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On the fluidization of Petri nets and marking homothecy[☆]

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ABSTRACT

The analysis of Discrete Event Dynamic Systems suffers from the well known *state explosion problem*. A classical technique to overcome it is to relax the behavior by partially removing the integrality constraints and thus to deal with hybrid or continuous systems. In the Petri nets framework, continuous net systems (technically hybrid systems) are the result of removing the integrality constraint in the firing of transitions. This relaxation may highly reduce the complexity of analysis techniques but may not preserve important properties of the original system. This paper deals with the basic operation of fluidization. More precisely, it aims at establishing conditions that a discrete system must satisfy so that a given property is preserved by the continuous relaxation. These conditions will be mainly based on the *marking homothetic behavior* of the system. The focus will be on logical properties as boundedness, B-fairness, deadlock-freeness, liveness and reversibility. Furthermore, testing homothetic monotonicity of some properties in the discrete systems is also studied, as well as techniques to improve the quality of the fluid relaxation by removing spurious solutions.

1. Introduction

Petri nets [1,2], as other formalisms for Discrete Event Dynamic Systems (DEDS), suffer from the *state explosion problem*. Such a problem may render analysis techniques based on exhaustive enumeration computationally infeasible, particularly for large population systems. A promising approach to overcome this difficulty is to relax the original discrete model by explicitly removing the integrality constraint in the firing of transitions. This process is known as *fluidization*, being its result a continuous Petri net (PN) in which both the firing amounts of transitions and the marking of places are non-negative real quantities (see [3,4]).

Continuous PNs allow the use of some polynomial time complexity techniques for several analysis purposes [4]. Unfortunately, continuous nets may not always preserve important properties of the discrete model (first pointed out in [5]). For this reason, it is crucial to study which discrete PN systems can be "successfully" fluidified and which ones not. Moreover, some techniques can be used to improve the fluidization.

At first glance, the simple way in which the basic definitions of discrete models are extended to continuous ones may make us naively think that their behavior will be similar. However, the behavior of the continuous model can be completely different just because the integrality constraint has been dropped. In other words, not all DEDS can be satisfactorily fluidified. Consider, for instance, the net system in Fig. 1(a). If considered as discrete, the system is deadlock-free: from $\boldsymbol{m}_0 = (3, 0)$, both t_2 and t_1 can be fired alternatively, and no deadlock can be reached. However, if considered as continuous, transition t_2 can be fired in an amount of 1.5 from $\boldsymbol{m}_0$, leading to a deadlock marking $\boldsymbol{m}_d = (0, 1.5)$.

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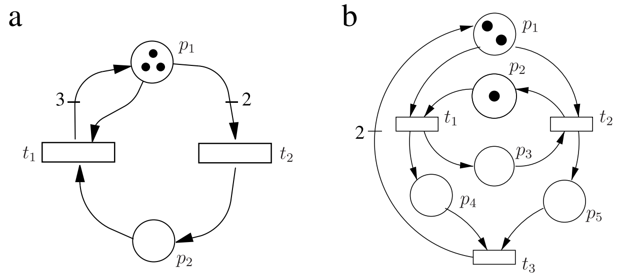

Fig. 1. (a) Not homothetically deadlock-free PN system [5]; (b) homothetic deadlock-free PN system.

Notice that deadlock-freeness of the discrete system in Fig. 1(a) highly depends on its initial marking. In fact, if the initial marking is doubled, i.e., if we consider $\boldsymbol{m}'_0 = (6, 0)$, then the system deadlocks by firing t_2 an amount of 3.

Let us now consider the PN in Fig. 1(b), which exhibits a different behavior. Considered as discrete, it is deadlock-free for $\boldsymbol{m}_0 = (2, 1, 0, 0, 0)$. Moreover, it is deadlock-free for any initial marking proportional to $\boldsymbol{m}_0$, i.e., $\boldsymbol{m}'_0 = k \cdot \boldsymbol{m}_0$, with $k \in \mathbb{N}_{>0}$.

When the PN system is fluidified, i.e., the PN system in Fig. 1(b) is considered as a continuous system, it preserves deadlock-freeness. We will exploit this idea to extract conditions for the preservation of properties.

The present paper explores the kind of features that a discrete net system must exhibit so that a given property is preserved when it is fluidified. It focuses on classical properties as boundedness, B-fairness, deadlock-freeness, liveness and reversibility. The main ideas used here are: (a) the property of homothecy of continuous firing sequences (needed for Lemma 16); (b) the fact that every real number could be approximated by a rational number (used in Lemma 17). Properties preservation is built over these two ideas. Furthermore, homothetic monotonicity of boundedness, B-fairness and deadlock-freeness properties in discrete Petri nets is studied, as well as property preservation for some net system subclasses. Some techniques to improve the fluidization are also considered, where the spurious deadlocks are *removed* with the addition of some *implicit* places.

This work is organized as follows. Section 2 recalls some definitions that will be used in the rest of the paper. Section 3 sets the main results concerning homothetic properties in a discrete net system and its relations with the fluid counterpart. In Section 4, some results about homothetic boundedness and homothetic B-fairness of discrete PN are presented. Section 5 studies whether a discrete PN is homothetically deadlock-free and some techniques for the elimination of spurious deadlocks. Finally, an application example is presented in Section 6, while Section 7 deals with some conclusions.

2. Preliminary concepts and definitions

Some concepts used in the rest of the paper are defined here. In the following, it is assumed that the reader is familiar with discrete Petri nets (see [1,2] for a gentle introduction).

2.1. Petri nets

Definition 1. A PN is a tuple $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post}\rangle$ where $P = \{p_1, p_2, \ldots, p_n\}$ and $T = \{t_1, t_2, \ldots, t_m\}$ are disjoint and finite sets of places and transitions, and $\boldsymbol{Pre}, \boldsymbol{Post}$ are $|P| \times |T|$ sized, natural valued, incidence matrices.

Given a Petri net and a marking, the discrete Petri net system is defined.

Definition 2. A discrete PN system is a tuple $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ where $\mathcal{N}$ is the structure and $\boldsymbol{m}_0 \in \mathbb{N}^{|P|}$ is the initial marking.

In discrete PN systems, a transition t is *enabled* at $\boldsymbol{m}$ if for every $p \in {}^\bullet t$, $m[p] \geq Pre[p, t]$. An enabled transition t can be fired in any amount $\alpha \in \mathbb{N}$ such that $0 < \alpha \leq enab(t, \boldsymbol{m})$, where $enab(t, \boldsymbol{m}) = \min_{p \in {}^\bullet t} \lfloor \frac{m[p]}{Pre[p,t]} \rfloor$.

The main difference between discrete and continuous PNs is in the firing amounts and consequently in the marking, which in *discrete* PNs are restricted to be in the naturals, while in *continuous* PNs are relaxed into the non-negative real numbers [3,4]. Thus, a continuous PN system is understood as a relaxation of a discrete one.

Definition 3. A continuous PN system is a tuple $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ where $\mathcal{N}$ is the structure and $\boldsymbol{m}_0 \in \mathbb{R}^{|P|}_{\geq 0}$ is the initial marking.

In continuous systems, a transition t is *enabled* at $\boldsymbol{m}$ if for every $p \in {}^\bullet t$, $m[p] > 0$. It can be fired in any amount $\alpha \in \mathbb{R}$ such that $0 < \alpha \leq enab(t, \boldsymbol{m})$, where $enab(t, \boldsymbol{m}) = \min_{p \in {}^\bullet t}\{\frac{m[p]}{Pre[p,t]}\}$.

In both discrete and continuous PN systems, the firing of t in a certain amount α leads to a new marking $\boldsymbol{m}'$, and it is denoted as $\boldsymbol{m} \xrightarrow{\alpha t} \boldsymbol{m}'$. It holds $\boldsymbol{m}' = \boldsymbol{m} + \alpha \cdot \boldsymbol{C}[P, t]$, where $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the token flow matrix (incidence matrix if $\mathcal{N}$ is self-loop free) and $\boldsymbol{C}[P, t]$ denotes the column t of the matrix $\boldsymbol{C}$. The state (or fundamental) equation, $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}$, summarizes the way the marking evolves, where $\boldsymbol{\sigma}$ is the firing count vector (also known as the Parikh vector) associated with the fired sequence σ.

Right and left natural annullers of the token flow matrix are called T- and P-semiflows, respectively. When $\exists \mathbf{y} > \mathbf{0}$, $\mathbf{y} \cdot \boldsymbol{C} = \mathbf{0}$, the net is said to be *conservative*, and when $\exists \mathbf{x} > \mathbf{0}$, $\boldsymbol{C} \cdot \mathbf{x} = \mathbf{0}$, the net is said to be *consistent*. A set of places Θ is a *trap* if $\Theta^\bullet \subseteq {}^\bullet\Theta$, while a set of places Σ is a *siphon* if ${}^\bullet\Sigma \subseteq \Sigma^\bullet$. Finally, let us define $\|\mathbf{v}\|$ as the infinite norm (or maximum norm) of the vector $\mathbf{v}$: $\|\mathbf{v}\| = \max\{|v_1|, \ldots, |v_n|\}$. It will be used to compare two markings.

The set of all the reachable markings of a discrete system $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is denoted as the reachability set, $\mathrm{RS}_D(\mathcal{N}, \boldsymbol{m}_0)$.

Definition 4. $\mathrm{RS}_D(\mathcal{N}, \boldsymbol{m}_0) = \{\boldsymbol{m} \mid \exists \sigma = t_{\gamma_1} \cdots t_{\gamma_k}$ such that $\boldsymbol{m}_0 \xrightarrow{t_{\gamma_1}} \boldsymbol{m}_1 \xrightarrow{t_{\gamma_2}} \boldsymbol{m}_2 \cdots \xrightarrow{t_{\gamma_k}} \boldsymbol{m}_k = \boldsymbol{m}\}$.

A *spurious* marking $\boldsymbol{m}$ of a discrete system $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is a solution of the state equation, i.e., $\exists \boldsymbol{\sigma} \in \mathbb{N}^{|T|}$ s.t. $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}$ but is not reachable in the discrete system: $\nexists \sigma$ fireable in $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$. Notice that $\boldsymbol{m}$ could be reachable in the continuous system.

In continuous PN systems, two sets of reachable markings are considered: one denoted as $\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$, that contains all the markings that are reachable with *finite* firing sequences, and the *lim-reachability* set, denoted as lim-$\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$, that contains all the markings that are reachable either with a finite or with an *infinite* firing sequence.

Definition 5. $\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0) = \{\boldsymbol{m} \mid \exists \sigma = \alpha_1 t_{\gamma_1} \ldots \alpha_k t_{\gamma_k}$ s.t. $\boldsymbol{m}_0 \xrightarrow{\alpha_1 t_{\gamma_1}} \boldsymbol{m}_1 \xrightarrow{\alpha_2 t_{\gamma_2}} \boldsymbol{m}_2 \cdots \xrightarrow{\alpha_k t_{\gamma_k}} \boldsymbol{m}_k = \boldsymbol{m}$ where $\alpha_i \in \mathbb{R}_{>0}$, $\forall i \in \{1..k\}\}$.

Definition 6. lim-$\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0) = \{\boldsymbol{m} \mid \exists \sigma = \alpha_1 t_{\gamma_1} \ldots \alpha_i t_{\gamma_i} \ldots$ s.t. $\boldsymbol{m}_0 \xrightarrow{\alpha_1 t_{\gamma_1}} \boldsymbol{m}_1 \xrightarrow{\alpha_2 t_{\gamma_2}} \boldsymbol{m}_2 \cdots \boldsymbol{m}_{i-1} \xrightarrow{\alpha_i t_{\gamma_i}} \boldsymbol{m}_i \cdots$ and $\lim_{i\to\infty} \boldsymbol{m}_i = \boldsymbol{m}$ where $\alpha_i \in \mathbb{R}_{>0}$, $\forall i > 0\}$.

Notice that it holds $\mathrm{RS}_D(\mathcal{N}, \boldsymbol{m}_0) \subseteq \mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0) \subseteq$ lim-$\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$. An immediate consequence of the definition of continuous firings is the following homothetic property [5].

Proposition 7. *If* $\boldsymbol{m} \in \mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$ *then* $\alpha \cdot \boldsymbol{m} \in \mathrm{RS}_C(\mathcal{N}, \alpha \cdot \boldsymbol{m}_0)$, $\forall \alpha \in \mathbb{R}_{>0}$.

2.2. *Petri net properties*

Some interesting properties, often required for real systems, are recalled below. They are well known in discrete systems [1,2], and redefined here for continuous systems. First, two safety properties Π_S: boundedness (B) and B-fairness (BF), which is chosen as a representative of synchronic properties [6,7], and then some other classical behavioral properties Π_L: deadlock-freeness (DF), liveness (L) and reversibility (R) are defined. Their lim-counterparts are also defined: lim-B, lim-BF, lim-DF, lim-L and lim-R.

Definition 8 (*(*lim-*)boundedness (B)*)**.**

- A place p is (lim-)bounded if $\exists b \in \mathbb{R}_{>0}$ such that for all $\boldsymbol{m} \in$ (lim-) $\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$, $m[p] \le b$.
- A system $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is (lim-)bounded if every $p \in P$ is (lim-)bounded.

Definition 9 (*(*lim-*)B-fairness (BF)*)**.**

- Two transitions t, t' are in (lim-)B-fair relation if $\exists b \in \mathbb{R}_{>0}$ such that for all $\boldsymbol{m} \in$ (lim-)$\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$, for every finite (infinite) firing sequence σ fireable from $\boldsymbol{m}$, it holds that if $\boldsymbol{\sigma}[t] = 0$ then $\boldsymbol{\sigma}[t'] \le b$, and if $\boldsymbol{\sigma}[t] = 0$ then $\boldsymbol{\sigma}[t] \le b$.
- A system $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is (lim-)B-fair if every pair of transitions $t, t' \in T$ is in (lim-)B-fair relation.

Definition 10 (*(*lim-*)deadlock-freeness (*DF*)*)**.** A system $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is (lim-)deadlock-free if $\forall \boldsymbol{m} \in$ (lim-)$\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$, $\exists t \in T$ such that t is enabled at $\boldsymbol{m}$.

Definition 11 (*(*lim-*)liveness (L)*)**.** A system $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is (lim-)live if for every transition t and for every marking $\boldsymbol{m} \in$ (lim-)$\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$ there exists $\boldsymbol{m}' \in$ (lim-)$\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m})$ such that t is enabled at $\boldsymbol{m}'$.

Definition 12 (*(*lim-*)reversibility (R)*)**.** A system $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is (lim-) reversible if for any marking $\boldsymbol{m} \in$ (lim-)$\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$ it holds that $\boldsymbol{m}_0 \in$ (lim-)$\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m})$.

Due to the fact that $\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0) \subseteq$ lim-$\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$, a direct implication is that if $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is lim-bounded (resp. lim-B-fair, lim-deadlock-free), then it is also bounded (resp. B-fair, resp. deadlock-free) [5].

A net $\mathcal{N}$ is *structurally* bounded (resp. *structurally* B-fair) if $\forall \boldsymbol{m}_0$, $\langle \mathcal{N}, \boldsymbol{m}_0\rangle$ is bounded (resp. B-fair). A net $\mathcal{N}$ is *structurally* deadlock-free (resp. *structurally* live; resp. *structurally* reversible) if $\exists \boldsymbol{m}_0$ s.t. $\langle \mathcal{N}, \boldsymbol{m}_0\rangle$ is deadlock-free (resp. live; resp. reversible).

Finally, *marking monotonicity* and *marking homothetic monotonicity* are defined below for Π (where Π represents one of the defined properties: B, BF, DF, L or R). In this work, we will limit ourselves to the use of those concepts with respect to the marking. However, *monotonicity* with respect to other properties of interest are considered in other works, such as performance monotonicity w.r.t. the firing rates in timed PN systems [8].

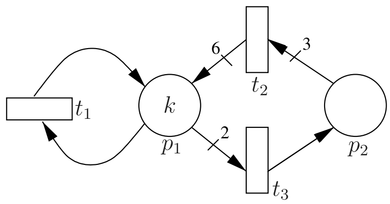

Fig. 2. Non-monotonic deadlock-free PN system.

Definition 13 (*Monotonicity*)**.** Given a system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$, a property Π is *monotonic* w.r.t. $\boldsymbol{m}_0$ if:
Π holds in $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D \Longrightarrow \Pi$ holds in $\langle \mathcal{N}, \boldsymbol{m}'_0 \rangle_D$ for every $\boldsymbol{m}'_0 \geq \boldsymbol{m}_0$.

For example, in the PN system in Fig. 2, DF is not monotonic for $k = 1$, because it is deadlock-free for $k = 1$, $k = 3$, and $k \geq 5$; but it deadlocks for $k = 2$ and for $k = 4$. Moreover, it is live for $k \geq 6$.

Definition 14 (*Homothetic Monotonicity*)**.** Given a system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$, a property Π is *homothetically monotonic* (for short, *homothetic*) w.r.t. $\boldsymbol{m}_0$ if:
Π holds in $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D \Longrightarrow \Pi$ holds in $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0 \rangle_D$, $\forall k \in \mathbb{N}_{>0}$.

Homothetic monotonicity of DF can be illustrated with the example in Fig. 1(b). The discrete net system is DF for $\boldsymbol{m}_0 = (2, 1, 0, 0, 0)$, and for any proportional initial marking $k \cdot \boldsymbol{m}_0$, i.e., it is homothetically DF. Nevertheless, the system is not *monotonically* DF for $\boldsymbol{m}_0$, for example, for $\boldsymbol{m}'_0 = (2, 2, 0, 0, 0)$ it deadlocks, where $\boldsymbol{m}'_0 \geq \boldsymbol{m}_0$.

Notice that monotonicity is more restrictive than homothetic monotonicity, i.e., if Π is monotonic then Π is also homothetically monotonic. Some classical results on studying monotonicity of certain properties such as liveness are the rank theorems [9], which give necessary or sufficient conditions from the structure of the net (with polynomial complexity), or the siphon–trap property [10], which gives a necessary and sufficient condition for the behavioral property (with higher complexity).

3. Homothetic monotonicity and property preservation by fluidization

The aim of this section is to set certain conditions that a discrete PN system has to fulfill to preserve a certain property after being fluidified to a continuous PN system. It will be proved that, given a property Π which exhibits homothetic monotonicity in $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$, Π is preserved by fluidization (i.e. in $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$). We will focus on the well-known properties considered in Section 2.2. First, two technical results (Lemmas 16 and 17) about reachability are presented.

3.1. Reachability

Let us introduce an additional reachability set to be used in this work, the *rational reachability set* ($\mathrm{RS}_Q(\mathcal{N}, \boldsymbol{m}_0)$): the set of markings that can be reached from $\boldsymbol{m}_0$ considering only firings in the set of rational numbers ($\mathbb{Q}$). We will denote as $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_Q$ the net system in which only rational amounts are fired by the transitions.

Definition 15. $\mathrm{RS}_Q(\mathcal{N}, \boldsymbol{m}_0) = \{\boldsymbol{m} \mid \exists \sigma = \alpha_1 t_{\gamma_1} \cdots \alpha_k t_{\gamma_k}$ s.t. $\boldsymbol{m}_0 \xrightarrow{\alpha_1 t_{\gamma_1}} \boldsymbol{m}_1 \cdots \xrightarrow{\alpha_k t_{\gamma_k}} \boldsymbol{m}_k = \boldsymbol{m}$ where $\alpha_i \in \mathbb{Q}_{>0}, \forall i \in \{1..k\}\}$.

The following lemma states that for any marking $\boldsymbol{m}$ reachable in $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_Q$, there exists a $k \in \mathbb{N}$ such that a scaled marking $k \cdot \boldsymbol{m}$ is reachable in $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0 \rangle_D$.

Lemma 16. *Given a* PN *structure* $\mathcal{N}$ *and an initial marking* $\boldsymbol{m}_0 \in \mathbb{N}^{|P|}$, $\boldsymbol{m} \in \mathrm{RS}_Q(\mathcal{N}, \boldsymbol{m}_0) \Longrightarrow \exists k \in \mathbb{N}_{>0}$ *such that* $k \cdot \boldsymbol{m} \in \mathrm{RS}_D(\mathcal{N}, k \cdot \boldsymbol{m}_0)$.

Proof. Let us suppose $\boldsymbol{m} \in \mathrm{RS}_Q(\mathcal{N}, \boldsymbol{m}_0)$, i.e., $\boldsymbol{m}_0 \xrightarrow{\sigma'} \boldsymbol{m}$, where $\sigma' = \alpha'_1 t_{\gamma_1} \cdots \alpha'_n t_{\gamma_n}$, and $\alpha'_i \in \mathbb{Q}, \forall i \in \{1 \cdots n\}$. Because each α'_i is a rational amount, it can be considered as its irreducible fraction: $\alpha'_i = \frac{n_i}{d_i}$.

We can multiply the rational sequence σ' by the l.c.m. (least common multiple) of the denominators of the irreducible fractions, to obtain a sequence σ'' in the naturals: $\sigma'' = k \cdot \sigma'$, where $k = \text{l.c.m.}(d_i \mid \frac{n_i}{d_i} = \alpha'_i, \forall \alpha'_i \in \sigma')$.

For every firing amount α''_i which appears in the sequence σ'', it holds that $\alpha''_i \in \mathbb{N}$. Because of the properties of the continuous PN (see Proposition 7), the initial marking ($\boldsymbol{m}_0$), the firing sequence (σ') and the resulting marking ($\boldsymbol{m}$) can be scaled by k in the continuous PN: $k \cdot \boldsymbol{m}_0 \xrightarrow{k \cdot \sigma'} k \cdot \boldsymbol{m}$. Because it is a natural sequence fireable in the continuous PN, $\sigma'' = k \cdot \sigma'$ is also fireable from $k \cdot \boldsymbol{m}_0$ in the discrete system: $k \cdot \boldsymbol{m}_0 \xrightarrow{\sigma''} k \cdot \boldsymbol{m}$. ■

Now it is proved that, for any marking $\boldsymbol{m}$ reachable with a *real* firing sequence, another marking $\boldsymbol{m}'$ exists that is reachable with *rational* firings, such that it is as close to $\boldsymbol{m}$ as desired, and the set of empty places coincide.

Lemma 17. *For every $\sigma = \alpha_1 t_{\gamma_1} \dots \alpha_i t_{\gamma_i}$, with $\alpha_j \in \mathbb{R}_{>0}, j \in 1..i$, s.t. $\boldsymbol{m}_0 \xrightarrow{\sigma} \boldsymbol{m}$, with $\boldsymbol{m}_0 \in \mathbb{N}^{|P|}$, $\boldsymbol{m} \in RS_C(\mathcal{N}, \boldsymbol{m}_0)$, and every $\varepsilon, \varepsilon' > 0$, there exists $\sigma' = \alpha'_1 t_{\gamma_1} \dots \alpha'_i t_{\gamma_i}$ s.t. $\boldsymbol{m}_0 \xrightarrow{\sigma'} \boldsymbol{m}'$, with $\boldsymbol{m}' \in RS_Q(\mathcal{N}, \boldsymbol{m}_0)$ such that:*

- $\|\boldsymbol{m}' - \boldsymbol{m}\| < \varepsilon$
- $m'[p] = 0 \Leftrightarrow m[p] = 0$
- $\forall j \le i, |\alpha'_j - \alpha_j| < \varepsilon'$.

Proof. Given that $\sigma = \alpha_1 t_{\gamma_1} \dots \alpha_i t_{\gamma_i}$, with $\alpha_j \in \mathbb{R}_{>0}, \forall j \in \{1..i\}$ such that $\boldsymbol{m}_0 \xrightarrow{\sigma} \boldsymbol{m}$, for any $\varepsilon, \varepsilon' > 0$, we will build the firing sequence $\sigma' = \alpha'_1 t_{\gamma_1} \dots \alpha'_i t_{\gamma_i}$, $\alpha'_j \in \mathbb{Q}, \forall j \in \{1..i\}$, such that $\boldsymbol{m}_0 \xrightarrow{\sigma'} \boldsymbol{m}'$, where $\|\boldsymbol{m}' - \boldsymbol{m}\| < \varepsilon$, $|\alpha'_j - \alpha_j| < \varepsilon'$ and $(m'[p] = 0 \Leftrightarrow m[p] = 0)$. It will be proved by induction on the length of the sequence σ: $|\sigma| = i$.

- Base case ($|\sigma| = 1$).
  Let $\sigma = \alpha_1 t_{\gamma_1}$. Then, $\alpha'_1 \in \mathbb{Q}$ has to be chosen. The firing of $\alpha_1 t_{\gamma_1}$ yields $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C}[P, t_{\gamma_1}]\alpha_1$, and the firing of a given $\alpha'_1 t_{\gamma_1}$ yields $\boldsymbol{m}' = \boldsymbol{m}_0 + \boldsymbol{C}[P, t_{\gamma_1}]\alpha'_1$. Subtracting both equations and considering its norm, we obtain $\|\boldsymbol{m}' - \boldsymbol{m}\| = \|\boldsymbol{C}[P, t_{\gamma_1}]\ (\alpha'_1 - \alpha_1)\|$. Since all the elements in $\boldsymbol{C}$ are finite numbers, a rational $\alpha'_1 \in \mathbb{Q}$ close enough to α_1 can be chosen to satisfy $\|\boldsymbol{m}' - \boldsymbol{m}\| < \varepsilon$ and $|\alpha'_1 - \alpha_1| < \varepsilon'$. Moreover, since $\boldsymbol{m}_0 \in \mathbb{N}^{|P|}$, if the firing of α_1 emptied some places, then $\alpha_1 \in \mathbb{Q}$ and $\alpha'_1 = \alpha_1$ can be chosen. Otherwise (if no place has been emptied), then $\alpha'_1 \in \mathbb{Q}$ as close as desired to α_1 can be chosen that does not empty places.
- Inductive hypothesis ($|\sigma| = i$)
  Given $\sigma = \alpha_1 t_{\gamma_1} \dots \alpha_i t_{\gamma_i}$, such that $\boldsymbol{m}_0 \xrightarrow{\sigma} \boldsymbol{m}_i$, there exists $\sigma' = \alpha'_1 t_{\gamma_1} \cdots \alpha'_i t_{\gamma_i}$, such that $\alpha'_j \in \mathbb{Q}, \forall j \in \{1..i\}$ and $\boldsymbol{m}_0 \xrightarrow{\sigma'} \boldsymbol{m}'_i$, where $\|\boldsymbol{m}'_i - \boldsymbol{m}_i\| < \varepsilon$, $|\alpha'_i - \alpha_i| < \varepsilon'$ and $(m'_i[p] = 0 \Leftrightarrow m[p] = 0)$.
- Inductive step ($|\sigma| = i + 1$)
  Let us consider the $i + 1$ firing. We can distinguish two cases:
  (a) The firing of $\alpha_{i+1} t_{\gamma_{i+1}}$ does not empty places in $^\bullet t_{\gamma_{i+1}}$. Then, it holds that $\boldsymbol{m} = \boldsymbol{m}_i + \boldsymbol{C}[P, t_{\gamma_{i+1}}]\alpha_{i+1}$, and $\boldsymbol{m}' = \boldsymbol{m}'_i + \boldsymbol{C}[P, t_{\gamma_{i+1}}]\alpha'_{i+1}$. Again, subtracting both equations and considering its norm, we obtain $\|\boldsymbol{m}' - \boldsymbol{m}\| = \|(\boldsymbol{m}'_i - \boldsymbol{m}_i) + \boldsymbol{C}[P, t_{\gamma_{i+1}}](\alpha'_{i+1} - \alpha_{i+1})\|$.
  We have to force that $\|\boldsymbol{m}' - \boldsymbol{m}\| < \varepsilon$ and $|\alpha'_{i+1} - \alpha_{i+1}| < \varepsilon'$. Given that $\boldsymbol{m}_i$ and $\boldsymbol{m}'_i$ fulfill the inductive hypothesis, the quantity $\|\boldsymbol{m}'_i - \boldsymbol{m}_i\|$ can be as small as desired. Moreover, since the elements of the matrix $\boldsymbol{C}$ are finite numbers, a rational α'_{i+1} close to α_{i+1} can be chosen such that $\|\boldsymbol{C}[P, t_{\gamma_{i+1}}](\alpha'_{i+1} - \alpha_{i+1})\|$ is as small as desired and no places in $^\bullet t_{\gamma_{i+1}}$ are emptied.
  (b) The firing of $\alpha_{i+1} t_{\gamma_{i+1}}$ empties places in $^\bullet t_{\gamma_{i+1}}$. Then, $\alpha'_{i+1} = enab(t_{\gamma_{i+1}}, \boldsymbol{m}'_i)$ is chosen, in order to empty the same input places. The amount α'_{i+1} is in $\mathbb{Q}$, because $\boldsymbol{m}'_i$ (and hence the enabling degree) is rational. Since $\boldsymbol{m}_i$ and $\boldsymbol{m}'_i$ fulfill the inductive hypothesis, they can be as close as desired. Thus, the firing of α'_{i+1} empties the same places than α_{i+1}, and $\boldsymbol{m}_{i+1}$ and $\boldsymbol{m}'_{i+1}$ can be as close as desired, as well as α_{i+1} and α'_{i+1}. ■

Lemmas 16 and 17 will help to prove some properties related to the preservation of B, BF; and DF, L, and R.

3.2. Synchronic properties: boundedness and B-fairness

Some properties are included in the general concept of synchronic properties [6], which are considered here. For a continuous PN system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ in which every transition can be fired (i.e., there are no empty siphons at $\boldsymbol{m}_0$), *behavioral* and *structural synchronic relations* coincide, as noticed in [11]. Moreover, here it is proved (Propositions 18 and 19) that in any PN system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$, a synchronic property Π_S (boundedness or B-fairness) is equivalent in the homothetic discrete system $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0 \rangle_D$ and in the continuous PN system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$. The corresponding properties in the limit (i.e., with infinite sequences) are considered in Section 4.

Proposition 18. *$\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$ is homothetically bounded $\Longleftrightarrow$ $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is bounded.*

Proof. ($\Longrightarrow$) Let us suppose that the $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is unbounded, i.e., $\forall b \in \mathbb{R}_{>0} \exists p \in P \exists \boldsymbol{m} \in \mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$ s.t. $m[p] > b$. If $\boldsymbol{m}$ is not in $RS_Q(\mathcal{N}, \boldsymbol{m}_0)$, but $m[p] > b$, because of Lemma 17, $\forall \varepsilon > 0$, $\exists \boldsymbol{m}'$ s.t. $\|\boldsymbol{m}' - \boldsymbol{m}\| < \varepsilon$, so we can find another $\boldsymbol{m}' \in RS_C(\mathcal{N}, \boldsymbol{m}_0)$ as near to $\boldsymbol{m}$ as desired, such that also $m'[p] > b$. And because of Lemma 16, if $\boldsymbol{m}' \in RS_Q(\mathcal{N}, \boldsymbol{m}_0)$, then $\exists k \in \mathbb{N}_{>0}$ s.t. $k \cdot \boldsymbol{m}' \in \mathrm{RS}_D(\mathcal{N}, k \cdot \boldsymbol{m}_0)$. Hence, in the discrete PN, $\forall b \in \mathbb{R}_{>0}$, $\exists k \cdot \boldsymbol{m} \in \mathrm{RS}_D(\mathcal{N}, k \cdot \boldsymbol{m}_0)$ s.t. $k \cdot m[p] > b$. Consequently, $\exists k \in \mathbb{N}_{>0}$ s.t. the discrete system $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0 \rangle_D$ is unbounded.

($\Longleftarrow$) Let us suppose $\exists k \in \mathbb{N}_{>0}$ s.t. the discrete system $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0 \rangle_D$ is unbounded. It means $\forall b \in \mathbb{R}^+$, $\exists p \in P\ \exists \boldsymbol{m} \in \mathrm{RS}_D(\mathcal{N}, k \cdot \boldsymbol{m}_0)$ s.t. $b < m[p]$. If $\boldsymbol{m} \in \mathrm{RS}_D(\mathcal{N}, k \cdot \boldsymbol{m}_0)$, then also $\boldsymbol{m} \in \mathrm{RS}_C(\mathcal{N}, k \cdot \boldsymbol{m}_0)$. Because of Proposition 7, for each marking $\boldsymbol{m} \in \mathrm{RS}_C(\mathcal{N}, k \cdot \boldsymbol{m}_0)$, the marking $\boldsymbol{m}' = \frac{\boldsymbol{m}}{k}$ is reachable in $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$. For every $c \in \mathbb{R}_{>0}$ s.t. it holds, also for $c \cdot k$ it holds (since it holds for every real): $\exists p \in P\ \exists \boldsymbol{m} \in \mathrm{RS}_D(\mathcal{N}, k \cdot \boldsymbol{m}_0)$ s.t. $m[p] > c \cdot k$. Consequently, for every real c, it holds that $\exists p \in P$ s.t. $m[p] > c \cdot k$. And it implies $m'[p] = \frac{m[p]}{k} > c$, where $m'[p] \in \mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$. Hence, $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is unbounded. ■

Using similar arguments, the following result considers B-fairness.

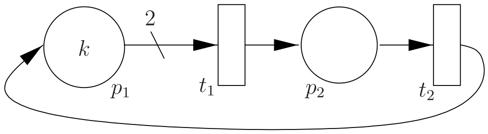

Fig. 3. PN system which deadlocks as discrete for any k. It is deadlock-free as continuous, but it lim-deadlocks ($\boldsymbol{m}_d = (0, 0)$ is lim-reachable).

Proposition 19. *$\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is homothetically B-fair $\Longleftrightarrow$ $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is B-fair.*

Proof. ($\Longrightarrow$) Let us suppose that $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is not B-fair; then $\exists \boldsymbol{m} \in \mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$, $\exists t, t'$ s.t. $\forall b \in \mathbb{R}_{>0}$, $\exists \sigma$ fireable from $\boldsymbol{m}$ s.t. $\boldsymbol{\sigma}[t] = 0$ and $\boldsymbol{\sigma}[t'] > b$. By Lemma 17, if $\boldsymbol{m}$ (or the firing amounts in σ) is not in $\mathbb{Q}^{|P|}$, then there exists another $\boldsymbol{m}'$ (or other σ') in $\mathbb{Q}^{|P|}$ with the properties shown in the lemma. Then, by Lemma 16, $\exists k \in \mathbb{N}_{>0}$ s.t. $k \cdot \boldsymbol{m}' \in \mathrm{RS}_D(\mathcal{N}, k \cdot \boldsymbol{m}_0)$. And, by applying again Lemma 16, from $k \cdot \boldsymbol{m}'$ it is also possible to fire $\sigma'' = k \cdot \sigma'$ such that $\boldsymbol{\sigma}''[t] = 0$ and it makes $\boldsymbol{\sigma}''[t'] > k \cdot b \geq b$, $\forall b \in \mathbb{R}_{>0}$. Consequently, $\exists k \in \mathbb{N}_{>0}$ s.t. the discrete system $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0\rangle_D$ is not B-fair.

($\Longleftarrow$) Let us suppose $\exists k \in \mathbb{N}_{>0}$ s.t. the discrete system $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0\rangle_D$ is not B-fair. It means $\exists \boldsymbol{m} \in \mathrm{RS}_D(\mathcal{N}, k \cdot \boldsymbol{m}_0)$, $\exists t, t'$ s.t. $\forall b \in \mathbb{R}_{>0}$, $\exists \sigma$ fireable from $\boldsymbol{m}$ s.t. $\boldsymbol{\sigma}[t] = 0$ and $\boldsymbol{\sigma}[t'] > b$. Due to the fact that $\boldsymbol{m} \in \mathrm{RS}_D(\mathcal{N}, k \cdot \boldsymbol{m}_0)$, then a marking $\boldsymbol{m}' = \frac{1}{k}\boldsymbol{m}$ is reachable in $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$, from which $\sigma' = \frac{\sigma}{k}$ can be fired. It holds that $\boldsymbol{\sigma}'[t] = \frac{\boldsymbol{\sigma}[t]}{k} = 0$ and $\boldsymbol{\sigma}'[t'] = \frac{\boldsymbol{\sigma}[t']}{k} > \frac{b}{k}$. This reasoning can be done for every $b \in \mathbb{R}_{>0}$. Hence, $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is not B-fair either. ■

3.3. Deadlock-freeness

Some results about the preservation of a homothetic property Π_L (deadlock-freeness, liveness and reversibility) when the system is fluidified are presented in the sequel. The results and their proofs are analogous for deadlock-freeness, liveness and reversibility. For a didactic purpose, in this section the results are explained for DF, and in the following section they are extended to liveness and reversibility.

As previously defined (Section 2.2), DF in continuous PNs only considers the markings that are reachable with *finite* firing sequences (reachability), while lim-DF considers also *infinite* firing sequences (lim-reachability). Both concepts will be considered here.

A technical result is presented first. It sets that, given a reachable deadlock marking $\boldsymbol{m}_d$ (the subscript $_d$ denotes *deadlock*) in $RS_C(\mathcal{N}, \boldsymbol{m}_0)$, either its firing sequence is in $\mathbb{Q}$ (so it is in $RS_Q(\mathcal{N}, \boldsymbol{m}_0)$) or it is in $\mathbb{R} \setminus \mathbb{Q}$ and then there exists another "close" deadlock that is in $\mathbb{Q}$ (also in $RS_Q(\mathcal{N}, \boldsymbol{m}_0)$). In summary, $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is DF if and only if $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_Q$ is DF.

Lemma 20. *$\boldsymbol{m}_d \in RS_C(\mathcal{N}, \boldsymbol{m}_0)$ is a deadlock $\Longleftrightarrow$ $\forall \varepsilon > 0$, $\exists \boldsymbol{m}'_d \in RS_Q(\mathcal{N}, \boldsymbol{m}_0)$ s.t. $\|\boldsymbol{m}'_d - \boldsymbol{m}_d\| < \varepsilon$ and $\boldsymbol{m}'_d$ is a deadlock.*

Proof. ($\Longrightarrow$) Assume that $\boldsymbol{m}_d \in RS_C(\mathcal{N}, \boldsymbol{m}_0) \setminus RS_Q(\mathcal{N}, \boldsymbol{m}_0)$. Because of Lemma 17, $\forall \varepsilon > 0$ another $\boldsymbol{m}'_d \in RS_Q(\mathcal{N}, \boldsymbol{m}_0)$ exists such that $\|\boldsymbol{m}'_d - \boldsymbol{m}_d\| < \varepsilon$, and $\forall p$ s.t. $m_d[p] = 0$, also $m'_d[p] = 0$. Since $\forall t \in T$, t is not enabled in $\boldsymbol{m}_d$, then also $\forall t \in T$, t is not enabled in $\boldsymbol{m}'_d$. Hence, $\exists \boldsymbol{m}'_d \in RS_Q(\mathcal{N}, \boldsymbol{m}_0)$ that is a deadlock in the continuous system.

($\Longleftarrow$) It trivially holds: if $\exists \boldsymbol{m}'_d \in RS_Q(\mathcal{N}, \boldsymbol{m}_0)$, then $\boldsymbol{m}'_d$ is also reachable in $RS_C(\mathcal{N}, \boldsymbol{m}_0)$ and it is also a deadlock. ■

Let us now prove that, if a discrete PN is homothetically DF, it will also be DF as continuous.

Proposition 21. *$\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is homothetically* DF $\Longrightarrow$ *$\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is* DF.

Proof. Let us suppose $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ deadlocks. It means $\exists \boldsymbol{m} \in RS_C(\mathcal{N}, \boldsymbol{m}_0)$ that is a deadlock. Because of Lemma 20, if $\boldsymbol{m}$ is a deadlock, then there exists $\boldsymbol{m}' \in RS_Q(\mathcal{N}, \boldsymbol{m}_0)$ that is a deadlock. Because of Lemma 16, $\exists k \in \mathbb{N}$ s.t. $\boldsymbol{m}'' = k \cdot \boldsymbol{m}'$, where $\boldsymbol{m}'' \in RS_D(\mathcal{N}, k \cdot \boldsymbol{m}_0)$. Since $\forall t \in T$, $\exists p \in {}^\bullet t$, $m'[p] = 0$, then also $\forall t \in T$, $\exists p \in {}^\bullet t$, $k \cdot \boldsymbol{m}''[p] = 0$, and consequently $\boldsymbol{m}''$ it is also deadlock: $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0\rangle_D$ deadlocks. ■

Proposition 21 can be illustrated by the example in Fig. 1(b). However, in general, $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is DF $\not\Longrightarrow \langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is homothetically DF, as the PN in Fig. 3 shows. The net system is DF when considered as continuous (illustrated in [5]); but $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0\rangle_D$ deadlocks for every k when considered as discrete.

The previous results deal with DF. What happens if lim-DF is considered?

A continuous system which is lim-DF could be homothetically DF as discrete. When this is not the case, a minimum value of k can be considered for homothetic monotonicity. We will denote that a property Π_L is *homothetic from n* if $\exists n \in \mathbb{N}$, s.t. $\forall k \geq n$, with $k \in \mathbb{N}$, Π_L holds in $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0\rangle_D$. Now, the implication can be formulated (in "some sense" it is the inverse of Proposition 21).

Proposition 22. *$\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is* lim $-DF$ $\Longrightarrow$ *$\exists n \in \mathbb{N}$ s.t. $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is homothetically* DF *from n.*

Proof. Let us suppose that $\forall n \in \mathbb{N}\ \exists k \geq n$, $k \in \mathbb{N}$, such that the discrete system $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0\rangle_D$ deadlocks. It means there exists an infinite ordered set $A = \{a_1, a_2, a_3 \ldots\}$, such that $\forall a_i \in A$, $a_i < a_{i+1}$ and $\langle \mathcal{N}, a_i \cdot \boldsymbol{m}_0\rangle_D$ deadlocks.

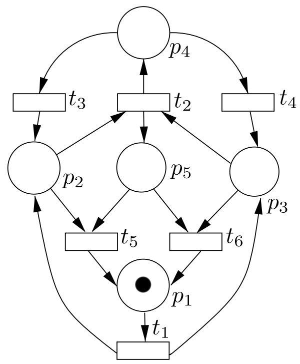

Fig. 4. PN system which is live as discrete for any $k \cdot \boldsymbol{m}_0$, with $k \in \mathbb{N}_{>0}$ (homotheticallydeadlock-free). It is deadlock-free as continuous, but not lim-deadlock-free ($\boldsymbol{m}_d = (0, 0, 0, 0, 2)$ is lim-reachable).

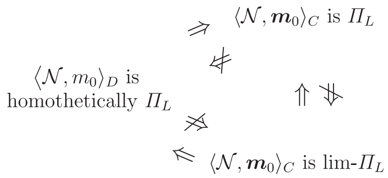

Fig. 5. Relations w.r.t. a property $\Pi_L \in \{\mathrm{DF}, L, R\}$.

For each a_i for which it deadlocks, $\exists \boldsymbol{m}_d \in \mathrm{RS}_D(\mathcal{N}, a_i \cdot \boldsymbol{m}_0)$ s.t. $\boldsymbol{m}_d$ is a deadlock. It holds that $\forall t \in T, \exists p \in {}^\bullet t, m_d[p] < Pre[p, t]$. Because of the definitions of continuous firings (Proposition 7), marking $\frac{\boldsymbol{m}_d}{a_i}$ is reachable in $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$.

Given that a_i tends to infinity, making $a_i \to \infty$, then $\frac{m_d}{a_i}[p] \to 0$, and it will reach a deadlock in the limit. Consequently, the continuous $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is not lim-deadlock-free. ■

An interesting topic is to compute which is the minimal n s.t. $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is homothetically DF from n. This value depends on the net structure and on the initial marking of the system. Initially, it could seem that it is similar to a minimum initial marking problem in which DF is required. However, in the mentioned problem there are two important differences: (1) there are $|P|$ degrees of freedom, one per place, while in our case $\boldsymbol{m}_0$ is fixed and the only parameter is n; (2) monotonicity w.r.t. DF of the obtained $\boldsymbol{m}_0$ is not guaranteed, which is essential here. Notice that the second difference does not appear in net subclasses in which DF is monotonic w.r.t. $\boldsymbol{m}_0$ (for example, equal conflict nets). This is an interesting problem to be considered as future work.

The reverse proposition is not true (the stated Proposition 21 does not hold for lim-DF): $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is homothetically DF; $\not\Longrightarrow \langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is lim-DF. Considered as discrete, the net system in Fig. 4 is homothetically DF for $\boldsymbol{m}_0 = (1, 0, 0, 0, 0)$. Even more, it can easily be proved that the discrete system is fully monotonic DF for $\boldsymbol{m}_0$ because every siphon contains a marked trap (see, for example, [10]). However, when the net system is considered as continuous, the infinite firing sequence $\sigma = t_1\, t_2\, \frac{1}{2}t_3\, \frac{1}{2}t_4\, \frac{1}{2}t_2\, \frac{1}{4}t_3\, \frac{1}{4}t_4\, \frac{1}{4}t_2\, \frac{1}{8}t_3\, \frac{1}{8}t_4 \ldots$ can be fired, leading to the deadlock marking $\boldsymbol{m}_d = (0, 0, 0, 0, 2)$: the continuous system reaches a deadlock in the limit (as already noticed in [5]).

Observe that $\boldsymbol{m}_d$ empties the trap and siphon $\{p_1, p_2, p_3, p_4\}$. Emptying a trap in a continuous net system can only be done considering an *infinitely* long firing sequence. A trap cannot be emptied in a discrete system, and thus $\boldsymbol{m}_d$ is a *spurious* solution in the discrete net system; it is a deadlock, and hence it is a *killing spurious* solution.

3.4. Liveness and reversibility

The lemmas and properties presented here are analogous to the ones presented for DF (Section 3.3); even the proofs are technically analogous. Fig. 5 summarizes the relations among a certain property Π_L (DF, L or R), when considered in the discrete system (homothetically Π_L), in the continuous system (Π_L) and in the limit (lim-Π_L).

Analogously to Lemma 20:

Lemma 23. *$\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is $\Pi_L \iff \langle \mathcal{N}, \boldsymbol{m}_0\rangle_Q$ is Π_L.*

The proof of this lemma when $\Pi_L = L$ is similar to the one of Lemma 20, but instead of considering a marking $\boldsymbol{m}_d$ which is a deadlock, a marking $\boldsymbol{m}_l$ from which $\exists t$ s.t. t cannot be enabled from $\boldsymbol{m}_l$ must be considered. In the lemma when $\Pi_L = R$, a marking $\boldsymbol{m}_r$ from which the initial marking is not reachable must be considered.

Given these lemmas, a general result about the preservation of a homothetic property by fluidization can be formulated (similar to Proposition 21).

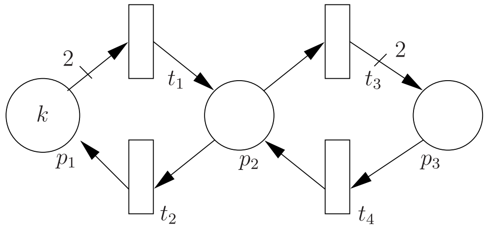

Fig. 6. If $k \geq 2$, non-reversible, non-live discrete PN system. It is reversible and live as continuous.

Theorem 24. *$\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is homothetically $\Pi_L \Longrightarrow \langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is Π_L.*

The proofs of the theorem for liveness ($\Pi_L = L$) and for reversibility ($\Pi_L = R$) would be similar to the proof of Proposition 21. The lemmas obtained from Lemma 23 for $\Pi_L = L$ and $\Pi_L = R$ would also be used in these proofs.

As an illustrative example, consider the Petri net example in Fig. 1(b). It is homothetically live and homothetically reversible. Thus it preserves these properties when fluidified.

If the opposite implication is considered, again, even in the case of considering not *every k* but a *big enough k*, the implication is not true.

$\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is $\Pi_L \not\Longrightarrow \exists n \in \mathbb{N}$ such that $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is homothetically Π_L from n. Let us consider the example in Fig. 6. As continuous, it is live and reversible for $\boldsymbol{m}_0 = (1, 0, 0)$: the marking can decrease by firing t_1 and t_2, but it can also be increased in the same amount by firing t_3 and t_4. However, if the net system is discrete, it is neither live nor reversible for $\boldsymbol{m}_0 = (1, 0, 0)$, for any proportional initial marking $k \cdot \boldsymbol{m}_0$. This is true because for any value of k, transitions t_1 and t_2 can be fired until $m[p_1] < 2$. Then, no transition is enabled; it is deadlocked and the system is neither live nor reversible.

Let us now consider the properties in the limit, i.e., lim-Π_L. In this case, it holds (similar to Proposition 22).

Theorem 25. *$\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is $\Pi_L \Longrightarrow \exists n \in \mathbb{N}$ s.t. $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is homothetically Π_L from n.*

The proofs of Theorem 25 for $\Pi_L = L$ and for $\Pi_L = R$ would be analogous to that of Proposition 22, in which also Lemma 16 would be used.

Analogously to lim-DF, the reverse is not true:

$\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is homothetically $\Pi_L \not\Longrightarrow \langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is lim-Π_L.

Again, the PN system in Fig. 4 is live and reversible for $\boldsymbol{m}_0 = (1, 0, 0, 0, 0)$ if it is considered as discrete. However, when considered continuous, the infinite firing sequence $\sigma = t_1\ t_2\ \frac{1}{2}t_3\ \frac{1}{2}t_4\ \frac{1}{2}t_2\ \frac{1}{4}t_3\ \frac{1}{4}t_4\ \frac{1}{4}t_2\ \frac{1}{8}t_3\ \frac{1}{8}t_4 \ldots$ would reach the deadlock marking $\boldsymbol{m}_d = (0, 0, 0, 0, 2)$ in the limit, so the system is not lim-live and not lim-reversible from $\boldsymbol{m}_d$.

4. Homothetic boundedness and homothetic B-fairness in discrete PN systems

The aim of this brief section is to propose a characterization (necessary and sufficient condition) of homothetic boundedness and homothetic B-fairness for discrete PN systems.

By definition, if a net $\mathcal{N}$ is structurally bounded (structurally B-fair), then $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is homothetically bounded (homothetically B-fair).

However, the opposite is not true: $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is homothetically bounded $\not\Rightarrow$ $\mathcal{N}$ is structurally bounded. For instance, if there is an empty siphon in $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$, some transitions can never be fired. Thus, the system can be bounded and homothetically bounded for that $\boldsymbol{m}_0$, but $\mathcal{N}$ can be unbounded for a different initial marking.

Furthermore, if $\nexists$ empty siphon in $\boldsymbol{m}_0$ (a very reasonable condition for real systems), then every transition can be fired sooner or later from $\boldsymbol{m}_0$ or from a given $k \cdot \boldsymbol{m}_0$, and then homothetic boundedness implies structural boundedness. It is analogous when B-fairness is considered.

Theorem 26. *Let $\mathcal{N}$ be a net system and $\boldsymbol{m}_0$ be a marking in which every siphon of the net is marked. The following statements are equivalent:*

(1) *$\mathcal{N}$ is structurally bounded*
(2) *$\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is homothetically bounded*
(3) *$\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is bounded*
(4) *$\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is lim-bounded*

Proof. (2) $\Leftrightarrow$ (3). Because of Proposition 18.

(1) $\Leftrightarrow$ (3). Proved in [5].

(3) $\Leftrightarrow$ (4). Stated in [5]. ∎

Theorem 27. *Let $\mathcal{N}$ be a net system and $\boldsymbol{m}_0$ be a marking in which every siphon of the net is marked. The following statements are equivalent:*

(1) *$\mathcal{N}$ is structurally B-fair*
(2) *$\langle\mathcal{N}, \boldsymbol{m}_0\rangle_D$ is homothetically B-fair*
(3) *$\langle\mathcal{N}, \boldsymbol{m}_0\rangle_C$ is B-fair*
(4) *$\langle\mathcal{N}, \boldsymbol{m}_0\rangle_C$ is lim-B-fair*

Proof. (2) ⇔ (3). Because of Proposition 19.

(1) ⇔ (3). Only (3) ⇒ (1) needs to be proven. If every siphon is initially marked, a strictly positive marking can be reached, then every T-semiflow can be fired. From the fireability of the minimal T-semiflows, it is deduced that behavioral and structural relations coincide [5]. Hence B-fairness in $\langle\mathcal{N}, \boldsymbol{m}_0\rangle_C$ is equivalent to structural B-fairness in $\mathcal{N}$.

(3) ⇔ (4). Only (3) ⇒ (4) needs to be proven. This result is analogous to the equivalence between boundedness and lim-boundedness. Every pair $t, t' \in T$ is in B-fair relation, i.e., for every $\boldsymbol{m} \in RS(\mathcal{N}, \boldsymbol{m}_0)$, for every finite firing sequence $\sigma_j = \alpha_1 t_{\gamma_1} \ldots \alpha_j t_{\gamma_j}$ of length j which is fireable from $\boldsymbol{m}$, it holds that $\exists b \in \mathbb{R}_{>0}$ s.t. if $\boldsymbol{\sigma}_j(t) = 0$ then $\boldsymbol{\sigma}_j(t') \le b$ and if $\boldsymbol{\sigma}_j(t') = 0$ then $\boldsymbol{\sigma}[t]_j \le b$. Then, considering an infinitely long sequence, $\sigma = \alpha_1 t_{\gamma_1} \ldots \alpha_j t_{\gamma_j} \alpha_{j+1} t_{\gamma_{j+1}} \ldots$, it holds that for every finite subsequence of σ as long as desired it is also true. Hence, if $\boldsymbol{\sigma}[t] = 0$, then $\boldsymbol{\sigma}[t']$ converges to $\lim_{j\to\infty} \boldsymbol{\sigma}_j(t') \le b$, and if $\boldsymbol{\sigma}[t'] = 0$, then $\boldsymbol{\sigma}[t]$ converges to $\lim_{j\to\infty} \boldsymbol{\sigma}_j(t) \le b$. Hence, t, t' are in lim-B-fair relation. ■

The existence of empty siphons at a given marking, structural boundedness, and structural B-fairness of a PN system can be checked in polynomial time (see [4,9,7]). Consequently, boundedness and B-fairness of a continuous system $\langle\mathcal{N}, \boldsymbol{m}_0\rangle_C$ can be checked in polynomial time.

5. Homothetic deadlock-freeness in structurally bounded discrete systems

The aim of this section is to characterize homothetic deadlock-freeness for structurally bounded Petri nets.

A first method to check homothetic DF of $\langle\mathcal{N}, \boldsymbol{m}_0\rangle_D$ can be to check monotonic DF, which implies homothetic DF. As already said, monotonic DF can be checked with the siphon–trap property [10,12]: if every siphon of $\mathcal{N}$ contains a marked trap which is marked at $\boldsymbol{m}_0$, then $\langle\mathcal{N}, \boldsymbol{m}_0\rangle_D$ is monotonic DF (with some marking restrictions in the case of non-ordinary PNs). However, checking this property is NP-complete, even for ordinary nets [13].

In this section, a linear technique is presented to provide a sufficient condition for homothetic DF. For this purpose, a technique for the study of DF in discrete PN systems considered in [9] is recalled. It will allow us to analyze not only DF of a given system $\langle\mathcal{N}, \boldsymbol{m}_0\rangle_D$, but also homothetic DF (for any scaled initial marking $k \cdot \boldsymbol{m}_0$).

5.1. Towards a linear characterization of DF in discrete net systems

The following general sufficient condition for DF, based on the state equation, exploits the definition: "a deadlock corresponds to a marking in which no transition is fireable".

Let $\langle\mathcal{N}, \boldsymbol{m}_0\rangle_D$ be a PN system. If there does not exist any solution $(\boldsymbol{m}, \boldsymbol{\sigma})$ to the following system, then $\langle\mathcal{N}, \boldsymbol{m}_0\rangle_D$ is deadlock-free.

$$\begin{aligned}
&\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma} \\
&\boldsymbol{m} \ge \boldsymbol{0}, \quad \boldsymbol{\sigma} \ge \boldsymbol{0}, \\
&\bigvee_{p \in {}^\bullet t} m[p] \le Pre[p, t] - 1, \quad \forall t \in T
\end{aligned} \tag{1}$$

Nevertheless, notice that the system above contains $|T|$ "complex conditions" (one for each transition) which are non-linear, due to the "$\vee$" connective. Thus, (1) can be handled by solving independently a set of $\prod_{t\in T} |{}^\bullet t|$ systems of linear inequalities, a quantity that grows exponentially: the number of linear systems is multiplied by $|{}^\bullet t|$ for each *join* transition.

Let us illustrate the key idea with the example in Fig. 1(b). Initially, the system that characterizes the sufficient condition for DF is: if there does not exist a solution to the following system, then the net system is DF (thus $2^3 = 8$ linear systems should be explored):

$$\begin{aligned}
&\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}, \\
&\boldsymbol{m} \ge \boldsymbol{0}, \quad \boldsymbol{\sigma} \ge \boldsymbol{0}, \\
&(m[p_1] = 0 \vee m[p_2] = 0), \quad \{t_1 \textit{ is not enabled}\} \\
&(m[p_1] = 0 \vee m[p_3] = 0), \quad \{t_2 \textit{ is not enabled}\} \\
&(m[p_4] = 0 \vee m[p_5] = 0) \quad \{t_3 \textit{ is not enabled}\}
\end{aligned} \tag{2}$$

In [9], some transformation rules are considered in order to reduce the number of systems generated by (1). Furthermore, in Theorem 34 of [9], it was proved that the system (1) can be rewritten as a single system of linear inequalities for every structurally bounded PN system. The structural bound of a place p, SB(p), can be computed as $\mathrm{SB}(p) = \max\{m[p] \mid \boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}, \boldsymbol{m}, \boldsymbol{\sigma} \ge 0\}$.

System (2) is not linear; however, by applying the transformation and the reduction rules in [9], it is converted to the single linear system (3). First, the PN is transformed to a PN in which every transition has at most one input place whose SB

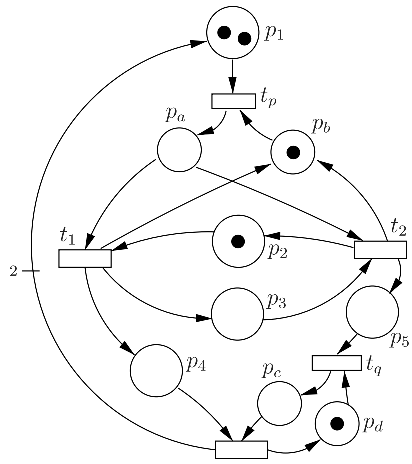

Fig. 7. PN system obtained from the transformation of the PN system in Fig. 1(b) in order to study DF with a single linear system.

is larger than the weight of its input arc. Then, the following rule from [9] is applied, which preserves the solutions of the original system.

Reduction rule. Let t be a transition s.t. ${}^\bullet t = \pi \cup \{p'\}$, where $\text{SB}(p) > 0$ and $\text{SB}(p) \leq Pre[p, t]$ for every $p \in \pi$. Then, the set of integer solutions is preserved of (2) is preserved if the disabledness condition corresponding to t is replaced by the following one:

$$\text{SB}(p') \cdot \sum_{p \in \pi} m[p] + m[p'] \leq \text{SB}(p') \cdot Pre[p, t] + Pre[p', t] - 1.$$

The PN system (Fig. 1(a)) is first transformed to the one in Fig. 7, a transformation that preserves the reachable sequences (hence, it preserves DF): place p_1 is transformed to "p_1, t_p, p_a, p_b". Then, a term related to transition t_p needs to be added in order to express a deadlock: $m[p_1] = 0 \vee m[p_b] = 0$. Now, the rule can be applied to t_1 (with $p' = p_1$ and $\pi = \{p_b\}$) because $\text{SB}(p_1) = 2$ and $\text{SB}(p_b) = 1$. The term is transformed to $\text{SB}(p_1) \cdot m[p_b] + m[p_1] \leq \text{SB}(p_1) \cdot Pre[p_b, t] + Pre[p_1, t] - 1$, i.e., $m[p_1] + 2 \cdot m[p_b] \leq 2$. Moreover, $\text{SB}(p_a) = \text{SB}(p_2) = 1$ and non-enabledness of t_1, $(m[p_1] = 0 \vee m[p_2] = 0)$, is reduced to $m[p_a] + m[p_2] \leq 1$. The other terms are analogously reduced.

The resulting system is:

$$\begin{array}{ll} \boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}, & \\ \boldsymbol{m} \geq \boldsymbol{0}, \quad \boldsymbol{\sigma} \geq \boldsymbol{0}, & \\ m[p_a] + m[p_2] \leq 1, & \{t_1 \text{ is not enabled}\} \\ m[p_a] + m[p_3] \leq 1, & \{t_2 \text{ is not enabled}\} \\ m[p_c] + m[p_4] \leq 1, & \{t_3 \text{ is not enabled}\} \\ m[p_1] + 2 \cdot m[p_b] \leq 2, & \{t_p \text{ is not enabled}\} \\ m[p_d] + m[p_5] \leq 1 & \{t_q \text{ is not enabled}\} \end{array} \tag{3}$$

In this example, (3) has no solution, so the PN system is DF. In general, if a solution exists, it may be a reachable deadlock or a *spurious* marking of the discrete net system (a *killing spurious* marking). In other words, system (3) only provides a sufficient condition for DF of discrete PN systems (semidecision).

5.2. Characterization of homothetic deadlock-freeness of discrete systems

Because (3) is a linear system, the absence of real solutions in (3) for a given $\boldsymbol{m}_0$ guarantees the absence of real solutions in an analogous system for $k \cdot \boldsymbol{m}_0$. Hence, if (3) has not real solutions, then the system is *homothetically* DF. Therefore, the continuous net system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is deadlock-free (Proposition 21). For example, system (3) obtained for the PN in Fig. 7(b) has no solution in the real domain. Consequently, $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$ is homothetically DF, and $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is also DF as continuous.

As already said, if the inequalities system (in the example, system (3)) has a solution $\boldsymbol{m}$, it may be a *spurious* marking. Therefore, two questions may appear:

- do there exist net subclasses for which no spurious deadlock can appear? (otherwise stated, for which ones the inequalities system, system (3) in the example, provides a necessary and sufficient condition?)
- if spurious deadlocks can appear, how to try to remove them?

For the first question (addressed in Section 5.3), live and bounded *equal conflict* net systems are shown not to have *spurious* deadlocks, in contrast to other net system subclasses. For the second question (Section 5.4), techniques to remove *spurious* deadlocks in the discrete system by means of the addition of implicit places can be used. A more classical one is recalled in the Appendix.

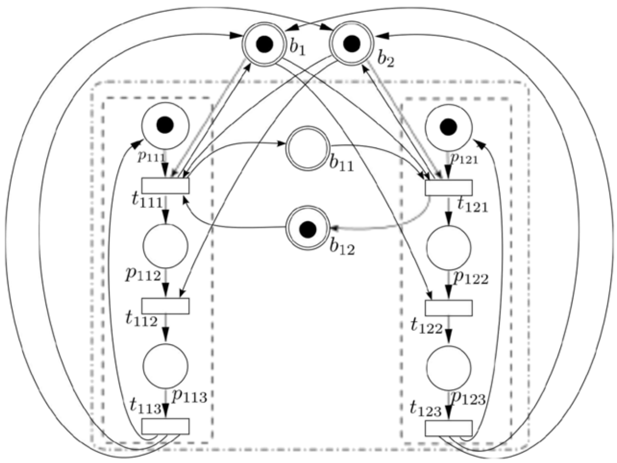

Fig. 8. A (DS)*SP system with a spurious deadlock ([15], p. 54).

5.3. About the existence of spurious deadlocks in PN system subclasses

The objective of this section is to study the existence of spurious deadlocks in some net system subclasses, what is highly related to deadlock-freeness and liveness preservation. First, it is stated that live and bounded equal conflict net systems [14] do not have spurious deadlocks, and consequently the technique in Section 5.1 does not report *spurious* deadlocks (i.e., it gives a necessary and sufficient condition). Since equal conflict nets are a superclass of choice-free, weighted-T-systems and marked graphs, the results obtained for equal conflict also hold for these subclasses. Equal Conflict nets are defined as follows.

Definition 28. Equal Conflict (EQ) is a subclass of PN in which conflicts are equal, i.e., for all $t, t' \in T$, $^\bullet t \cap {}^\bullet t' \neq \emptyset \Rightarrow \boldsymbol{Pre}[P, t] = \boldsymbol{Pre}[P, t']$.

Let us recall two classical results for EQ systems. The first one, Theorem 30 in [14], gives a necessary and sufficient condition for boundedness and liveness in discrete EQ systems. From this theorem, Proposition 29 can be directly obtained, which states necessary structural conditions for boundedness and liveness of a discrete EQ system. In the following results, *SEQS* denotes the *set of equal conflict sets* which can be computed from the net structure.

Proposition 29. *An EQ system* $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ *is live and bounded* $\Rightarrow \mathcal{N}$ *is consistent, conservative, and* $\mathrm{rank}(C) = |SEQS| - 1$.

The second result gives a necessary and sufficient condition for boundedness and lim-liveness of a continuous EQ system, considering the structure of the net and its initial marking.

Theorem 30 (*Theorem 11 in [5]*)**.** *An EQ system* $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ *is lim-live and bounded* $\Leftrightarrow \mathcal{N}$ *is consistent, conservative, rank*$(C) = |SEQS| - 1$*, and the support of every p-semiflow is marked at* $\boldsymbol{m}_0$*, i.e.,* $\nexists\, \mathbf{y} \geq \mathbf{0}$ *s.t.* $\mathbf{y} \cdot \boldsymbol{C} = \mathbf{0}, \mathbf{y} \cdot \boldsymbol{m}_0 = 0$.

From the two previous results, the following implication is straightforwardly obtained.

Proposition 30. *An EQ system* $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ *is live and bounded* $\Rightarrow \langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ *is lim-live and bounded.*

Proof. Given $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ which is live and bounded, then $\mathcal{N}$ is consistent, conservative, $rank(C) = |SEQS| - 1$ (Proposition 30). Moreover, because the discrete system is live, the support of every p-semiflow of $\mathcal{N}$ is marked at $\boldsymbol{m}_0$ (otherwise, some places will never be marked, and some transitions will never be fired). Given that $\mathcal{N}$ is consistent, conservative, $\mathrm{rank}(C) = |SEQS| - 1$ and every p-semiflow is marked at $\boldsymbol{m}_0$, then $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ is lim-live and bounded (Theorem 30). ■

A direct implication of Proposition 30 is the absence of *killing spurious* solutions in the state equation in $\mathbb{R}^+$. Hence, the technique presented in Section 5 provides not only a sufficient but also necessary condition for deadlock-freeness. Alternatively, liveness preservation (considering finite firing sequences) can also be deduced from monotonicity of liveness in bounded discrete EQ systems (Theorem 15 in [14]): because it is monotonically live, then it is also homothetically live. Hence, it is live when fluidified (Theorem 24).

However, when more general subclasses of net systems are considered, the state equation can contain some *killing spurious* solutions. It is the case of the following two examples, shown in Figs. 8 and 9.

The PN example in Fig. 8 is deadlock-free as discrete. However, it has a *killing spurious* solution which becomes (lim-)reachable in the fluidified net system. The system belongs to (DS)*SP [16], a subclass of PN systems which models intricate *cooperation* relations.

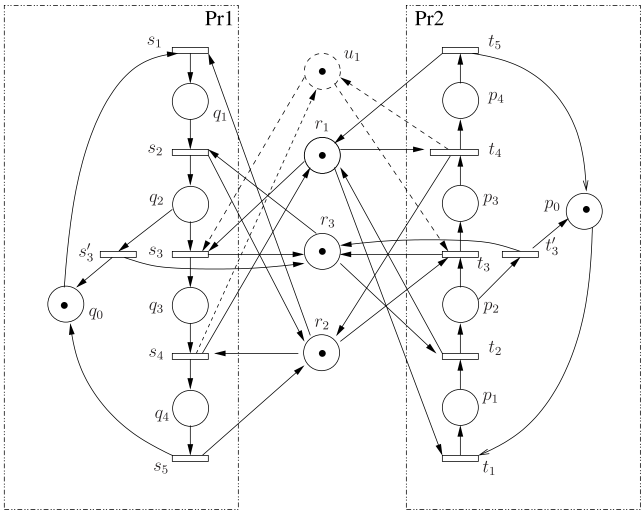

Fig. 9. Without the dotted place u_1 and the related arcs, it is a deadlock-free S^{3}PR discrete system. It deadlocks as continuous, reaching $\boldsymbol{m}_d = (0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1)$, which was a *spurious* deadlock of the state equation ([17], p. 111).

Considering the initial marking depicted in the figure, the infinite firing sequence $\sigma_d = \frac{1}{2}t_{111}\ \frac{1}{2}t_{121}\ \frac{1}{4}t_{111}\ \frac{1}{4}t_{121}\ \frac{1}{8}t_{111}\ \frac{1}{8}t_{121}\ \ldots$ can be fired, which converges to a marking $\boldsymbol{m}_d$ in which $m_d[p_{112}] = m_d[p_{122}] = m_d[b_{12}] = 1$, and the other places are empty [15]. Notice that this marking is a deadlock (i.e., a *killing spurious* marking in the discrete system which becomes lim-reachable in the continuous one).

Other example of not preserving deadlock-freeness, even with finite sequences, is presented in Fig. 9 [17]. Without the dotted part, it models a system in which two sequential processes Pr_1 and Pr_2 share resources r_1, r_2 and r_3. It belongs to the subclass S^{3}PR [18], characterized by the *competition* of processes. The PN system is live as discrete from the initial marking $\boldsymbol{m}_0$, where $m_0[r_1] = m_0[r_2] = m_0[r_3] = m_0[q_0] = m_0[p_0] = 1$ and all the other places are empty.

However, when the system is fluidified, it can reach a deadlock: when the sequence $\sigma_d = \frac{1}{2}s_1\frac{1}{2}t_1\frac{1}{2}s_2\frac{1}{2}t_2\frac{1}{2}s_3\frac{1}{2}t_3\frac{1}{2}s_1\frac{1}{2}t_1\frac{1}{2}s_2\frac{1}{2}t_2\frac{1}{2}s_3\frac{1}{2}t_3$ is fired, a deadlock marking $\boldsymbol{m}_d$ is reached, where $m_d[r_3] = m_d[q_3] = m_d[p_3] = 1$, and the marking of all the other places is 0. Notice that $\boldsymbol{m}_d$ is a *killing spurious* marking in the discrete system, which can be reached by the continuous system with a *finite* firing sequence. A method to remove this kind of *killing spurious* solutions is presented in the next section.

5.4. Removing spurious deadlocks

As previously explained, a *spurious* marking in the discrete system is a solution of the state equation (see system of equations (3) in Section 5.1 for an example) which is not reachable in $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$. However, due to the relaxation, it can become reachable by the continuous system $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$, either by considering an infinite or a finite firing sequence.

A technique developed for removing spurious solutions which are due to the emptying of an initially marked trap is considered in [9] for discrete PN systems (applied to continuous PN systems in [4]). This kind of spurious solutions, which could be reached in the continuous system by firing an infinite firing sequence, are removed by adding some places which are *implicit* in the discrete model. Let us recall that a place p' is *implicit* if its removal does not modify the set of fireable sequences (hence, reachable markings). In other words, p' is never the only place that prevents the firing of a transition ([9] for discrete; [4] for continuous).

Let θ be a trap which is marked at $\boldsymbol{m}_0$. If $\boldsymbol{m}$, a solution of the state equation does not mark θ, then $\boldsymbol{m}$ is a *spurious* marking in the discrete system. The technique consists in using a trap generator [19], what allows the checking of implicit places in polynomial time (a sufficient condition). Then, a monitor *implicit* place is added to the system such that it adds an invariant (a p-semiflow) to the net, that forces the trap to remain marked. The technique is recalled in the Appendix and applied to an example.

Here, we propose another technique to remove the spurious solutions which can be reached by firing even a *finite* firing sequence in the corresponding continuous PN system. This technique removes the spurious solutions by preventing certain siphons from being emptied, with the addition of some places which are *implicit* in the discrete model. An important difference with respect to the previous technique is that some of the *deadlock* markings that now can be removed may be reachable by the original discrete system if it was not deadlock-free. Consequently, this technique should be applied to remove solutions that *we know* that are spurious.

The first step is to identify the existence of a siphon that is initially marked but it is emptied in a marking $\boldsymbol{m}_d$ (which is a solution of the state equation). It can be characterized by a set of linear inequations, which consists in a *siphon generator* [19] and the expression that an initially marked siphon becomes empty.

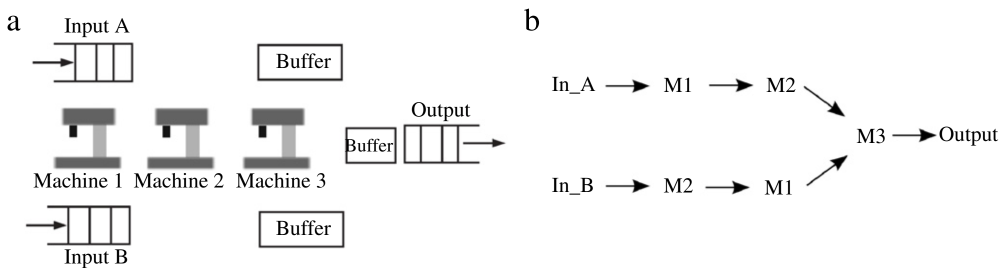

Fig. 10. (a) Logical layout of a manufacturing system, and (b) its production cycles.

Let us define $\mathbf{Pre}_\Sigma$ and $\mathbf{Post}_\Sigma$ as $|P| \times |T|$ sized matrices such that:

- $Pre_\Sigma[p,t] = |t^\bullet|$ if $Pre[p,t] > 0$, $Pre_\Sigma[p,t] = 0$ otherwise
- $\mathrm{Post}_\Sigma[p,t] = 1$ if $\mathrm{Post}[p,t] > 0$, $\mathrm{Post}_\Sigma[p,t] = 0$ otherwise.

Equations $\{\mathbf{y}^T \cdot \boldsymbol{C}_\Sigma \geq \mathbf{0},\ \mathbf{y} \geq \mathbf{0}\}$ where $\boldsymbol{C}_\Sigma = \mathbf{Post}_\Sigma - \mathbf{Pre}_\Sigma$ define a generator of siphons (Σ is a siphon iff $\exists \mathbf{y} \geq \mathbf{0}$ such that $\Sigma = \|\mathbf{y}\|$, $\mathbf{y}^T \cdot \boldsymbol{C}_\Sigma \geq \mathbf{0}$, analogous to the generator of traps [19,9]). Hence, given $\boldsymbol{m}_d$ a solution of the state equation reported as a deadlock, we can check in polynomial time if a minimal siphon marked at $\boldsymbol{m}_0$ is unmarked at $\boldsymbol{m}_d$ and report which is such siphon.

Proposition 31. *Given* $\boldsymbol{m}_d \in \mathbb{R}^{|P|}_{\geq 0}$ ($\boldsymbol{m}_d = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \sigma$, $\boldsymbol{m}_d, \sigma \geq \mathbf{0}$)*, if*

- $\mathbf{y}^T \cdot \boldsymbol{C}_\Sigma \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}$, *{siphon generator}*
- $\mathbf{y}^T \cdot \boldsymbol{m}_0 \geq 1$, *{initially marked siphon}*
- $\mathbf{y}^T \cdot \boldsymbol{m} = 0$, *{siphon empty at* $\boldsymbol{m}_d$*}*

has solution, then a siphon Σ *marked at* $\boldsymbol{m}_0$ *is emptied at* $\boldsymbol{m}_d$.

Once a siphon which has been emptied is identified, it can be forced to remain marked by the addition of a place which is *implicit* in the discrete model.

Let us illustrate this idea with an example. Consider again the S^3PR system in Fig. 9, which reaches the continuous deadlock $\boldsymbol{m}_d$, where $m_d[r_3] = m_d[q_3] = m_d[p_3] = 1$, and $m_d[q_i] = m_d[p_i] = m_d[r_i] = 0\ \forall i \in \{0,1,2,4\}$. This deadlock occurs due to the fact that siphon $\Sigma_1 = \{q_1, q_4, p_1, p_4, r_1, r_2\}$ has been emptied (notice that once a siphon is emptied, it is never marked again).

The technique presented here adds a monitor place to prevent this siphon from being emptied. In order to keep Σ_1 marked in the continuous net system, the following inequality should be forced: $m[q_1] + m[q_4] + m[p_1] + m[p_4] + m[r_1] + m[r_2] \geq 1$. It can be forced by the addition of a slack variable u_1, i.e., a *cutting* implicit place, to force the following invariant relation: $m[q_1] + m[q_4] + m[p_1] + m[p_4] + m[r_1] + m[r_2] - m[u_1] = 1$. Since $m[u_1] \geq 0$, siphon Σ_1 will remain marked. Place u_1 is shown in the dotted part in Fig. 9; it has s_4 and t_4 as input transitions and s_3 and t_3 as output transitions and its initial marking can be calculated from the invariant: $m_0[u_1] = m_0[q_1] + m_0[q_4] + m_0[p_1] + m_0[p_4] + m_0[r_1] + m_0[r_2] - 1 = 0 + 0 + 0 + 0 + 1 + 1 - 1 = 1$.

Observe that the added place u_1 is *implicit* in the discrete PN system and makes the system deadlock-free as continuous.

6. An example. A model of a flexible manufacturing system

In this section, we apply some of the results already presented in the previous sections to an example. Let us consider a flexible manufacturing system (see Fig. 10) composed of two production lines with three machines M1, M2 and M3. The PN which models the system and its initial state $\boldsymbol{m}_0$ are depicted in Fig. 11.

Parts of type A are processed in machine M1 and then in machine M2, with intermediate products stored in buffers B_1A and B_2A. Parts of type B are first processed in M2 and then in M1, with intermediate products stored in buffers B_1B and B_2B. Finally, machine M3 assembles part A and part B, obtaining the final product, which is stored in buffer B_3 until its removal. Places Max_B_1A and Max_B_1B initially have only one token, so there can be at most one part of type A and one part of type B either in B_1A and B_1B, or being processed by M1 and M2. Parts A and B are moved in pallets all along the process, and there are 20 pallets of type A and 15 pallets of type B. Place Max_B_3 has initially one token, so only one final product can be stored in the buffer B_3 until its removal.

Typical *competition* and *cooperation* relations that often appear in manufacturing systems are introduced by means of the movement of parts inside the system. For instance, machine M1 and machine M2 are shared for processing parts A and B; therefore, these activities are in mutual exclusion (*mutex*). Final products can be assembled only when both intermediate produces of types A and B are available (i.e., buffer B_2A and B_2B are not empty) (*rendez-vous*).

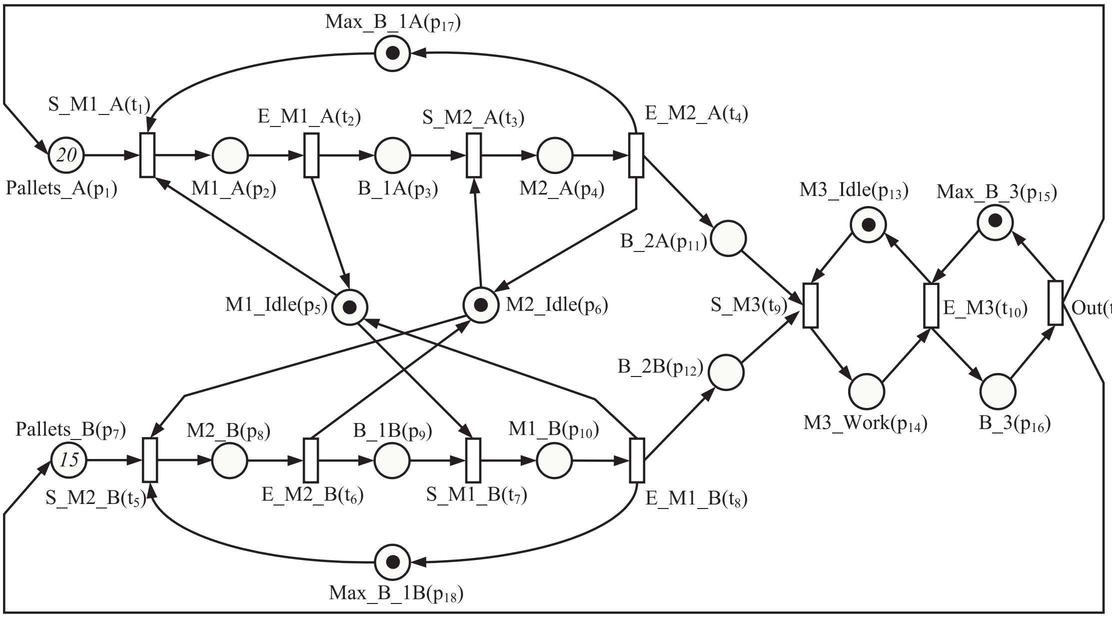

Fig. 11. The PN system that models the manufacturing system described in Fig. 10.

Some of the properties characterized in the previous sections can be checked in this example. Notice that checking these properties with the enumeration of the state space of the discrete PN system would be more expensive than checking the properties in the continuous system. For this initial marking, the state space of $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ has 15,455 reachable markings.

According to Section 4, given that the PN system has no empty siphons at $\boldsymbol{m}_0$, boundedness (resp. B-fairness) is equivalent to structurally boundedness (resp. structurally B-fairness). In this case, because $\exists\, \mathbf{y} > \mathbf{0}$ s.t. $\mathbf{y}^T \cdot \boldsymbol{C} = \mathbf{0}$, the net is conservative; hence it is structurally bounded. The continuous system is bounded for every $\boldsymbol{m}_0 \in \mathbb{R}^{|P|}_{\geq 0}$. In structurally bounded nets, structural B-fairness is equivalent to consistency and existence of a unique T-semiflow [7]. In this case, the net is consistent and has a unique T-semiflow (it is $\mathbf{x} = \mathbf{1} > \mathbf{0}$, s.t. $\boldsymbol{C} \cdot \mathbf{x} = \mathbf{0}$). Consequently, it is B-fair as continuous for every $\boldsymbol{m}_0 \in \mathbb{R}^{|P|}_{\geq 0}$.

Let us now consider the method presented in Section 5 to check homothetic deadlock-freeness. The system (1) can be written as follows:

$$\begin{cases} \boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}, & \\ \boldsymbol{m} \geq \mathbf{0}, \boldsymbol{\sigma} \geq \mathbf{0}, & \\ m[p_2] = m[p_4] = m[p_8] = 0, m[p_{10}] = m[p_{16}] = 0, & (\text{for } t_2, t_4, t_6, t_8, t_{11}) \\ m[p_1] = 0 \vee m[p_5] = 0 \vee m[p_{17}] = 0, & (\text{for } t_1) \\ m[p_3] = 0 \vee m[p_6] = 0, & (\text{for } t_3) \\ m[p_7] = 0 \vee m[p_6] = 0 \vee m[p_{18}] = 0, & (\text{for } t_5) \\ m[p_5] = 0 \vee m[p_9] = 0, & (\text{for } t_7) \\ m[p_{11}] = 0 \vee m[p_{12}] = 0 \vee m[p_{13}] = 0, & (\text{for } t_9) \\ m[p_{14}] = 0 \vee m[p_{15}] = 0 & (\text{for } t_{10}) \end{cases} \tag{4}$$

As observed in Section 5.1, the system (4) contains some "complex conditions" due to the "$\vee$" connective. However, since the PN system is structurally bounded, it may be simplified and rewritten as a system of linear inequalities. In order to do that, the structural bound of every place has to be computed.

Now, let us consider, for example, the condition of disabling transition t_3. The SB of places p_3 and p_6 is 1. Hence, in the discrete model $m[p_3] = 0 \vee m[p_6] = 0 \Leftrightarrow m[p_3] + m[p_6] \leq 1$. This rule can be applied to the term corresponding to t when $\forall p_i \in {}^\bullet t$, $\mathrm{SB}(p_i) \leq Pre[p_i, t]$.

A similar reduction rule can be applied when there are $|{}^\bullet t| - 1$ input places of t that satisfy the following: $\mathrm{SB}(p_i) \leq Pre[p_i, t]$, $p_i \in {}^\bullet t$, which is true for t_1. For this initial marking, $\mathrm{SB}(p_5) = \mathrm{SB}(p_{17}) = 1$, while $\mathrm{SB}(p_1) = 20$. Therefore, $m[p_1] = 0 \vee m[p_5] = 0 \vee m[p_{17}] = 0 \Leftrightarrow m[p_1] + (m[p_5] + m[p_{17}]) \cdot 20 \leq 40$.

If more than one input place of t does not fulfill $\mathrm{SB}(p_i) \leq Pre[p_i, t]$, $p_i \in {}^\bullet t$, a pre-transformation is needed. Consider the term $m[p_{11}] = 0 \vee m[p_{12}] = 0 \vee m[p_{13}] = 0$, which corresponds to transition t_9. The structural bounds of p_{11}, p_{12} and p_{13} are $\mathrm{SB}(p_{11}) = 20$, $\mathrm{SB}(p_{12}) = 15$ and $\mathrm{SB}(p_{13}) = 1$. Consequently, the rule used for t_1 is not directly applicable to t_9, and the pre-transformation depicted in Fig. 12, analogous to the transformation shown in Fig. 7(b), needs to be applied first. Then, the term corresponding to t_9 is written as: $m[p_{12}] + (m[p_{13}] + m[p_{20}]) \cdot 15 \leq 30$, and the inequality $m[p_{11}] + m[p_{19}] \cdot 20 \leq 20$ is also added (due to the transition t_{12}).

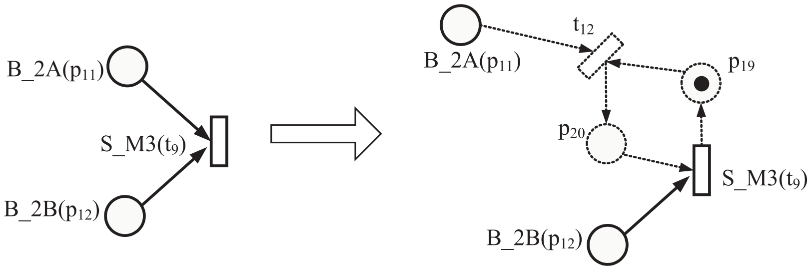

Fig. 12. Transformation related to t_9 in Fig. 11.

The linear system obtained after the transformation of the non-linear one (4) is as follows:

$$\begin{cases} \boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}, & \\ \boldsymbol{m} \geq \boldsymbol{0}, \boldsymbol{\sigma} \geq \boldsymbol{0}, & \\ m[p_2] = m[p_4] = m[p_8] = 0, m[p_{10}] = m[p_{16}] = 0, & \text{(for } t_2, t_4, t_6, t_8, t_{11}\text{)} \\ m[p_1] + (m[p_5] + m[p_{17}]) \cdot 20 \leq 40, & \text{(for } t_1\text{)} \\ m[p_3] + m[p_6] \leq 1, & \text{(for } t_3\text{)} \\ m[p_7] + (m[p_6] + m[p_{18}]) \cdot 15 \leq 30, & \text{(for } t_5\text{)} \\ m[p_5] + m[p_9] \leq 1, & \text{(for } t_7\text{)} \\ m[p_{12}] + (m[p_{13}] + m[p_{20}]) \cdot 15 \leq 30, & \text{(for } t_9\text{)} \\ m[p_{11}] + m[p_{19}] \cdot 20 \leq 20, & \text{(added, for } t_{12}\text{)} \\ m[p_{14}] + m[p_{15}] \leq 1 & \text{(for } t_{10}\text{)} \end{cases} \quad (5)$$

There does not exist a feasible solution for (5), consequently the discrete net system is homothetically deadlock-free; thus it is also DF as continuous.

Let us notice that the PN system is conservative, consistent and it has a unique T-semiflow; therefore (lim-)DF is equivalent to (lim-)liveness [20]. It is consistent and lim-live; hence it is lim-reversible [5].

7. Conclusions

In this paper, the fluidization of autonomous Petri net systems has been considered. The preservation of basic properties such as boundedness, B-fairness, deadlock-freeness, liveness, and reversibility has been studied: if one of those properties, Π, has a *homothetic* behavior in a discrete PN system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$, Π will be preserved when the system is fluidified (i.e., in $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$). Two basic kind of properties (or facts) are used to achieve these results: that every real number can be approximated by a rational one; and the properties of homothecy and monotonicity of firing sequences in the continuous PN systems.

When boundedness or B-fairness are considered, homothetic Π_S is equivalent to Π_S of the continuous system, and also equivalent to lim-Π_S. Moreover, under some general conditions (every transition is fireable at least once) it is also equivalent to structural Π_S. However, when deadlock-freeness, liveness or reversibility, are considered, homothetic Π_L of the discrete system implies Π_L in the continuous, but it does not imply lim-Π_L. In contrast, lim-Π_L implies homothetic Π_L in the discrete system (see Fig. 5).

Some techniques are recalled from the discrete PN system analysis and adapted to study homothetic deadlock-freeness and (lim-)deadlock-freeness preservation. Moreover, the preservation of (lim-)liveness for some subclasses has been studied, as well as some subclasses for which, in general, deadlock-freeness or liveness is not preserved by fluidization. Finally, "dual" in some sense to a well-known technique to remove spurious solutions, a new one has been introduced with the same purpose, but requiring a priori knowledge of non-reachability of the marking being investigated.

Appendix. Removing deadlock spurious markings reachable in which a marked trap is emptied

In this Appendix, a technique to remove spurious solutions is recalled from [9] and applied to an example to obtain liveness and lim-liveness preservation of the system by fluidization. It consists on the identification of those spurious solutions which empty a trap, and the addition of implicit places to remove them.

An example with this type of spurious deadlocks is the PN in Fig. A.1 (without considering the dotted part) with initial marking $\boldsymbol{m}_0 = (2, 2, 0)$. This PN system is live as discrete. However, when considered as continuous, it may reach a spurious deadlock $\boldsymbol{m}_d = (0, 4, 0)$ (see the *reachability graph* of the net system in Fig. A.1(b), where the shaded markings correspond to spurious solutions). It is done by firing the infinite sequence $\sigma = 1t_1 1t_3 1t_3 1t_2 \frac{1}{2}t_1 \frac{1}{2}t_3 \frac{1}{2}t_3 \frac{1}{2}t_2\ \frac{1}{4}t_1 \frac{1}{4}t_3 \frac{1}{4}t_3 \frac{1}{4}t_2 \ldots$. Therefore, the trap $\Theta_1 = \{p_1, p_3\}$ is emptied, but it was initially marked ($m_0[p_1] + m_0[p_3] = 2$).

The existence of a trap that is initially marked but can be emptied in the limit can be characterized with a set of linear inequations, which consists in a *trap generator* and the expression that an initially marked trap becomes empty. Let us define $\mathbf{Pre}_\Theta$ and $\mathbf{Post}_\Theta$ as $|P| \times |T|$ sized matrices such that:

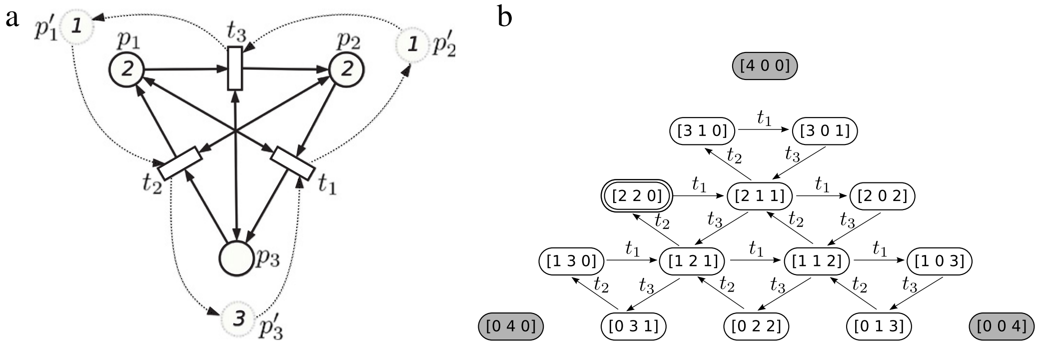

Fig. A.1. (a) Without the dotted part, PN system that deadlocks with an infinite firing sequence as continuous, and (b), its reachability graph as discrete, where the shaded markings correspond to spurious solutions, all isolated deadlocks.

- $\mathbf{Pre}_\Theta[p,t] = 1$ if $\mathbf{Pre}[p,t] > 0$, $\mathbf{Pre}_\Theta[p,t] = 0$ otherwise
- $\mathbf{Post}_\Theta[p,t] = |{}^\bullet t|$ if $\mathbf{Post}[p,t] > 0$, $\mathbf{Post}_\Theta[p,t] = 0$ otherwise.

Equations $\{\mathbf{y}^T \cdot \boldsymbol{C}_\Theta \geq \mathbf{0},\ \mathbf{y} \geq \mathbf{0}\}$ where $\boldsymbol{C}_\Theta = \mathbf{Post}_\Theta - \mathbf{Pre}_\Theta$ define a generator of traps (Θ is a trap iff $\exists\, \mathbf{y} \geq \mathbf{0}$ such that $\Theta = \|\mathbf{y}\|, \mathbf{y}^T \cdot \boldsymbol{C}_\Theta \geq \mathbf{0}$) [19,9]. Hence, given $\boldsymbol{m}$ a solution of the state equation, we can check in polynomial time a sufficient condition for being spurious.

Proposition 32. *Given* $\boldsymbol{m} \in \mathbb{N}^{|P|}$ ($\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma},\ \boldsymbol{m}, \boldsymbol{\sigma} \geq 0$), *if*

- $\mathbf{y}^T \cdot \boldsymbol{C}_\Theta \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}$, *{trap generator}*
- $\mathbf{y}^T \cdot \boldsymbol{m}_0 \geq 1$, *{initially marked trap}*
- $\mathbf{y}^T \cdot \boldsymbol{m} = 0$, *{trap empty at* $\boldsymbol{m}$*}*

has solution, then $\boldsymbol{m}$ *is a spurious solution in the discrete PN system.*

First proposed for discrete PN [9], the technique presented in [9,4] removes this kind of spurious deadlocks by adding some *implicit* places in the discrete model. Let us consider the trap $\Theta_1 = \{p_1, p_3\}$ in the net in Fig. A.1(a). Since it is initially marked, in the discrete model its marking must satisfy $m[p_1]+m[p_3] \geq 1$. Considering the token conservation low obtained from the P-semiflow, $m[p_1] + m[p_2] + m[p_3] = 4$, it leads to $m[p_2] \leq 3$. This last inequality can be forced by adding a slack variable to the system, i.e., a *cutting implicit place* p'_2 (shown in Fig. A.1(a)), such that $m[p_2] + m[p'_2] = 3$. The initial marking of p'_2 can be simply set as $m_0[p'_2] = 3 - m_0[p_2] = 1$. By adding p'_2, the spurious deadlock marking $\boldsymbol{m}_d = (0, 4, 0)$, in which trap $\Theta_1 = \{p_1, p_3\}$ is empty, is removed. Similarly, by adding p'_1, p'_3, the spurious markings which empty traps $\Theta_2 = \{p_2, p_3\}$ and $\Theta_3 = \{p_1, p_2\}$ are also removed.

It is interesting to remark that, by removing spurious deadlocks (in fact, any spurious marking), the approximation of the performance of the discrete net system, provided by the timed relaxation, is also improved. This is true even if the deadlock is not reached in the timed continuous model. In any case, removing spurious solutions represents an improvement of the fluidization, being specially important when those solutions are deadlocks or non-live steady states.

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