Control of continuous Petri nets using ON/OFF based method

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Abstract: Continuous Petri Nets (CPN) can be used to approximate classical discrete Petri nets which suffer from the state explosion problem. In this paper we focus on the control of timed CPN (TCPN), aiming to drive the system from an initial state to a desired final one. This problem is similar to the set-point control problem in a general continuous-state system. In a previous work, a simple and efficient ON/OFF controller was proposed for structurally persistent nets, and it is proved to be minimum-time. In this work the ON/OFF controller is extended to general TCPN, but in this case, the minimum-time evolution is not guaranteed. Three extensions are proposed, all of them are based on the ON/OFF strategy. Some comparisons of those controllers are given in terms of their applications to an assembly system.

1. INTRODUCTION

Petri Nets (PN) is a well known paradigm used for modeling, analysis, and synthesis of *discrete event systems* (DES). With strong facility to depict sequences, concurrency, conflicts and synchronizations, it is widely applied in the industry for the analysis of manufacturing, traffic, or software systems, for example. Similarly to other modeling formalisms for DES, it also suffers from the *state explosion* problem. To overcome it, a classical relaxation technique called *fluidization* can be used.

Continuous PN (CPN) [1, 9] are fluid approximations of classical *discrete PN* obtained by removing the integrality constraints, which means that the firing count vector and consequently the marking are no longer restricted to be in the naturals but relaxed into the non-negative real numbers. An important advantage of this relaxation is that more efficient algorithms are available for their analysis. In [10], the control methods are first achieved in the fludified continuous model, after that they are applied back to control its original discrete system.

One of the important objectives in the control of CPN is to drive the system from an initial state $\boldsymbol{m}_0$ to a desired final state $\boldsymbol{m}_f$, which is similar to the *set-point* control problem in a general continuous-state system. By considering the CPN as a relaxation of discrete systems, the continuous state can be viewed as the approximation of the average state in its original discrete system. Several approaches can be found in the literature for handling this control problem, for example, in [2, 5, 7]. Many of these works are based on *infinite server semantics*. For a broad class of nets, it has been proved that this semantics provides a better approximation of discrete systems than *finite sever semantics* under some general conditions [6]. In the case of systems with uncontrollable transitions, the control problem may become much more complex [4, 11].

A minimum-time ON/OFF controller has been proposed for structurally persistent PN [12]. The essential problem of this standard ON/OFF controller is that when there is conflict, this "greedy" strategy of firing transitions may bring the system to a "blocked" situation (see Ex.2 for a example). In this work, the ON/OFF control scheme is further investigated and three heuristic extensions are presented, ensuring that the final state is reached in finite time, even if the minimum time is not guaranteed. By forcing the conflicting transitions firing proportionally, we obtain the *ON/OFF-plus* (ON/OFF+) controller. But the drawback of this method is obvious: the firing speeds of transitions in a conflict relation are decided by the slower ones, and the overall system may be highly slowed down. Therefore, the second extension, *balanced ON/OFF* (B-ON/OFF) controller is proposed, trying to balance the fast and slow transitions before applying the pure ON/OFF+ controller. The third method is a combination of *model predictive control* (MPC) and ON/OFF strategy: solving the conflicts using MPC and firing other transition using ON/OFF strategy. The first two methods have very low computational complexity, while using the *ON/OFF-MPC* controller we may reach the final state faster, but with considerable higher computational complexity. Some comparisons are made by using different control methods and parameters.

This paper is organized as follows: Section 2 briefly recalls some basic concepts of CPN. In Section 3 the standard ON/OFF controller is recalled and its main drawback is stated. Three ON/OFF strategies are proposed in Section 4. Section 5 compares these control methods by using an assembly system. Some conclusions are given in section 6.

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2. BASIC CONCEPTS AND NOTATIONS

2.1 Continuous Petri Nets

The reader is assumed to be familiar with basic concepts of continuous Petri nets (see [1, 9] for a gentle introduction).

Definition 1. A continuous PN system is a pair $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ where $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post} \rangle$ is a net structure where:

- P and T are the sets of places and transitions respectively.
- $\boldsymbol{Pre}, \boldsymbol{Post} \in \mathbb{Q}_{\geq 0}^{|P| \times |T|}$ are the pre and post incidence matrices.
- $\boldsymbol{m}_0 \in \mathbb{R}_{\geq 0}^{|P|}$ is the initial marking (state).

For $v \in P \cup T$, the sets of its input and output nodes are denoted as ${}^\bullet v$ and $v^\bullet$, respectively. Let p_i, $i = 1, \ldots, |P|$ and $t_j, j = 1, \ldots, |T|$ denote the places and transitions. Each place can contain a non-negative real number of tokens, its marking. The distribution of tokens in places is denoted by $\boldsymbol{m}$. The enabling degree of a transition $t_j \in T$ is given by:

$$enab(t_j, \boldsymbol{m}) = \min_{p_i \in {}^\bullet t_j} \left\{ \frac{\boldsymbol{m}(p_i)}{\boldsymbol{Pre}(p_i, t_j)} \right\}$$

which represents the maximum amount in which t_j can fire. Transition t_j is called *k-enabled* at marking $\boldsymbol{m}$, if $enab(t, \boldsymbol{m}) = k$, being enabled if $k > 0$. An enabled transition t_j can fire in any real amount α, with $0 < \alpha \leq enab(t_j, \boldsymbol{m})$ leading to a new state $\boldsymbol{m'} = \boldsymbol{m} + \alpha \cdot \boldsymbol{C}(\cdot, t_j)$ where $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the *token flow matrix* and $\boldsymbol{C}(\cdot, j)$ is its j^{th} column.

Non negative left and right natural annullers of the token flow matrix $\boldsymbol{C}$ are called P-*semiflows* (denoted by $\boldsymbol{y}$) and T-semiflows (denoted by $\boldsymbol{x}$), respectively. If $\exists \boldsymbol{y} > 0$, $\boldsymbol{y} \cdot \boldsymbol{C} = 0$, then the net is said to be conservative. If $\exists \boldsymbol{x} > 0$, $\boldsymbol{C} \cdot \boldsymbol{x} = 0$ it is said to be consistent.

A PN system is bounded when every place is bounded, i.e., its token content is less than some bounds at every reachable marking. It is live when every transition is live, i.e., it can ultimately occur from every reachable marking.

If $\boldsymbol{m}$ is reachable from $\boldsymbol{m}_0$ through a finite sequence σ, the state (or fundamental) equation is satisfied: $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma} \in \mathbb{R}_{\geq 0}^{|T|}$ is the *firing count vector*, i.e., $\boldsymbol{\sigma}(t_j)$ is the cumulative amount of firings of t_j of the sequence σ. A firing count vector $\boldsymbol{\sigma}$ is said to be *minimal* if for any T-*semiflow* $\boldsymbol{x}$, $||\boldsymbol{x}|| \not\subseteq ||\boldsymbol{\sigma}||$, where $||\cdot||$ stands for the support of a vector, i.e., the index of the elements different than zero. A minimal firing count vector $\boldsymbol{\sigma}$, driving the system from $\boldsymbol{m}_0$ to $\boldsymbol{m}_f$ can be computed by solving the following linear programming problem (LPP):

$$\begin{aligned} &min\ \mathbf{1}^T \cdot \boldsymbol{\sigma} \\ &\text{s.t.}\ \ \boldsymbol{m}_f = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma} \\ &\qquad \boldsymbol{\sigma} \geq 0 \end{aligned} \tag{1}$$

If for all $p \in P$, $|p^\bullet| \leq 1$ then $\mathcal{N}$ is called *Choice-Free PN* (CFPN). A CFPN is *structurally persistent* in the sense that independently of the initial marking, the net has no conflict.

In timed continuous PN (TCPN) the state equation has an explicit dependence on time: $\boldsymbol{m}(\tau) = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}(\tau)$ which through time differentiation becomes $\dot{\boldsymbol{m}}(\tau) = \boldsymbol{C} \cdot \dot{\boldsymbol{\sigma}}(\tau)$. The derivative of the firing count $\boldsymbol{f}(\tau) = \dot{\boldsymbol{\sigma}}(\tau)$ is called the *firing flow*. Depending on how the flow is defined, many firing server semantics appear, being the most used ones *infinite* (or variable speed) and *finite* (or constant speed) server semantics [1, 9], for which a firing rate $\lambda_j \in \mathbb{R}_{>0}$ is associated to transition t_j. This paper deals with *infinite server semantics* for which the flow of a transition t_j at time τ is the product of its firing rate, λ_j, and its enabling degree at $\boldsymbol{m}(\tau)$:

$$f(t_j, \tau) = \lambda_j \cdot enab(t_j, \boldsymbol{m}(\tau)) = \lambda_j \cdot \min_{p_i \in {}^\bullet t_j} \left\{ \frac{\boldsymbol{m}(p_i, \tau)}{\boldsymbol{Pre}(p_i, t_j)} \right\}$$

2.2 Control Problem

In this paper the net system is considered to be subject to external control actions, and it is assumed that the only admissible control law consists in *slowing down* the firing speed of transitions [9]. Under this assumption, the controlled flow of a TCPN system is denoted as: $\boldsymbol{w}(\tau) = \boldsymbol{f}(\tau) - \boldsymbol{u}(\tau)$, with $0 \leq \boldsymbol{u}(\tau) \leq \boldsymbol{f}(\tau)$. The overall behavior of the system is ruled by: $\dot{\boldsymbol{m}}(\boldsymbol{\tau}) = \boldsymbol{C} \cdot (\boldsymbol{f}(\tau) - \boldsymbol{u}(\tau))$. In this paper, it is assumed that every transition is *controllable* (t_j is uncontrollable if the only control that can be applied is $u(t_j) = 0$).

The control problem addressed here is to design a control action $\boldsymbol{u}$ that drives the system from the initial marking $\boldsymbol{m}_0$ to the desired final marking $\boldsymbol{m}_f$.

3. ON/OFF CONTROLLER AND ITS PROBLEMS

By sampling the continuous-time TCPN system with a *sampling period* Θ, we obtain the discrete-time TCPN [5] given by:

$$\begin{aligned} &\boldsymbol{m}_{k+1} = \boldsymbol{m}_k + \boldsymbol{C} \cdot \boldsymbol{w}_k \cdot \Theta \\ &0 \leq \boldsymbol{w}_k \leq \boldsymbol{f}_k \end{aligned} \tag{2}$$

Here $\boldsymbol{m}_k$ and $\boldsymbol{w}_k$ are the marking and controlled flow at sampling instant k, i.e., at $\tau = k \cdot \Theta$.

It is proved in [5] that if the sampling period satisfies (3), the reachability spaces of discrete-time and continuous-time PN systems are the same, excepting at borders.

$$\forall p \in P : \sum_{t_j \in p^\bullet} \lambda_j \cdot \Theta < 1 \tag{3}$$

In this paper, we assume that the sampling period Θ satisfies (3).

An ON/OFF controller is proposed in [12] for structurally persistent PN, where every transition is fired as fast as possible at any time step until an upper bound, the minimal firing count vector $\boldsymbol{\sigma}$, is reached. Alg. 1 computes the firing flow of the ON/OFF controller for each time step. In the case of CFPN, using the ON/OFF controller based on the minimum firing count vector, the desired marking is reached in minimum time. However, in the case of general nets this is not true in general.

Let us notice that, different from CFPN, in a PN with general structure, its minimal firing count vector may be not unique for given initial and final states.

The main advantage of the ON/OFF control strategy is its low computational complexity. Given a (minimal)

Algorithm 1 ON/OFF controller

Input: $\boldsymbol{m}_0$, $\boldsymbol{m}_f$, $\boldsymbol{\sigma}$, $\boldsymbol{C}$, $\boldsymbol{\lambda}$, Θ

Output: $\boldsymbol{w}_0$, $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, ...

$k = 0$

while $\sum_{i=0}^{k-1} \boldsymbol{w}_i \cdot \Theta \neq \boldsymbol{\sigma}$ **do**

  Solve the following LPP :

$$\begin{aligned} & max\ \mathbf{1}^T \cdot \boldsymbol{w}_k \\ & \textbf{s.t.}\ \ \boldsymbol{m}_{k+1} = \boldsymbol{m}_k + \boldsymbol{C} \cdot \boldsymbol{w}_k \cdot \Theta \\ & \qquad \mathbf{0} \leq \boldsymbol{w}_k \cdot \Theta \leq \boldsymbol{\sigma} - \sum_{i=0}^{k-1} \boldsymbol{w}_i \cdot \Theta \\ & \qquad \boldsymbol{w}_k(t_j) \leq \lambda_j \cdot enab(t_j, \boldsymbol{m}_k), \forall t_j \in T \\ & \qquad \boldsymbol{m}_{k+1} \geq 0 \end{aligned} \tag{4}$$

  Apply $\boldsymbol{w}_k : \boldsymbol{m}_{k+1} = \boldsymbol{m}_k + \boldsymbol{C} \cdot \boldsymbol{w}_k \cdot \Theta$

  $k := k + 1$

end while

return $\boldsymbol{w}_0$, $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, ...

firing count vector (which can be computed in polynomial time), the control actions can be obtained by solving a simple LPP in each time step. But when the system is not CF, the convergence of the final state may not be ensured. Moreover, in the case of non-CFPN, conflicts ($|p^\bullet| > 1$) may appear and by applying the ON/OFF strategy, firing one transition faster may reduce the firing of another transition, and the overall time for reaching $\boldsymbol{m}_f$ may not be minimal. The following example shows a live and bounded system, but by applying the ON/OFF strategy, the final state is not reached.

Example 2. Assume we want to drive the system in Fig.1 to final state $\boldsymbol{m}_f = [0.5\ 0.5\ 0.5\ 0.5\ 0.2\ 0.4\ 1.4]^T$, and the firing rate of t_3 is 100, while the firing rates of other transitions are all set to 1. $\boldsymbol{\sigma} = [0.8\ 1.3\ 0.5\ 0\ 1\ 0\ 0]^T$ is a minimal firing count vector driving $\boldsymbol{m}_0$ (shown in the figure) to $\boldsymbol{m}_f$. Using this setting and applying the ON/OFF controller, $\boldsymbol{m}_f$ is not reached. This is because t_3 is fired much faster than t_2 ($\lambda_3 \gg \lambda_2$), and consequently all the tokens in p_5 will go to p_7, leading p_6 to be emptied in the limit. If p_6 gets emptied, t_5 can not be fired, and p_5 is also emptied in the limit. Notice that, if t_7 could be fired, p_6 could get some tokens, but according to the control law, it is not allowed because $\boldsymbol{\sigma}(t_7) = 0$.

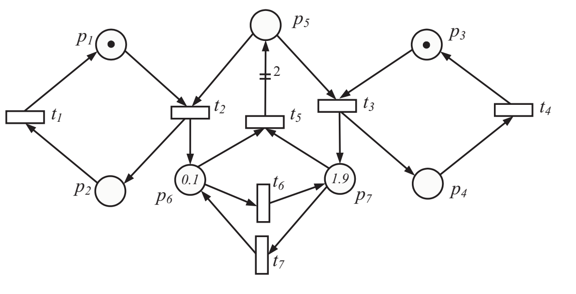


Fig. 1. A live and bounded CPN system

4. EXTENDED ON/OFF CONTROLLER

In this section, the ON/OFF controller that cannot be directly applied to general TCPN (see Ex.2 for an example when desired final marking is not reached with standard ON/OFF) is extended. Three extensions are proposed: ON/OFF+ controller, B-ON/OFF controller and ON/OFF-MPC controller.

4.1 ON/OFF-plus (ON/OFF+) controller

The problem of the ON/OFF controller may arise from the incorrect manner of solving the conflicts (e.g., between t_2 and t_3 in Fig.1). Two transitions t_a and t_b are in a (structural) conflict relation if ${}^\bullet t_a \cap {}^\bullet t_b \neq \emptyset$. Here let us define the *coupled conflict* relation as its transitive closure.

In order to overcome this problem, we will force the flows of transitions that are in coupled conflict relation to be proportional to the given firing count vector, while for the other transitions the ON/OFF strategy is applied.

The modified ON/OFF controller is shown in Alg.2 and we will call it ON/OFF+ controller.

Algorithm 2 ON/OFF+ controller

Input: $\boldsymbol{m}_0$, $\boldsymbol{m}_f$, $\boldsymbol{\sigma}$, $\boldsymbol{C}$, $\boldsymbol{\lambda}$, Θ

Output: $\boldsymbol{w}_0$, $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, ...

$k = 0$

while $\sum_{i=0}^{k-1} \boldsymbol{w}_i \cdot \Theta \neq \boldsymbol{\sigma}$ **do**

  Solve the following LPP :

$$\begin{aligned} & max\ \mathbf{1}^T \cdot \boldsymbol{w}_k \\ & \textbf{s.t.}\ \ \boldsymbol{m}_{k+1} = \boldsymbol{m}_k + \boldsymbol{C} \cdot \boldsymbol{w}_k \cdot \Theta \\ & \qquad \mathbf{0} \leq \boldsymbol{w}_k \cdot \Theta \leq \boldsymbol{\sigma} - \sum_{i=0}^{k-1} \boldsymbol{w}_i \cdot \Theta \\ & \qquad \boldsymbol{w}_k(t_j) \leq \lambda_j \cdot enab(t_j, \boldsymbol{m}_k), \forall t_j \in T \\ & \qquad \boldsymbol{m}_{k+1} \geq 0 \\ & \qquad \boldsymbol{w}_k(t_a) \cdot \boldsymbol{\sigma}(t_b) = \boldsymbol{w}_k(t_b) \cdot \boldsymbol{\sigma}(t_a) \\ & \qquad\quad \forall p_a, p_b, {}^\bullet p_a \cap {}^\bullet p_b \neq \emptyset \ \textbf{and} \\ & \qquad\quad \boldsymbol{\sigma}(t_a) > 0, \boldsymbol{\sigma}(t_b) > 0 \end{aligned} \tag{5}$$

  Apply $\boldsymbol{w}_k : \boldsymbol{m}_{k+1} = \boldsymbol{m}_k + \boldsymbol{C} \cdot \boldsymbol{w}_k \cdot \Theta$

  $k := k + 1$

end while

return $\boldsymbol{w}_0$, $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, ...

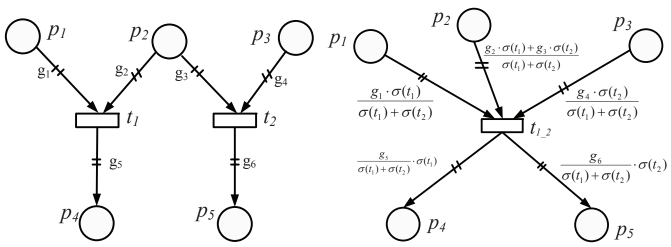
The procedure of ON/OFF+ controller is similar to the one of standard ON/OFF, except the last constraint of LPP (5), which means that, in any time step k, if transitions t_a and t_b are in conflict, the following will be forced: $\frac{\boldsymbol{w}_k(t_a)}{\boldsymbol{w}_k(t_b)} = \frac{\boldsymbol{\sigma}(t_a)}{\boldsymbol{\sigma}(t_b)}$. Notice that, only transitions with positive values in the corresponding firing count vector should be considered. In the following, it is assumed that transitions in a coupled conflict relation are enabled. Notice that this condition is rather weak since it is, for instance, verified by any system that can reach a positive marking.

In order to prove the convergence of Alg.2, it is first shown that the original system with the ON/OFF+ controller is equivalent to a CFPN system with a particular controller, i.e., the same state trajectory can be obtained. It is clear that $\boldsymbol{m}_f$ is reached in the CFPN system, implying that it is also reached in the original one.

Reduction Rule. Let $T_j = \{t_1, t_2, ..., t_n\}$ be a set of transitions of net $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post} \rangle$ that are in coupled conflict relation. These transitions will be fired proportionally according to a given firing count vector $\boldsymbol{\sigma}$, i.e., for any $t_a, t_b \in T_j$, $\boldsymbol{\sigma}(t_a), \boldsymbol{\sigma}(t_b) > 0$, if t_a is fired in an amount s_a, simultaneously, t_b is fired in an amount s_b, such that $\frac{s_a}{s_b} = \frac{\boldsymbol{\sigma}(t_a)}{\boldsymbol{\sigma}(t_b}$. Let $\bar{\sigma} = \sum_{t \in T_j} \boldsymbol{\sigma}(t)$, $\mathcal{N}$ is transformed to $\mathcal{N}' = \langle P, T', \boldsymbol{Pre}', \boldsymbol{Post}' \rangle$ in the following way:

(1) $T' = T \setminus T_j$
(2) Merge T_j to a new transition t_j, $T' = T' \cup \{t_j\}$
(3) $\forall p \in {}^\bullet T_j$, $\boldsymbol{Pre}'(p, t_j) = \sum\limits_{t \in p^\bullet} \boldsymbol{Pre}(p, t) \cdot \boldsymbol{\sigma}(t)/\bar{\sigma}$
(4) $\forall p \in T_j{}^\bullet$, $\boldsymbol{Post}'(p, t_j) = \sum\limits_{t \in {}^\bullet p} \boldsymbol{Post}(p, t) \cdot \boldsymbol{\sigma}(t)/\bar{\sigma}$

***Example** 3.* Let $\boldsymbol{m} > 0$ and $\boldsymbol{\sigma}(t_1) > 0$, $\boldsymbol{\sigma}(t_2) > 0$. Fig. 2 shows how to merge two conflicting transitions t_1 and t_2 to t_{1_2}.



(a) Original system (b) Transformed system

Fig. 2. Reduction rule: merging t_1 and t_2

***Proposition** 4.* Let $\mathcal{S} = \langle \mathcal{N}, \boldsymbol{m}_0 \rangle$, and $\mathcal{S}' = \langle \mathcal{N}', \boldsymbol{m}_0 \rangle$ be the transformed system from $\mathcal{S}$ by merging $T_j = \{t_1, t_2, ..., t_n\}$ to t_j by using the reduction rule. If in $\mathcal{S}$, the transitions in T_j are fired proportionally according to a given firing count vector $\boldsymbol{\sigma}$, and in $\mathcal{S}'$, transition t_j is fired in an amount equal to the sum of the firing amounts of transitions in T_j, then the same marking is reached in $\mathcal{S}$ and $\mathcal{S}'$.

Proof: It follows immediately by the definition of the reduction rule. ■

For example, let us consider place p_2 in Fig.2, and let t_1, t_2 be fired in amounts $s_1 = \alpha \cdot \boldsymbol{\sigma}(t_1)$, $s_2 = \alpha \cdot \boldsymbol{\sigma}(t_2)$, $\alpha > 0$. If $t_1(s_1)t_2(s_2)$ is fired in the original system the new marking of p_2 is:

$$\boldsymbol{m}_1(p_2) = \boldsymbol{m}_0(p_2) - g_2 \cdot \alpha \cdot \boldsymbol{\sigma}(t_1) - g_3 \cdot \alpha \cdot \boldsymbol{\sigma}(t_2)$$

In the transformed system, if $t_{1\_2}(s_1 + s_2)$ is fired, the new making of p_2 is:

$$\begin{aligned} \boldsymbol{m}_1'(p_2) &= \boldsymbol{m}_0(p_2) - (s_1 + s_2) \cdot \frac{g_2 \cdot \boldsymbol{\sigma}(t_1) + g_3 \cdot \boldsymbol{\sigma}(t_2)}{\boldsymbol{\sigma}(t_1) + \boldsymbol{\sigma}(t_2)} \\ &= \boldsymbol{m}_0(p_2) - \alpha \cdot (\boldsymbol{\sigma}(t_1) + \boldsymbol{\sigma}(t_2)) \cdot \frac{g_2 \cdot \boldsymbol{\sigma}(t_1) + g_3 \cdot \boldsymbol{\sigma}(t_2)}{\boldsymbol{\sigma}(t_1) + \boldsymbol{\sigma}(t_2)} \\ &= \boldsymbol{m}_1(p_2). \end{aligned}$$

Similarly, for the places p_1 and p_3, they also follow the equality of the markings in both systems.

***Corollary** 5.* If $\boldsymbol{m}_f > 0$ is reachable in $\mathcal{S}$ by firing $\boldsymbol{\sigma}$ from $\boldsymbol{m}_0 > 0$, then $\boldsymbol{m}_f$ is reachable in $\mathcal{S}'$ by firing $\boldsymbol{\sigma}'$, where:

$$\boldsymbol{\sigma}'(t_j) = \begin{cases} \sum\limits_{t \in T_j} \boldsymbol{\sigma}(t) & \text{if } t_j \text{ is obtained by merging} \\ & \text{a set of transitions } T_j \\ \boldsymbol{\sigma}(t_j) & \text{otherwise} \end{cases}$$

***Proposition** 6.* Let $\mathcal{S} = \langle \mathcal{N}, \boldsymbol{\lambda}, \Theta, \boldsymbol{m}_0 \rangle$ be a discrete-time TCPN system, with $\boldsymbol{m}_0 > 0$. Let $\boldsymbol{m}_f > 0$ be a reachable final marking, such that $\boldsymbol{m}_f = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}$. By applying the ON/OFF+ controller, $\boldsymbol{m}_f$ is reached in finite time.

Proof: Let $\mathcal{S}' = \langle \mathcal{N}', \boldsymbol{\lambda}', \Theta, \boldsymbol{m}_0 \rangle$ be the system transformed from $\mathcal{S}$ by merging all the conflicting transitions using the reduction rule (therefore $\mathcal{S}'$ is CFPN).

Assume there exists a controller $\mathcal{A}$ applied to $\mathcal{S}'$, with $\boldsymbol{w}_k'(t_j)$ the controlled flow in each time step k, such that: (1) if t_j is obtained by merging a set of transitions T_j in a coupled conflict relation, we have $\boldsymbol{w}_k'(t_j) = \sum_{t \in T_j} \boldsymbol{w}_k(t)$; (2) otherwise $\boldsymbol{w}_k'(t) = \boldsymbol{w}_k(t)$, where $\boldsymbol{w}_k(t)$ is flow of transition t in $\mathcal{S}$ that is controlled by using ON/OFF+ controller. Then, according to Proposition 4, the state trajectory of $\mathcal{S}'$ obtained by applying controller $\mathcal{A}$ is the same as in $\mathcal{S}$ obtained by applying ON/OFF+ controller. Therefore it is equivalent to prove that by applying controller $\mathcal{A}$ to $\mathcal{S}'$, $\boldsymbol{m}_f$ is reached in finite time.

This controller $\mathcal{A}$ always exists, because if the firing rate of t_j is big enough, case (1) can always be satisfied, using a positive control input $\boldsymbol{u}_k(t_j)$. For case (2) we simply use the ON/OFF strategy and the same firing rate as in $\mathcal{S}$.

Finally, let us notice that $\mathcal{S}'$ is a CFPN, so for sure controller $\mathcal{A}$ can drive $\mathcal{S}'$ to its final state in finite time [12], implying that by applying ON/OFF+ controller to $\mathcal{S}$, the final state is also reached. ■

***Remark** 7.* The results of Proposition 6 can be naturally extended to continuous-time TCPN by taking sampling period Θ tending to 0.

It should be noticed that for continuous timed system under infinite server semantics, once a place is marked it will take infinite time to be emptied (like the theoretical discharging of a capacitor in an electrical RC-circuit). Therefore, if there exist places that must be emptied during the trajectory to $\boldsymbol{m}_f$, the final marking is reached at the limit, i.e., in infinite time. If $\boldsymbol{m}_f > 0$ and using the proposed control method, this situation does not happen.

4.2 Balanced ON/OFF Controller (B-ON/OFF)

Using the ON/OFF+ controller, we can ensure that the final state $\boldsymbol{m}_f > 0$ is reached from $\boldsymbol{m}_0 > 0$ in finite time. But the main drawback of this method is also obvious. Since a set of transitions in coupled conflict relation are forced to be fired proportionally, the number of steps needed for firing $\boldsymbol{\sigma}$ is decided by the slower ones. Therefore, in extreme cases, when some of these transitions have very small flows, the whole system may also be slowed down.

***Example** 8.* Let us consider the simple (sub-)system in Fig.3, assuming that t_1, t_2 have the same firing rate equal to 1. Moreover, they are forced by a given $\boldsymbol{\sigma}$ to fire in the same amounts. It is obvious that the flow of t_2 is 100 times faster than the flow of t_1, but if t_1 and t_2 should fire proportionally according to $\boldsymbol{\sigma}$, t_2 is slowed down.

To overcome these extreme bad cases, first we fire the fast transitions and stop the slow ones for some time periods, expecting that the flows (speeds) of the slow transitions are increased, i.e., we will try to *balance* the fast and slow transitions. After that, the pure ON/OFF+ controller is applied until the final state is reached.

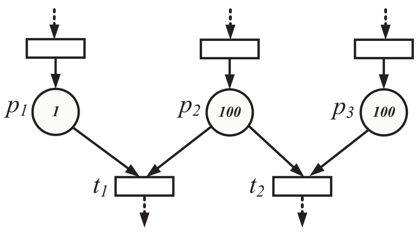


Fig. 3. Fast transitions may be slowed down

We will first shown how to classify the slow and fast transitions, then this balancing strategy is prensented.

Assume that the system is at marking $\boldsymbol{m}$, $\boldsymbol{w}$ is the corresponding flow, and let $\boldsymbol{\sigma}$ be the firing count vector that should be fired to reach $\boldsymbol{m}_f$. Then $s_j = \lceil \frac{\boldsymbol{\sigma}(t_j)}{\boldsymbol{w}(t_j)\cdot\Theta} \rceil$ can be viewed as the estimation of number of steps that transition t_j needs to be fired. For two transitions t_a and t_b, if $s_a > s_b$, then t_a is slower than t_b.

Let the estimation of the number of steps for t_j at $\boldsymbol{m}_0$ defined by:

$$s_j^0 = \lceil \frac{\boldsymbol{\sigma}(t_j)}{\lambda_j \cdot enab(t_j, \boldsymbol{m}_0) \cdot \Theta} \rceil \tag{6}$$

If $enab(t_j, \boldsymbol{m}_0) = 0$ then $s_j^0 = \infty$.

Let us consider again the system in Ex.8 and let $\boldsymbol{\sigma}(t_1) = \boldsymbol{\sigma}(t_2) = 10$, $\Theta = 0.01$. The initial estimation of the steps number is: $s_1^0 = 1000$, $s_2^0 = 10$.

Based on this initial estimation, for any given set of transitions T_c that are in coupled conflict relation, we will partition it into two subsets, the fast ones T_1 and the slow ones T_2, such that:

$$\begin{cases} T_1 \cap T_2 = \emptyset, T_1 \cup T_2 = T_c \\ \forall t_a \in T_1, t_b \in T_2, s_b^0/s_a^0 > d \\ \forall t_{a1}, t_{a2} \in T_1, 1/d \leq s_{a1}^0/s_{a2}^0 \leq d \end{cases} \tag{7}$$

where $d \geq 1$ is a design parameter used to classify slow and fast transitions.

From (7), the estimations of number of steps that the transitions in T_2 need to fire, are at least d times greater than the ones of transitions in T_1. If we fire the transitions in T_1 and T_2 proportionally, transitions in T_1 are obviously slowed down by the ones T_2.

Notice that, if the value of d is too big, all the transitions are put into T_1, then it is equivalent to applying ON/OFF+ controller directly. On the other side, if d is too small, all the transitions are put into T_2, then they are all stopped.

In the system shown in Ex.8, we can choose, for example, $d = 10$. Then the conflicting transition set $T_c = \{t_1, t_2\}$ is partitioned to $T_1 = \{t_2\}$ and $T_2 = \{t_1\}$.

Now let us consider that the system is in k^{th} time step at $\boldsymbol{m}_k$, and the firing count vector that has been fired $\boldsymbol{\sigma}'$, i.e., $\boldsymbol{m}_k = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}'$. The remained firing count vector should be fired is $\boldsymbol{\sigma}_k = \boldsymbol{\sigma} - \boldsymbol{\sigma}' \geq 0$. The estimation of number of steps for transition $t_j \in T_c$ at $\boldsymbol{m}_k$ is defined by:

$$s_j^k = \begin{cases} \lceil \frac{\boldsymbol{\sigma}_k(t_j)}{\boldsymbol{w}_k(t_j)\cdot\Theta} \rceil, & \text{if } t_j \in T_1 \\ \lceil \frac{\boldsymbol{\sigma}_k(t_j)}{\lambda_j \cdot enab(t_j, \boldsymbol{m}_k)\cdot\Theta} \rceil, & \text{if } t_j \in T_2 \end{cases}$$

where $\boldsymbol{w}_k(t_j)$ is the flow of transition t_j when the ON/OFF+ strategy is applied. Notice that, since the transitions in T_1 are fired proportionally, for any $t_j \in T_1$, the same estimation s_j^k is obtained, denoted by h^k.

For any $t_b \in T_2$, let $D_b^k = s_b^k / h^k$, it reflects the difference of the estimations between t_b and the faster transitions.

Let T_p be the set of persistent transitions (those transitions that are not in a coupled conflict relation), and T_c^i, $i = 1, 2, 3, ..., l$ be the sets of transitions in coupled conflict. Alg. 3 gives the control method: for transitions in T_p, ON/OFF strategy is always applied; for any $T_c^i = T_1^i \cup T_2^i$, the fast transitions T_1^i are fired proportionally using ON/OFF+ strategy, while every slow transition t_b in T_2^i is stopped until the following condition (C1) or (C2) is satisfied, then it is moved to T_1^i and also fired using ON/OFF+ strategy.

(C1) $D_b^k \leq d$.
(C2) $D_b^k \geq D_b^{k-1}$.

By stopping t_b while firing other transitions, tokens may be put into the input places of t_b, consequently increasing its flow, then t_b may become more balanced with those fast transitions, i.e., D_b^k is decreased. If D_b^k is keeping decreased, for sure in finite time, we will have condition (C1) satisfied, it means that t_b is already balanced with the fast transitions. If at one moment, D_b^k can not be decreased anymore, then condition (C2) is satisfied, i.e., transition t_b can not become more balanced with the fast ones. Therefore, one of this conditions will be satisfied in finite time. After that, there is no need to stop t_b and we should start to fire it using the pure ON/OFF+ strategy.

Now we will prove the convergence of this B-ON/OFF controller, i.e., the final state is reached in finite time.

Proposition 9. Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m}_0 \rangle$ be a TCPN system, with $\boldsymbol{m}_0 > 0$. Let $\boldsymbol{m}_f > 0$ be a reachable final marking, such that $\boldsymbol{m}_f = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}$. By applying the B-ON/OFF controller, $\boldsymbol{m}_f$ is reached in finite time.

Proof: For all the slow transitions, condition (C1) or (C2) will be satisfied in a finite number of steps, then the pure ON/OFF+ strategy is applied. Therefore, we only need to prove that when the pure ON/OFF+ controller starts to be applied, the system is in a state $\boldsymbol{m} > 0$ and $\boldsymbol{m}_f$ is reachable from $\boldsymbol{m}$.

Since the accumulative firing counts of transitions are upper bounded by $\boldsymbol{\sigma}$, then we have $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}'$, $0 \leq \boldsymbol{\sigma}' \leq \boldsymbol{\sigma}$. Since $\boldsymbol{\sigma} - \boldsymbol{\sigma}' \geq 0$ and $\boldsymbol{m}_f = \boldsymbol{m} + \boldsymbol{C} \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}')$, $\boldsymbol{m} > 0$, $\boldsymbol{m}_f$ is reachable from $\boldsymbol{m}$ [3]. ■

4.3 ON/OFF-MPC controller

MPC for TCPN is proposed in [5], where in each time step the following optimization problem is solved:

$$\begin{aligned} \min \quad & J(\boldsymbol{m}_k, N) \\ s.t.: \quad & \boldsymbol{m}_{k+j+1} = \boldsymbol{m}_{k+j} + \Theta \cdot C \cdot \boldsymbol{w}_{k+j}, & \text{(8a)} \\ & \boldsymbol{G} \cdot \begin{bmatrix} \boldsymbol{w}_{k+j} \\ \boldsymbol{m}_{k+j} \end{bmatrix} \leq 0, j = 0, ..., N-1 & \text{(8b)} \\ & \boldsymbol{w}_{k+j} \geq 0, j = 0, ..., N-1 & \text{(8c)} \end{aligned}$$

Algorithm 3 B-ON/OFF Controller

Input: $\boldsymbol{m}_0, \boldsymbol{m}_f, \boldsymbol{\sigma}, \boldsymbol{C}, \boldsymbol{\lambda}, \Theta,\ d\ , T_p,\ T_c^i,\ i=1,2,3...,l$
Output: $\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots$

Partition every T_c^i *into* T_1^i *and* T_2^i, $i=1,2,...,l$
$k=0$
while $\sum_{i=0}^{k-1} \boldsymbol{w}_i \cdot \Theta \neq \boldsymbol{\sigma}$ **do**
  Obtain $\boldsymbol{w}_k(t_j)|_{\forall t_j \in T_p}$: *applying ON/OFF to* T_p
  for $i=1$ **to** l **do**
    Stop transitions in T_2^i : $\boldsymbol{w}_k(t_j)|_{\forall t_j \in T_2^i} = 0$
    Obtain $\boldsymbol{w}_k(t_j)|_{\forall t_j \in T_1^i}$: *applying ON/OFF+*
  end for
  Apply $\boldsymbol{w}_k : \boldsymbol{m}_{k+1} = \boldsymbol{m}_k + \boldsymbol{C} \cdot \boldsymbol{w}_k \cdot \Theta$
  $\boldsymbol{\sigma}_{k+1} = \boldsymbol{\sigma} - \sum_{i=0}^{k} \boldsymbol{w}_i \cdot \Theta$
  for $i=1$ **to** l **do**
    if $T_2^i \neq \emptyset$ **then**
      Compute $\boldsymbol{w}_{k+1}(t_a), t_a \in T_1^i$
      $h^{k+1} = \boldsymbol{\sigma}_{k+1}(t_a)/(\boldsymbol{w}_{k+1}(t_a) \cdot \Theta)$
      for each $t_b \in T_2^i$ **do**
        $s_b^{k+1} = \boldsymbol{\sigma}(t_b)/(\lambda_b \cdot eanb(t_b, \boldsymbol{m}_{k+1}) \cdot \Theta)$
        $D_b^{k+1} = s_b^{k+1}/h^{k+1}$
        if $D_b^{k+1} \leq d$ **or** $D_b^{k+1} \geq D_b^k$ **then**
          $T_1^i = T_1^i \cup \{t_b\}$
          $T_2^i = T_2^i \setminus \{t_b\}$
        end if
      end for
    end if
  end for
  $k=k+1$
end while
return $\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots$

where $J(\boldsymbol{m}_k, N)$ may be a linear or quadratic objective function, while $\boldsymbol{G}$ is a particular matrix deduced from the net structure and (8b) gives the constraint on firing flows. For example, in the quadratic case, $J(\boldsymbol{m}_k, N)$ may be in the form of:

$$\begin{aligned} J(\boldsymbol{m}_k, N) = & (\boldsymbol{m}_{k+N} - \boldsymbol{m}_f)' \cdot \boldsymbol{Z} \cdot (\boldsymbol{m}_{k+N} - \boldsymbol{m}_f) \\ & + \sum_{j=0}^{N-1} [(\boldsymbol{m}_{k+j} - \boldsymbol{m}_f)' \cdot \boldsymbol{Q} \cdot (\boldsymbol{m}_{k+j} - \boldsymbol{m}_f) \\ & + (\boldsymbol{w}_{k+j} - \boldsymbol{w}_f)' \cdot \boldsymbol{R} \cdot (\boldsymbol{w}_{k+j} - \boldsymbol{w}_f)] \end{aligned} \quad (9)$$

MPC is usually used for optimizing trajectories satisfying certain objective functions. In our problem, the aim is to reach the desired state as soon as possible, i.e., minimizing the time. Even if it is difficult to obtain a minimum time control by using an MPC approach, we will consider this method for transitions in conflicts while for the others we will keep the ON/OFF controller. We will show that in some situations the number of steps to reach the desired final state is smaller than for ON/OFF+ or B-ON/OFF controller, but with higher computational complexity.

Let us denote by T_p the set of persistent transitions and T_c the set of transitions in any coupled conflict relation, $T_p \cap T_c = \emptyset, T_p \cup T_c = T$. The ON/OFF-MPC controller is shown in Alg.4.

The problem that should be solved in each time step k is:

Algorithm 4 ON/OFF-MPC controller

Input: $\boldsymbol{m}_0,\ \boldsymbol{m}_f,\ \boldsymbol{w}_f,\ \boldsymbol{\sigma},\ \boldsymbol{C},\ \boldsymbol{Z},\ \boldsymbol{Q},\ \boldsymbol{R}$
Output: $\boldsymbol{w}_0,\ \boldsymbol{w}_1,\ \boldsymbol{w}_2,\ \ldots$

Initiate: $T = T_p \cup T_c$
$k=0$
while $\boldsymbol{m}_k \neq \boldsymbol{m}_f$ **do**
  Solve LPP problem (10)
  Apply $\boldsymbol{w}_k : \boldsymbol{m}_{k+1} = \boldsymbol{m}_k + \boldsymbol{C} \cdot \boldsymbol{w}_k \cdot \Theta$
  $\boldsymbol{\sigma} = \boldsymbol{\sigma} - \boldsymbol{w}_k \cdot \Theta$
  $k=k+1$
end while
return $\boldsymbol{w}_0,\ \boldsymbol{w}_1,\ \boldsymbol{w}_2,\ \ldots$

$$\begin{aligned} \min \quad & J(\boldsymbol{m}_k, N) \\ s.t.: \quad & \boldsymbol{m}_{k+j+1} = \boldsymbol{m}_{k+j} + C \cdot \boldsymbol{w}_{k+j} \cdot \Theta, && (10a) \\ & \boldsymbol{G} \cdot \begin{bmatrix} \boldsymbol{w}_{k+j} \\ \boldsymbol{m}_{k+j} \end{bmatrix} \leq 0, j=0,...,N-1 && (10b) \\ & \boldsymbol{w}_{k+j} \geq 0, j=0,...,N-1 && (10c) \\ & \boldsymbol{w}_k(t_d) \leq \boldsymbol{\sigma}(t_d)/\Theta, \forall t_d \in T_c && (10d) \\ & \boldsymbol{w}_k(t_d) = \min\{\boldsymbol{\sigma}(t_d)/\Theta, \boldsymbol{f}_k(t_d)\}, \forall t_d \in T_p && (10e) \end{aligned}$$

where $\boldsymbol{f}_k(t_d)$ is the uncontrolled flow of transition t_d at $\boldsymbol{m}_k$. (10d) gives the upper bound of accumulative firing counts and (10e) makes sure that if a transition is persistent, it is fired using the ON/OFF strategy.

As defined in the unconstrained Linear Quadratic Regulation (LQR), let $\boldsymbol{K}$, $\boldsymbol{P}$ be the solution of (11) (see [8]), and let $\boldsymbol{Z} = \boldsymbol{P}$. Using results from the classical optimal control theory, we can guarantee the convergence to the desired state only if the set of feasible state and input vectors are bounded and the final state and input are interior points. If the final state/input is not an interior point, by forcing straight line trajectories, the asymptotic stability can also be achieved [5].

$$\begin{aligned} & \boldsymbol{K} = -(\boldsymbol{R} + \boldsymbol{B}^T \boldsymbol{P} \boldsymbol{B})^{-1} \boldsymbol{B}^T \boldsymbol{P} \boldsymbol{A} \\ & \boldsymbol{P} = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})^T \boldsymbol{P} (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}) + \boldsymbol{K}^T \boldsymbol{R} \boldsymbol{K} + \boldsymbol{Q} \end{aligned} \quad (11)$$

5. SIMULATIONS

In this section, the previous control methods are applied to a CPN model of an assembly system. The simulations are performed on a PC with Intel(R) Core(TM)2 Quad CPU Q9400 @ 2.66GHz, 3.24GB of RAM.

The system model in Fig. 4 represents an assembly system. There are two kinds of input raw materials stored in p_1 and p_2. The material A, B are first processed by $Proc\_A1$, then the obtained semi-products are further processed by $Proc\_A2$ and $Proc\_A3$. In the other processing line, material B is sequentially processed by $Proc\_B1$ and $Proc\_B2$. Then final produces are obtained after assembling all the semi-products.

It is assumed that the firing rate of t_2 is 4, while for the other transitions, are equal to 1. The simulations are performed under different setting, case:
1) $\Theta = 0.01$, $\boldsymbol{m_0} = [1\ 2\ 0.4\ 0.5\ 0\ 0\ 0\ 5\ 0]^T$, $\boldsymbol{\sigma} = [0.4\ 0\ 0.2\ 0.5\ 0.3\ 0.1\ 0]^T$, $\boldsymbol{m_f} = [0.6\ 1.8\ 0.7\ 0.2\ 0.1\ 0.4\ 0.2\ 4.7\ 0.3]^T$;
2) $\Theta = 0.01$, $\boldsymbol{m_0} = [1\ 2\ 0.001\ 0.5\ 0\ 0\ 0\ 5\ 0]^T$, $\boldsymbol{\sigma} = [0.4\ 0\ 0.2\ 0.5\ 0.3\ 0.1\ 0]^T$, $\boldsymbol{m_f} = [0.6\ 1.8\ 0.301\ 0.2\ 0.1\ 0.4\ 0.2\ 4.7\ 0.3]^T$;

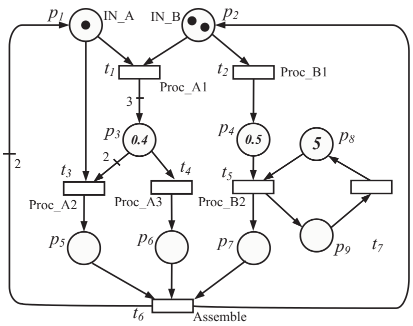


Fig. 4. The TCPN model of a assembly system.

3) $\Theta = 0.1$, $\boldsymbol{m}_0 = [1\ 2\ 0.4\ 0.5\ 0\ 0\ 0\ 5\ 0]^T$, $\boldsymbol{\sigma} = [2.1\ 1.7\ 1.9\ 2.2\ 2\ 1.8\ 0]^T$, $\boldsymbol{m}_f = [0.6\ 1.8\ 0.7\ 0.2\ 0.1\ 0.4\ 0.2\ 3\ 2]^T$.

Table.1 gives the number of time steps required for reaching $\boldsymbol{m}_f$ from $\boldsymbol{m}_0$ and the CPU times used for computing the corresponding control laws. The results of MPC are conducted using matrices $\boldsymbol{Q} = 1000 * \boldsymbol{I}_{|P|}$, $\boldsymbol{R} = 0.01 * \boldsymbol{I}_{|T|}$, and $\boldsymbol{Z} = \boldsymbol{P}$, with $\boldsymbol{P}$ solution of (11). In the B-ON/OFF controller, d is set to 10.

Table 1. Simulation Results

Methods	**Time steps**			**CPU time (ms)**		
Setting	1)	2)	3)	1)	2)	3)
ON/OFF+	124	954	85	60	368	51
B-ON/OFF	124	259	85	209	345	169
ON/OFF-MPC (N = 1)	120	158	203	1221	1418	1273
ON/OFF-MPC (N = 20)	118	149	90	139881	250248	86767

From the simulation results, it can be observed that the first two ON/OFF strategies have very law computational complexity. The B-ON/OFF controller does not improve the number of time steps in case 1) and 3), because the flows of conflicting transitions are similar, so it may make sense to directly fire them proportionally. But in the case 2), the B-ON/OFF controller is much better. As for the ON/OFF-MPC controller, the computational complexity is not very high when $N = 1$. Its number of time steps can be improved when greater horizon step is used, but consequently, the computing time is significantly increased. Finally, let us observe that, in this example, we may have "deadlock" when applying the standard ON/OFF controller.

Notice that we have shown the results of different methods for a particular example, but it does not indicate one method is definitely better than another in a general sense.

6. CONCLUSIONS

In this work, three ON/OFF strategy based extensions are presented. It is proved that they can drive a general CPN system to the desired final state in finite time. Some comparisons are also given. The advantage of these ON/OFF based controllers is the low computational complexity. As a future work, we will compare our methods with other control strategies, and consider how to identify the most suitable controller in different situations.

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