and t_2 is continuous, arc weights are modified to q.

In the steady state, the total cycle time Θ from $\boldsymbol{M}_0 =$ (k,0) is the addition of the average time to fire t_1 from $\boldsymbol{M}_0$ (which happens with a mean value of $\frac{1}{\lambda_1}$), the mean time to fire t_2 when $M[p_2] = k$ (which is $\frac{1}{k \cdot \lambda_2}$), the one when $M[p_2] = k - 1$ (which is $\frac{1}{(k-1) \cdot \lambda_2}$), etc. Thus, the average cycle time is $\Theta = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \cdot \sum_{i=1}^{k} \frac{1}{i}$. The throughput of t_1 is equal to $\frac{1}{\Theta}$ (see row "SPN" in Table 2):

$$\chi(t_1) = \frac{\lambda_1 \cdot \lambda_2}{\lambda_1 \cdot \sum_{i=1}^{k} \frac{1}{i} + \lambda_2} = \lambda_2 \frac{1}{\sum_{i=1}^{k} \frac{1}{i} + \frac{\lambda_2}{\lambda_1}} \qquad (2)$$

$\chi(t_1)$ is the product of a dimensionless coefficient depending on k and $\frac{\lambda_2}{\lambda_1}$, multiplied by λ_2 that defines a time scale. Thus, we can normalize $\lambda_2 = 1$. Consequently, the

Table 2. $\chi(t_1)$ in the PN in Fig. 2(a), with $\boldsymbol{\lambda} = (10, 1)$, for different methods.

Method	k = 1	k = 2	k = 3	k = 4	k = 5	k = 6	k = 7	k = 8	k = 9	k=10	k=50	k=100
SPN	0.909	0.625	0.517	0.458	0.420	0.392	0.371	0.355	0.341	0.330	0.217	0.189
TCPN	0.909	0.909	0.909	0.909	0.909	0.909	0.909	0.909	0.909	0.909	0.909	0.909
Meth1	0.909	0.729	0.607	0.522	0.459	0.412	0.374	0.343	0.318	0.296	0.090	0.051
Meth2, 2τ	0.475	0.475	0.475	0.475	0.475	0.475	0.475	0.475	0.475	0.475	0.475	0.475
Meth2, 3τ	0.324	0.324	0.324	0.324	0.324	0.324	0.324	0.324	0.324	0.324	0.324	0.324

normalized $\chi(t_1)$ depends on two parameters: k (in $\mathcal{N}$ and $\boldsymbol{m}_0$) and λ_1 (which depends on the relative firing rates).

3.2 First approaches to the BRP

After identifying the BRP, our aim is to find methods or techniques to approach it, i.e., to fluidify the net system to obtain a good approximation to the original discrete one. The techniques can range from fully continuous to hybrid.

Let us describe two very basic methods which can be used to approach the BRP. The first one is a continuous approximation with a firing semantics different from the usual ISS. The second one is a hybrid PN in which the transitions are continuous or discrete, but some arc weights must be modified. This kind of methods are also described in Zarnay and Silva (2010). They will be compared with the ρ-*semantics* proposed in Section 5.

- **(Meth1)Ad hoc continuous flow estimation.** It is based on the heuristic idea that the k tokens are considered to be independent. The probability of $M[p_1] = 1$ is "considered" to be $\frac{m[p_1]}{k}$, while the probability of $M[p_1] = k$ would be $(\frac{m[p_1]}{k})^k$. Based on this heuristic reasoning, a marking-dependent flow approximation for transition t_1 is defined as follows:
$$f(t_1) = \lambda_1 \left(\frac{m[p_1]}{k}\right)^k$$
With this semantics, the enabling degree of t_1 is *not linear* (see Fig. 5(b)). The flow given by this technique has the advantages of getting continuous fully differentiable models. However, with this semantics, some properties of ISS are lost, for example the flow evolution is not marking *homothetic* (given $\boldsymbol{f}$ the flow in $\langle \mathcal{N}, \boldsymbol{m}_0, \boldsymbol{\lambda}\rangle$, the one in $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0, \boldsymbol{\lambda}\rangle$ is not $k \cdot \boldsymbol{f}$).
A drawback of this approach is that although it provides a reasonable approximation of the SPN throughput for small (see k between 1 and 10 in Table 2), it is not so good for higher values of k (see k=50 or k=100 in Table 2).
- **(Meth2) Hybrid (with scaled arc weights).** Let us consider that every transition is continuous except the one which suffers from the BRP (in this case, t_1 discrete and t_2 continuous). Thus, a hybrid model is obtained. The time simulation of this hybrid net over time shows that filling place p_1 up to capacity k "lasts forever", so t_1 will never be enabled again after the first firing. It is a first order time invariant linear system without zero; for example, it corresponds to the loading of a capacitor in an electrical RC-circuit, described by a negative exponential function. This exponential is characterized by a parameter τ, which here is $\tau = \lambda_2$. It is a classical result that the 95% of k (the response time at 5%) is reached at time 3τ. Using this idea, we modify the arc weight of the input and output arcs of the hybrid transition from k to $q = 0.95 \cdot k$ (Fig. 2(b)). The enabling degree of the transition is modified to the one in Fig. 3(c). Equivalently, the 86.7% is reached at time 2τ and the 98.2% is reached at time 4τ.
Due to the fact that q depends proportionally on k, what is defined is a response time independent of the aptitude (k): at 5% if $q = 0.95 \cdot k$ (3τ), at 13,3% if $q = 0.867 \cdot k$ (2τ) or at 37,7% if $q = 0.623 \cdot k$ (τ). This is shown in Table 2. Another characteristic of this approach is the fact that the flow is discontinuous, being a hybrid net in the classical sense.

These two approaches are interesting, but just a first approach to the BRP. Hence, different techniques should be investigated.

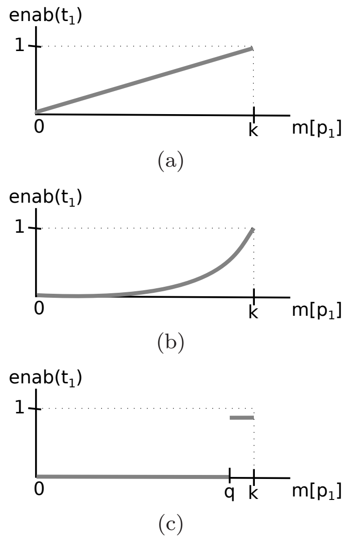

Fig. 3. Enabling degree of transition t_1 in Fig. 2 with different semantics: (a) TCPN under ISS; (b) Meth1; (c) Meth2. (a) and (b) are continuous and differentiable, (c) is discontinuous, a *hybrid* net in the classical sense.

4. A NEW SEMANTICS TO APPROACH THE BOUND REACHING PROBLEM

In this section, a new semantics for transitions is proposed to approach the BRP, denoted ρ-*semantics*. The preliminary idea which inspires this semantics is to try to simulate the "wait until there are enough tokens to fire" of the discrete net. This behaviour can be obtained with timed and immediate transitions and some additional places (Section 4.1). An *immediate* transitions which has some tokens in its input places, fires "immediatelly" (in 0 time units). Then, the desired behaviour is obtained by the definition of the new firing semantics for the transition, as explained in Subsection 4.2.

4.1 Simulating discrete behaviour with immediate transitions

Considering the PN in Fig. 2(a), a key difference between the behaviour of the SPN and the TCPN is that in the SPN, t_1 can fire only when the k tokens are in p_1; while in the continuous case, it is not needed to "wait until the k tokens" are in p_1 to fire t_1.

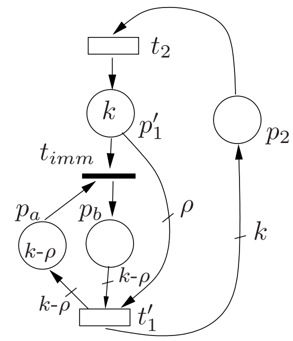

Fig. 4. Transformation of the PN in Fig. 2(a), as explained in Section 4. White transitions are continuous under ISS, while the thin black transition is *immediate*.

This is the idea exploited in this approach: to simulate the "wait" of t_1 until it has k tokens. As explained, waiting until p_1 has k tokens would last infinite time in a TCPN. Hence, it has no sense to wait until k, but until some other smaller value, such as $k - \rho$ (where ρ comes from "the rest"). This behaviour can be obtained by transforming p_1, t_1 (see Fig. 2(a)) to a subnet composed of p'_1, t'_1, p_a, p_b, t_{imm} (see Fig. 4), such that t'_1 is not enabled for "the first" $k - \rho$ tokens, and it is enabled for higher amounts.

Immediate transitions are difficult to handle in TCPN (Recalde et al., 2006). A first approximation can be to consider immediate transitions as timed transitions which are several orders of magnitude faster than the other transitions (for example, $\lambda_{imm} = 10000$ in the TCPN in Fig. 4). However, this has some disadvantages: If λ_{imm} is relatively not very high, then the steady state might not be the desired one (because for high populations, t_{imm} could be part of the bottleneck); while for very high values of λ_{imm}, *stiffness* problems can appear when integrating (1).

4.2 Defining the ρ-semantics

In this concrete construction, we can abstract the structure given by p'_1, t'_1, p_a, p_b, t_{imm} by a unique transition with a new semantics, which compacts the desired behaviour.

Let us detail the marking of the new structure (places p'_1, p_a, p_b) and the enabling degree of t'_1, with respect to the possible marking of p_1 in the original net, i.e., $m[p_1]$:

- $m[p_1] \leq k - \rho$. Then transition t_{imm} is enabled, the marking is moved *immediately* to p_b and there is no remaining token at p'_1. Hence, $m[p'_1] = 0$, $m[p_a] = k - \rho - m[p_1]$, $m[p_b] = m[p_1]$ and $enab(t'_1) = 0$.
- $k - \rho < m[p_1] \leq k$. Then p_a is empty, transition t_{imm} is disabled, and there is some remaining token at p'_1. Hence, $m[p_a] = 0$, $m[p_b] = k - \rho$, $m[p'_1] = m[p_1] - (k - \rho)$ and $enab(t'_1) = min\{\frac{m[p_b]}{k-\rho}, \frac{m[p'_1]}{\rho}\} = \frac{m[p'_1]}{\rho} = \frac{m[p_1]-(k-\rho)}{\rho}$.

The difference between the enabling of t_1 and t'_1 is depicted in Fig. 3 and Fig. 5: t_1 is enabled for $m[p_1] > 0$, but t'_1 is enabled for $m[p_1] > k - \rho$.

The new transition t'_1 has a specific firing semantics, different from ISS ($f_1 = \lambda_1 \cdot enab(t_1)$), obtained from $\lambda_1 \cdot enab(t'_1)$. The flow of a transition t_1 under the *ρ-semantics* is given by the following formula:

$$f_1 = \begin{cases} 0 & \text{if } m[p_1] \leq Pre[p_1, t_1]\text{-}\rho \\ \lambda_1 \cdot \frac{m[p_1]\text{-}(Pre[p_1,t_1]\text{-}\rho)}{\rho} & \text{otherwise} \end{cases} \quad (3)$$

The transient flow of t_1 is still a continuous function, but it is piecewise defined, introducing certain "hybridization" in the behaviour of the transition. The computation done by this approach is local t_1 and it is simple and fast.

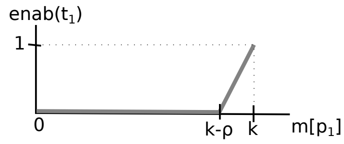

Fig. 5. Enabling degree of transition t_1 in Fig. 2(a) with the *ρ-semantics*. Certain "hybrid" behaviour is obtained in the firing of t_1, the flow is a piecewise function.

4.3 Selection of the appropriate ρ

With the proposed *ρ-semantics*, the throughput of the system can be "tuned" from 0 (when $\rho \sim 0$) to the throughput of the TCPN (when ρ is equal to $Pre[p_1, t_1]$).

The challenge is how to select ρ to approximate the steady state throughput of the SPN. Here we compute ρ first for the PN in Fig. 2(a), and then apply that *heuristics* on any PN system in which the transition t_1 which suffers from the BRP has the same structure: t_1 has only one input place, which has only one input and one output transition, $|{}^\bullet p_1| = |p_1{}^\bullet| = 1$.

Considering the *ρ-semantics* for t_1, and ISS for t_2, the throughput of t_1 at the steady state is (see the Appendix for a detailed explanation):

$$\chi_\rho(t_1) = \lambda_2 \cdot \frac{\rho}{k + \rho \cdot \frac{\lambda_2}{\lambda_1}} \quad (4)$$

Given (4) for the *ρ-semantics* and (2) for the SPN, forcing $\chi(t_1) = \chi_\rho(t_1)$, an analytical formula for the value of ρ is obtained, which is dependent on k (see the value of ρ for different values of k in Table 3):

$$\rho = \frac{k}{\sum_{i=1}^{k} \frac{1}{i}} \quad (5)$$

Interestingly, the value of ρ given by (5) is independent of $\boldsymbol{\lambda}$. We can think about the PN in Fig. 2(a) as a simplification of any net with analogous structure. Hence, it will be possible to use the formula of ρ calculated here as a heuristics for ρ in any system with the same structure.

Table 3. Optimal value of ρ, obtained from (5).

k	1	2	3	4	5	10	50
ρ	1	1.33	1.64	1.92	2.19	3.41	11.9

5. CASE STUDIES

In this section, the proposed *ρ-semantics* is applied to two case studies. It will be compared with Meth1 and Meth2, the two methods described in Section 3.2.

5.1 Example 2. A manufacturing system

Consider the PN in Fig. 6, which represents a manufacturing system in which tables are assembled and painted, in which cooperation and synchronization relations appear. Every transition has the same speed, $\boldsymbol{\lambda} = \mathbf{1}$.

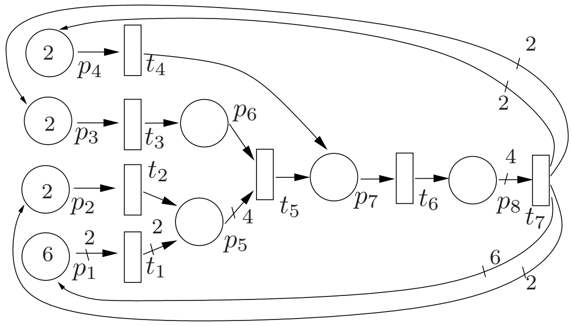

Fig. 6. Example 2. PN system modelling a manufacturing system, derived from Recalde and Silva (2001).

In this PN system, $BRS = \{p_8\}$, and $BRTS = \{t_7\}$. Hence, transition t_7 suffers from the BRP, and which the structure identified in Section 3.1. Applying the *ρ-semantics* to transition t_7. From (5), we set $\rho = \frac{4}{1+1/2+1/3+1/4} = 1.92$. The obtained throughput is shown in the last row in Table 4: $\chi(t_7) = 0.6443$, which in comparison with the other methods, is the best approximation of the original SPN system.

Table 4. Throughput of $t7$, $\chi(t_7)$, in Fig. 6 with $\boldsymbol{\lambda} = \mathbf{1}$. Comparative of different methods.

Method	throughput(t_7)
SPN	0.6573
TCPN	1.1429
Meth1	0.6030
Meth2, $q = 0.623 \cdot k$	0.4409
Meth2, $q = 0.867 \cdot k$	0.2636
Meth2, $q = 0.95 \cdot k$	0.1514
ρ-semantics	0.6443

Other methods can be also applied to the example as seen in Table 4. Meth1 and Meth2 for τ, 2τ and 3τ (with t_7 discrete and all the other transitions continuous, and weight arcs equal to $0.623 \cdot k$, $0.867 \cdot k$ and $0.95 \cdot k$ respectively) are better than a TCPN.

5.2 Example 3. A logic controller

The PN example in Fig. 7(a) is obtained after the decolourization of a net which models a Multi-Computer Programmable Logic Controller (MCPLC) in Zarnay and Silva (2010). It is a Generalized Stochastic PN (GSPN) (Balbo et al., 1987), which is a SPN enriched with immediate transitions (represented as thin black transitions).

As said, an interesting issue is how to model immediate transitions when a GSPN system is fluidified. Using the

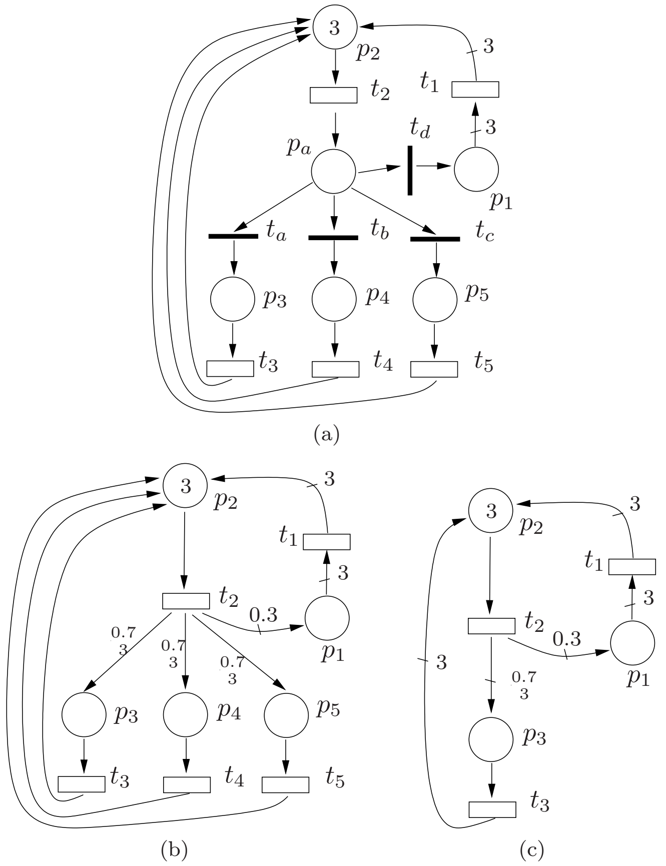

Fig. 7. Example 3. (a) Discrete GSPN system which models a MCPLC (Zarnay and Silva, 2010). (b) TCPN system obtained after the fluidization of (a). (c) Reduced TCPN system.

rules defined in Recalde et al. (2006), the immediate transitions in Fig. 7(a) can be reduced: First, transitions t_a, t_b, t_c and t_d, being in topologically equal conflict relation are merged into a fork transition. Then, that transition and the timed transition t_2 are transformed into a single transition, t_2 in Fig. 7(b). Moreover, the three symmetric branches can be merged (Meyer and Silva, 2012), obtaining the TCPN system in Fig. 7(c).

Table 5. Throughput of t_2, $\chi(t_2)$, of the PN in Fig. 7. Comparative of different methods.

Method	throughput(t_2) $\boldsymbol{\lambda}$=(2,10,5,5,5)	throughput(t_2) $\boldsymbol{\lambda}$=(100,1,10,10,10)
GSPN	4.666	1.525
TCPN	7.693	2.796
Meth1	4.756	2.427
Meth2, q=$0.623 \cdot k$	5.18	0.727
Meth2, q=$0.867 \cdot k$	3.86	0.607
Meth2, q=$0.95 \cdot k$	2.97	0.499
ρ-semantics	5.085	1.527

Let us first consider this example with the following vector of transition rates: $\boldsymbol{\lambda} = (2, 10, 5, 5, 5)$. For this $\boldsymbol{\lambda}$, the values of $\chi(t_1)$ are illustrated in the first column in Table 5. In this net system, $BRTS = \{t_3\}$, so the *ρ-semantics* is applied to t_3. It can be seen that the throughput when using *ρ-semantics*, in which $\rho = \frac{3}{1+1/2+1/3} = 1.6364$, is not as good as the one of Meth1.

However, for different combinations of $\boldsymbol{\lambda}$, for example $\boldsymbol{\lambda} = (100, 1, 10, 10, 10)$, the results are different (see the second column in Table 5). The throughput of t_2 for GSPN is equal to 1.525. In this case, Meth1 and Meth2 do not provide a good approximation for the GSPN. Nevertheless, the *ρ-semantics*, obtains good results for the approximation of the throughput of the system at the steady state, it is $\chi(t_2) = 1.527$.

6. CONCLUSIONS

This paper deals with a challenging problem which appears in the fludization of discrete PN with a localized part of not very high populations, which is denoted *bound reaching problem*: although the steady state throughput of a SPN system with high populations is well approximated by its TCPN counterpart, it is not the case for low populations. It is due to the relaxation of the integrality constraints of the original PN, which is specially relevant when the marking bound of a place coincides with the weight of one of its output arcs.

Different approaches for the (partial) fluidization of this kind of systems have been described. In particular, a new semantics has been proposed, the *ρ-semantics*, in which the transition suffering the BRP is enabled only if it has a certain amount of tokens in its input place. The *ρ-semantics* is discussed, and it is compared with the other methods. We are conscious of the fact that this is a first and partially heuristic consideration of a difficult problem that is essential to improve the quality of fluid approximation of discrete event systems.

The *bound reaching problem* may appear in more general schemas than the one addressed here, such as net systems in which the transition which suffers from the problem has several input places (i.e., it is a *join*), its input place has several output transitions (i.e., a *choice*) or the BRP appears in several points in the net system. Moreover, other probability distribution functions different from exponential can be considered for the firing of transitions of the discrete system. Future work will explore new methods, based on the *ρ-semantics* or in other continuous or partially hybrid ideas, to avoid the *bound reaching problem* in those cases.

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Appendix A. ANALYTICAL COMPUTATION OF THE STEADY STATE THROUGHPUT WITH THE *ρ-SEMANTICS*

Some notions for the analytical computation of equality (4) are presented here, i.e., $\chi_\rho(t_1)$ for the PN in Fig. 2(a). In that net system, t_1 has a *ρ-semantics*, while t_2 is a usual continuous transition, with ISS semantics.

Let $\boldsymbol{f}_{ss} = (f_{ss1}, f_{ss_2}) = (\chi_\rho(t_1), \chi_\rho(t_2))$ denote the steady state throughput of t_1 and t_2, and let $\boldsymbol{m}_{ss}$ denote the steady state marking. At steady state, $\boldsymbol{m}_{ss}$ keeps constant, and hence from (1) it holds $\boldsymbol{C} \cdot \boldsymbol{f}_{ss} = \boldsymbol{0}$. In this example:

$$f_{ss2} = k \cdot f_{ss1} \tag{A.1}$$

Given that the net system is live, it holds that $\boldsymbol{f}_{ss} > 0$. Then, by (3), the steady state throughput of t_1 is:

$$f_{ss1} = \lambda_1 \cdot \frac{m_{ss}[p_1] - (Pre[p_1, t_1] - \rho)}{\rho} \tag{A.2}$$

And by the ISS semantics, the steady state throughput of t_2 is:

$$f_{ss2} = \lambda_2 \cdot \frac{m_{ss}[p_2]}{Pre[p_2, t_2]} \tag{A.3}$$

From the net structure, the following relation is obtained:

$$m_{ss}[p_1] + m_{ss}[p_2] = k \tag{A.4}$$

From (A.1), (A.2), (A.3) and (A.4), the value of $\chi_\rho(t_1)$ is:

$$f_{ss1} = \frac{\lambda_1 \cdot \lambda_2 \cdot \rho}{\lambda_1 \cdot k + \lambda_2 \cdot \rho} = \lambda_2 \cdot \frac{\rho}{k + \rho \cdot \frac{\lambda_2}{\lambda_1}} \tag{A.5}$$