Marking homothetic monotonicity and fluidization of untimed Petri nets ⋆

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Abstract: The analysis of Discrete Event Systems suffer from the well known state explosion problem. A classical technique to overcome this problem is to relax the behaviour by partially removing the integrality constraints, thus dealing with hybrid or continuous systems. In the Petri nets framework, continuous net systems (technically hybrid systems) are the result of removing the integrality constraint in the firing of transitions. This relaxation may highly reduce the complexity of analysis techniques but may not preserve important properties of the original system. This paper deals with the basic operation of fluidization. More precisely, it aims at establishing conditions that a discrete system must satisfy so that a given property is preserved by the continuous system. These conditions will be mainly based on the here introduced *marking homothetic behaviours* of the system. The focus will be on logical properties as boundedness, deadlock-freeness, liveness and reversibility. Furthermore, testing homothetic montonicity of some properties in the discrete systems will be considered.

Keywords: fluidization; continuous Petri nets; homothetic monotonicity;

1. INTRODUCTION

Petri nets [1, 2], as other formalisms for Discrete Event Dynamic Systems (DEDs), suffer from the *state explosion problem*. Such a problem renders analysis techniques based on exhaustive enumeration computationally infeasible for large system populations. A promising approach to overcome this difficulty is to relax the original discrete model by explicitly removing the integrality constraint in the firing of transitions. This process is known as *fluidization* being its result a continuous Petri net (PN) in which both the firing amounts of transitions and the marking of places are non-negative real quantities [7, 9].

Continuous PNs allow the use of some polynomial time complexity techniques for several analysis purposes [9]. Unfortunately, continuous nets may not always preserve important properties of the discrete model [8]. For this reason, it is crucial to study which discrete PN systems can be "succesfully" fluidified and which ones not.

At first glance, the simple way in which the basic definitions of discrete models are extended to continuous ones may make us naively think that their behaviour will be similiar. However, the behaviour of the continuous model can be completely different just because the integrality constraint has been dropped. In other words, not all DEDs can be satisfactorily fluidified. Consider, for instance, the net system in Fig.1. If considered as discrete, the system is deadlock-free: from $\boldsymbol{m}_0 = (3, 0)$, both t_2 and t_1 can be fired

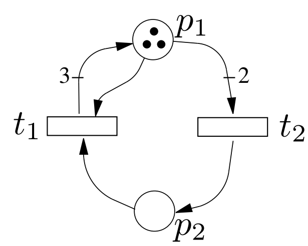

Fig. 1. Not homothetically deadlockfree PN [8]

alternatively, and no deadlock can be reached. However, if considered as continuous, transition t_2 can be fired in an amount of 1.5 from $\boldsymbol{m}_0$, leading to a deadlock marking $\boldsymbol{m}_d = (0, 1.5)$.

Notice that deadlock-freeness of the discrete system in Fig.1 highly depends on its initial marking. In fact, if the initial marking is doubled, i.e., we consider $\boldsymbol{m}'_0 = (6,\ 0)$, then the system deadlocks by firing t_2 an amount of 3.

Let us now consider the PN in Fig.2 (a), which exhibits a different behaviour. Considered as discrete, this PN is deadlockfree for $\boldsymbol{m}_0 = (2,\ 1,\ 0,\ 0,\ 0)$. Moreover, it is deadlockfree for an initial marking proportional to $\boldsymbol{m}_0$, i.e., $\boldsymbol{m}'_0 = k \cdot \boldsymbol{m}_0$, with $k \in \mathbb{N}$. We will say that the system is *homothetically* deadlockfree. When the PN system is fluidified, i.e., the PN system in Fig.2 (a) is a continuous system, it preserves the deadlock-freeness property.

The generalization of this fact is that, if a discrete system is deadlockfree with a certain $\boldsymbol{m}_0$, and it is also deadlockfree with a scaled initial marking ($k{\cdot}\boldsymbol{m}_0$ with $k \in \mathbb{N}^+$), then the system is said to be homothetically deadlock-free. We will exploit this idea to extract conditions for the preservation of properties when a discrete system is fluidified.

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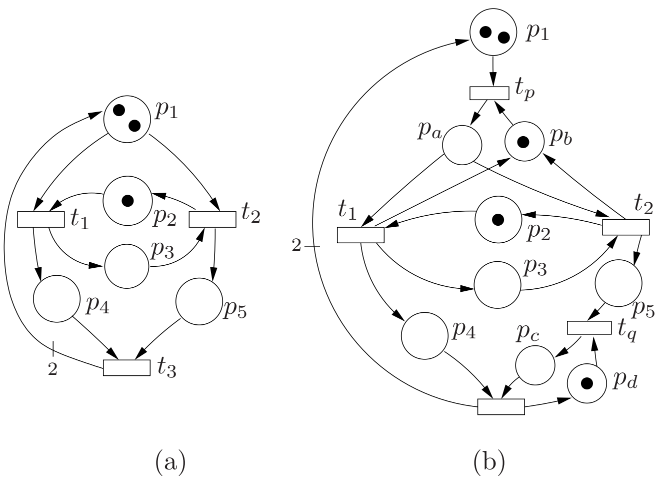

Fig. 2. Homothetically deadlockfree PN,$\boldsymbol{m}_0 = (2, 1, 0, 0, 0)$

This paper explores the kind of features a discrete net system must exhibit so that a given property is preserved when fluidization is applied. If focuses on classical properties as reversibility, boundedness, deadlock-freeness, liveness and reversibility. Furthermore, homothetic monotonicity of boundedness and deadlock-freeness properties in discrete Petri nets are studied.

This work is organized as follows. Section 2 recalls some definitions that will be used in the rest of the paper. Section 3 sets the main result: a property that exhibits homothetic monotonicity in a discrete PN system, will be preserved after fluidization. Section 5 studies whether a discrete PN is homothetic monotonous on deadlock-freeness property. Finally, Section 6 deals with some conclusions and open problems.

2. PRELIMINARY CONCEPTS AND DEFINITIONS

Some concepts used in the rest of the paper are defined here. In the following, it is assumed that the reader is familiar with Petri nets (see [1, 2] for a gentle introduction).

2.1 Continuous Petri nets

The Petri net structure is denoted $\mathcal{N}$:

Definition 1. A PN is a tuple $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post}\rangle$ where $P = \{p_1, p_2, ..., p_n\}$ and $T = \{t_1, t_2, ..., t_m\}$ are disjoint and finite sets of places and transitions, and $\boldsymbol{Pre}$, $\boldsymbol{Post}$ are $|P| \times |T|$ sized, natural valued, incidence matrices.

Given a Petri net and a marking, the discrete Petri net system can be defined:

Definition 2. A discrete PN system is a tuple $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ where $\mathcal{N}$ is the structure and $\boldsymbol{m}_0 \in \mathbb{N}^{|P|}$ is the initial marking.

In discrete PN systems, a transition t is *enabled* at $\boldsymbol{m}$ if for every $p \in {}^\bullet t$, $\boldsymbol{m}[p] \geq \boldsymbol{Pre}[p, t]$. It can be fired in any amount $\alpha \in \mathbb{N}$ such that $0 < \alpha \leq enab(t, m)$, where $enab(t, \boldsymbol{m}) = \min_{p\in {}^\bullet t} \lfloor \frac{\boldsymbol{m}[p]}{\boldsymbol{Pre}[p,t]} \rfloor$.

The main difference between continuous and discrete PNs is in the firing amounts and consequently in the marking, which in discrete PNs are restricted to be in the naturals, while in continuous PNs are relaxed into the non-negative real numbers. Thus, a *continuous* PN system is understood as a relaxation of a *discrete* one.

Definition 3. A continuous PN system is a tuple $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_C$ where $\mathcal{N}$ is the structure and $\boldsymbol{m}_0 \in \mathbb{R}^{|P|}_{\geq 0}$ is the initial marking.

In continuous systems, a transition t is *enabled* at $\boldsymbol{m}$ if for every $p \in {}^\bullet t$, $\boldsymbol{m}[p] > 0$. It can be fired in any amount $\alpha \in \mathbb{R}$ such that $0 < \alpha \leq enab(t, m)$, where $enab(t, \boldsymbol{m}) = \min_{p\in {}^\bullet t}\{\frac{\boldsymbol{m}[p]}{\boldsymbol{Pre}[p,t]}\}$.

In both discrete and continuous PN systems, the firing of t in a certain amount α leads to a new marking $\boldsymbol{m'}$, and denoted as $\boldsymbol{m} \xrightarrow{\alpha t} \boldsymbol{m'}$. It holds $\boldsymbol{m'} = \boldsymbol{m} + \alpha \cdot \boldsymbol{C}[P, t]$, where $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the token flow matrix (incidence matrix if $\mathcal{N}$ is self-loop free) and $C[P, t]$ denotes the column t of the matrix C. Hence, $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}$, the state (or fundamental) equation summarizes the way the marking evolves; where $\boldsymbol{\sigma}$ is the firing count vector associated to the fired sequence. Right and left natural annullers of the token flow matrix are called T- and P-semiflows, respectively. When $\exists \mathbf{y} > \mathbf{0}$, $\mathbf{y} \cdot \boldsymbol{C} = \mathbf{0}$, the net is said to be *conservative*, and when $\exists \mathbf{x} > \mathbf{0}$, $\boldsymbol{C} \cdot \mathbf{x} = \mathbf{0}$, the net is said to be *consistent*. A set of places Θ is a *trap* if $\Theta^\bullet \subseteq {}^\bullet\Theta$. While a set of places Σ is a *siphon* if ${}^\bullet\Sigma \subseteq \Sigma^\bullet$.

The set of all the reachable markings of $\langle \mathcal{N}, \boldsymbol{m}_0\rangle_D$ is denoted as reachability set, $\mathrm{RS}_D(\mathcal{N}, \boldsymbol{m}_0)$.

Definition 4. $\mathrm{RS}_D(\mathcal{N}, \boldsymbol{m}_0) = \{\boldsymbol{m} \mid \exists\ \sigma = t_{\gamma_1} \dots t_{\gamma_k}$ *such that* $\boldsymbol{m}_0 \xrightarrow{t_{\gamma_1}} \boldsymbol{m}_1 \xrightarrow{t_{\gamma_2}} \boldsymbol{m}_2 \cdots \xrightarrow{t_{\gamma_k}} \boldsymbol{m}_k = \boldsymbol{m}\}$.

In continuous PNs, two sets of reachable markings are considered: one denoted as $\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$, that contains all the markings that are reachable with finite firing sequences; and the lim-reachability set, denoted as lim-$\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$, that contains all the markings that are reachable either with a finite or with an infinite firing sequence.

Definition 5. $\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0) = \{\boldsymbol{m} \mid \exists\ \sigma = \alpha_1 t_{\gamma_1} \dots \alpha_k t_{\gamma_k}$ s.t. $\boldsymbol{m}_0 \xrightarrow{\alpha_1 t_{\gamma_1}} \boldsymbol{m}_1 \xrightarrow{\alpha_2 t_{\gamma_2}} \boldsymbol{m}_2 \cdots \xrightarrow{\alpha_k t_{\gamma_k}} \boldsymbol{m}_k = \boldsymbol{m}$ where $\alpha_i \in \mathbb{R}_{>0}, \forall i \in \{1..k\}\}$.

Definition 6. lim-$\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$=$\{\boldsymbol{m} \mid \exists\sigma = \alpha_1 t_{\gamma_1} ... \alpha_i t_{\gamma_i} ...$ s.t. $\boldsymbol{m}_0 \xrightarrow{\alpha_1 t_{\gamma_1}} \boldsymbol{m}_1 \xrightarrow{\alpha_2 t_{\gamma_2}} \boldsymbol{m}_2 \dots \boldsymbol{m}_{i-1} \xrightarrow{\alpha_i t_{\gamma_i}} \boldsymbol{m}_i \dots$ and $lim_{i\to\infty}\ \boldsymbol{m}_i = \boldsymbol{m}$ where $\alpha_i \in \mathbb{R}_{>0}, \forall i > 0\ \}$.

Notice that it holds $\mathrm{RS}_D(\mathcal{N}, \boldsymbol{m}_0) \subseteq \mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0) \subseteq$ lim-$\mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$.

An interesting consequence of the definition of continuous firings is Property 7, a well known result in [8].

Property 7. If $\boldsymbol{m} \in \mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$ then $\alpha \cdot \boldsymbol{m} \in \mathrm{RS}_C(\mathcal{N}, \alpha \cdot \boldsymbol{m}_0)$, $\forall\alpha \in \mathbb{R}_{>0}$.

Finally, let us define $||v||$ as the infinite norm (or maximum norm) of the vector v: $||v|| = \max\ \{|v_1|, \cdots, |v_n|\}$. It will be used to compare two markings.

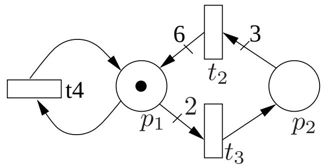

Fig. 3. No monotonic deadlockfree PN system, $\boldsymbol{m}_0 = (1, 0)$

2.2 Petri net properties

Some interesting properties, often required for real systems, are defined below for continuous PNs: boundedness (B); deadlock-freeness (DF), lim-deadlockfreeness (lim-DF); liveness (L), lim-liveness (lim-L), reversibility (R) and lim-reversibility (lim-R).

Definition 8. **(lim-)boundedness**. A system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is (lim-)bounded if there exists $b \in \mathbb{R}^+$ such that for all $\boldsymbol{m} \in$ (lim-)RS$_C(\mathcal{N}, \boldsymbol{m}_0)$, $\forall p \in P, \boldsymbol{m}[p] \leq b$.

Definition 9. **(lim-)deadlock-freeness**. A system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is (lim-)deadlockfree if $\forall \boldsymbol{m} \in$ (lim-)RS$_C(\mathcal{N}, \boldsymbol{m}_0)$, $\exists t \in T$ such that t is enabled at $\boldsymbol{m}$.

Definition 10. **(lim-)liveness**. A system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is (lim-)live if for every transition t and for every marking $\boldsymbol{m} \in$ (lim-)RS$_C(\mathcal{N}, \boldsymbol{m}_0)$ there exists $\boldsymbol{m}$' $\in$ (lim-)RS$_C(\mathcal{N}, \boldsymbol{m})$ such that t is enabled at $\boldsymbol{m}$'.

Definition 11. **(lim-)reversibility**. A system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is (lim-)reversible if for any marking $\boldsymbol{m} \in$ (lim-)RS$_C(\mathcal{N}, \boldsymbol{m}_0)$ it holds that $\boldsymbol{m}_0 \in$ (lim-)RS$_C(\mathcal{N}, \boldsymbol{m})$.

Due to the fact that RS$_C(\mathcal{N}, \boldsymbol{m}_0) \subseteq$ lim-RS$_C(\mathcal{N}, \boldsymbol{m}_0)$, it is straightforward that: (1) if a continuous system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is lim-DF, then it is also DF; (2) if $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is lim-L, then it is also L and (3) if $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is lim-R, then it is also R.

Finally, *marking monotonicity* and *marking homothetic monotonicity* are defined below for a behavioural property Π (where Π can be boundedness, deadlock-freeness, etc).

In this work, we will use the concepts of *marking monotonicity* and *marking homothetic monotonicity*, in which the initial marking of the PN system is scaled. Here, only *marking* monotonicity will be considered. However, different concepts of *monotonicity* are considered in other works, such us monotonicity w.r.t. the firing rates in timed PN systems.

Definition 12. **Monotonicity**. Given a system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$, a behavioural property Π is *monotonic* w.r.t. $\boldsymbol{m}_0$ if:
Π holds in $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D \Longrightarrow \Pi$ holds in $\langle \mathcal{N}, \boldsymbol{m}'_0 \rangle_D$ for every $\boldsymbol{m}'_0 \geq \boldsymbol{m}_0$.

For example, considering the PN system on Fig. 3, deadlock-freeness property is not monotonic for $\boldsymbol{m}_0 =$ (1,0), because it is deadlockfree for $\boldsymbol{m}_0 =$ (1,0), but it deadlocks for $\boldsymbol{m}'_0 =$ (2,0) and also for $\boldsymbol{m}'''_0 =$ (4,0); and it is deadlockfree for $\boldsymbol{m}''_0 =$ (3,0), $\boldsymbol{m}''_0 =$ (5,0) and higher initial markings.

Definition 13. **Homothetic monotonicity**. Given a system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$, a behavioural property Π is *homothetically monotonic* (for short, *homothetic*) w.r.t $\boldsymbol{m}_0$ if:
Π holds in $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D \Longrightarrow \Pi$ holds in $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0 \rangle_D$, $\forall k \in \mathbb{N}^+$.

Homothetic monotonicity of DF can be illustrated with the example in Fig. 2 (a). The discrete net system is DF for the initial marking $\boldsymbol{m}_0 = (2, 1, 0, 0, 0)$, and for any proportional initial marking $k \cdot \boldsymbol{m}_0$, i. e., it is homothetic DF. Nevertheless, the system is not *monotonic* DF for $\boldsymbol{m}_0$ (for example, for $\boldsymbol{m}'_0 = (2, 2, 0, 0, 0)$ it deadlocks, where $\boldsymbol{m}'_0 \geq \boldsymbol{m}_0$).

Notice that monotonicity is more restrictive than homothetic monotonicity, i.e., if Π is monotonic then Π is also homothetically monotonic.

Some classical results about the study of monotonicity of certain properties are the rank theorem [3] or properties defined in the traps and the siphons of the system [4]

3. PROPERTY PRESERVATION BY FLUIDIZATION

The aim of this section is to set certain conditions that a discrete PN system has to fulfill in order to preserve a certain property after being fluidified to a continuous PN system. It will be proved that, given a property Π which exhibits homothetic monotonicity in $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$, this property will be preserved when considered continuous (i.e. $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$). We will focus on the well-known properties considered in Section 2.2.

First, two technical results (Lemmas 15 and 16) about reachability are presented. They will be needed for the results on the following subsections.

3.1 Reachability

Let us introduce an additional reachability set that will be used in this work. It is the *rational reachability set* (RS$_Q(\mathcal{N}, \boldsymbol{m}_0)$), which is the set of markings that can be reached from the initial marking considering only firings in the set of rational numbers ($\mathbb{Q}$). Given this definition, we will denote $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_Q$ the net system in which only rational amounts are fired by the transitions.

Definition 14. RS$_Q(\mathcal{N}, \boldsymbol{m}_0) = \{\boldsymbol{m} \mid \exists\, \sigma = \alpha_1 t_{\gamma_1} \ldots \alpha_k t_{\gamma_k}$ s.t. $\boldsymbol{m}_0 \xrightarrow{\alpha_1 t_{\gamma_1}} \boldsymbol{m}_1 \cdots \xrightarrow{\alpha_k t_{\gamma_k}} \boldsymbol{m}_k = \boldsymbol{m}$ where $\alpha_i \in \mathbb{Q}_{>0}$, $\forall i \in \{1..k\}\}$

For any marking $\boldsymbol{m}$ reachable in the rational net system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_Q$, there exists a k such that a scaled marking $k \cdot \boldsymbol{m}$ is reachable in the discrete system $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0 \rangle_D$.

Lemma 15. Given a continuous and a discrete systems with the same structure $\mathcal{N}$ and the same initial marking $\boldsymbol{m}_0 \in \mathbb{N}$,
$\boldsymbol{m} \in$ RS$_Q(\mathcal{N}, \boldsymbol{m}_0) \Longrightarrow \exists k \in \mathbb{N} \mid k \cdot \boldsymbol{m} \in$ RS$_D(\mathcal{N}, k \cdot \boldsymbol{m}_0)$.

Proof. Let us suppose $\boldsymbol{m} \in$ RS$_Q(\mathcal{N}, \boldsymbol{m}_0)$, i.e., $\boldsymbol{m}_0 \xrightarrow{\sigma} \boldsymbol{m}$, where $\sigma = \alpha_1 t_{\gamma_1} \cdots \alpha_n t_{\gamma_n}$, and $\alpha_i \in \mathbb{Q}$, $\forall i \in \{1 \cdots n\}$. Because each α_i is a rational amount, it can be considered as its irreducible fraction: $\alpha_i = \frac{n_i}{d_i}$.

We can multiply the rational sequence σ by the l.c.m. (least common multiple) of the denominators of the irre-

ducible fractions, to obtain a sequence σ' in the naturals: $\sigma' = k \cdot \sigma$, where $k = l.c.m.(d_i \mid \frac{n_i}{d_i} = \alpha_i, \forall \alpha_i \in \sigma_m)$.
It holds $\alpha'_i \in \mathbb{N}$ for every α'_i in σ'. Because of the properties of the continuous PN, the initial marking ($\boldsymbol{m}_0$), the firing sequence (σ) and the resulting marking ($\boldsymbol{m}$) can be multiplied by k in the continuous PN: $k \cdot \boldsymbol{m}_0 \xrightarrow{k\cdot\sigma} k \cdot \boldsymbol{m}$. Because it is a natural sequence fireable in the continuous PN, $\sigma' = k \cdot \sigma$ is fireable from the same marking, $k \cdot \boldsymbol{m}_0$, also in the discrete system: $k \cdot \boldsymbol{m}_0 \xrightarrow{\sigma'} k \cdot \boldsymbol{m}$. □

Let us proof that, for any marking $\boldsymbol{m}$ reachable with a real firing sequence, another marking $\boldsymbol{m}'$ exists that is reachable with rational firings, such that it is as close to $\boldsymbol{m}$ as desired, and the set of empty places coincide.

Lemma 16. For every $\boldsymbol{m} \in RS_C(\mathcal{N}, \boldsymbol{m}_0)$, and every $\varepsilon > 0$, there exists $\boldsymbol{m}' \in RS_Q(\mathcal{N}, \boldsymbol{m}_0)$ such that:

- $||\boldsymbol{m}' - \boldsymbol{m}|| < \varepsilon$ and
- $(m'[p] = 0 \Leftrightarrow m[p] = 0)$.

Proof. Given $\sigma = \alpha_1 t_{\gamma_1} \alpha_2 t_{\gamma_2} \cdots \alpha_n t_{\gamma_n}$, with $\alpha_i \in \mathbb{R}, \forall i \in \{1..n\}$ such that $\boldsymbol{m}_0 \xrightarrow{\sigma} \boldsymbol{m}$; then for any $\varepsilon > 0$, we will build a firing sequence $\sigma' = \alpha'_1 t_{\gamma_1} \alpha'_2 t_{\gamma_2} \cdots \alpha'_n t_{\gamma_n}$, $\alpha'_i \in \mathbb{Q}$, $\forall i \in \{1..n\}$, such that $\boldsymbol{m}_0 \xrightarrow{\sigma'} \boldsymbol{m}'$. Where $||\boldsymbol{m}' - \boldsymbol{m}|| < \varepsilon$ and $(m'[p] = 0 \Leftrightarrow m[p] = 0)$. It will be proof by induction on the length of the sequence σ: $|\sigma| = k$.

- Base case ($|\sigma| = 1$).
Let $\sigma = \alpha_1 t_{\gamma_1}$. Then, a $\alpha'_1 \in \mathbb{Q}$, has to be chosen. The firing of $\alpha_1 t_{\gamma_1}$ yields $\boldsymbol{m} = \boldsymbol{m}_0 + C[P, t_{\gamma_1}]\alpha_1$; and the firing of a given $\alpha'_1 t_{\gamma_1}$ yields $\boldsymbol{m}' = \boldsymbol{m}_0 + C[P, t_{\gamma_1}]\alpha'_1$. Subtracting both equations and considering its norm, we obtain $||\boldsymbol{m} - \boldsymbol{m}'|| = ||C[P, t_{\gamma_1}](\alpha_1 - \alpha'_1)||$. Since all the elements in C are finite numbers, a rational $\alpha'_1 \in \mathbb{Q}$ close enough to α_1 can be chosen to satisfy $||\boldsymbol{m} - \boldsymbol{m}'|| < \varepsilon$. Moreover, since $\boldsymbol{m}_0 \in \mathbb{N}$, if the firing of α_1 emptied some places, then $\alpha_1 \in \mathbb{Q}$ and $\alpha'_1 = \alpha_1$ can be chosen. Otherwise (if no place has been emptied), then $\alpha'_1 \in \mathbb{Q}$ as close as desired to α_1 can be chosen that does not empty places.
- Inductive hypothesis ($|\sigma| = k$)
Given $\sigma = \alpha_1 t_{\gamma_1} \alpha_2 t_{\gamma_2} \cdots \alpha_k t_{\gamma_k}$, such that $\boldsymbol{m}_0 \xrightarrow{\sigma} \boldsymbol{m}_k$; there exists $\sigma' = \alpha'_1 t_{\gamma_1} \alpha'_2 t_{\gamma_2} \cdots \alpha'_k t_{\gamma_k}$, such that $\alpha'_i \in \mathbb{Q}, \forall i \in \{1..k\}$ and $\boldsymbol{m}_0 \xrightarrow{\sigma'} \boldsymbol{m}'_k$, where $||\boldsymbol{m}'_k - \boldsymbol{m}_k|| < \varepsilon$ and $(m'[p] = 0 \Leftrightarrow m[p] = 0)$.
- Inductive step ($|\sigma| = k + 1$)
Let consider the $k + 1$ firing. We can distinguish two cases:
(a) The firing of $\alpha_{k+1} t_{\gamma_{k+1}}$ does not empty places in $^{\bullet}t_{\gamma_{k+1}}$. Then, it holds that $\boldsymbol{m} = \boldsymbol{m}_k + C[P, t_{\gamma_{k+1}}]\alpha_{k+1}$, and $\boldsymbol{m}' = \boldsymbol{m}'_k + C[P, t_{\gamma_{k+1}}]\alpha'_{k+1}$. Again, subtracting both equations and considering its norm, we obtain $||\boldsymbol{m} - \boldsymbol{m}'|| = ||(\boldsymbol{m}_k - \boldsymbol{m}'_k) + C[P, t_{\gamma_{k+1}}](\alpha_{k+1} - \alpha'_{k+1})||$. We have to force that $||\boldsymbol{m} - \boldsymbol{m}'|| < \varepsilon$. Given that $\boldsymbol{m}_k$ and $\boldsymbol{m}'_k$ fulfill the inductive hypothesis, the quantity $||\boldsymbol{m}_k - \boldsymbol{m}'_k||$ can be as small as desired. Moreove, since the elements of the matrix C are finite numbers, a rational α'_{k+1} close to α_{k+1} can be chosen such that $||C[P, t_{\gamma_{k+1}}](\alpha_{k+1} - \alpha'_{k+1})||$ is as small as desired and no places in $^{\bullet}t_{\gamma_{k+1}}$ are emptied.
(b) The firing of $\alpha_{k+1} t_{\gamma_{k+1}}$ empties places in $^{\bullet}t_{\gamma_{k+1}}$. Then, we take $\alpha'_{k+1} = enab(t_{\gamma_{k+1}}, \boldsymbol{m}'_k)$, in order to empty the same input places. The amount α'_{k+1} is in $\mathbb{Q}$, because $\boldsymbol{m}'_k$ (and hence the enabling degree) is rational. Since $\boldsymbol{m}_k$ and $\boldsymbol{m}'_k$ fulfill the inductive hypothesis, they can be as close as desired. Thus, the firing of α'_{k+1} empties the same places than α_{k+1}, and $\boldsymbol{m}_{k+1}$ and $\boldsymbol{m}'_{k+1}$ can be as close as desired. □

Given Lemmas 15 and 16, some theorems are obtained for the preservation of boundedness; and deadlock-freeness, liveness, and reversibility.

3.2 Boundedness

In this section it is proved that given an initial marking, a PN system is homothetically bounded as discrete iff it is bounded as continuous.

Theorem 17. $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$ is homotethic bounded $\Longleftrightarrow$ $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is bounded.

Proof. ($\Longrightarrow$) Let us suppose the $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is unbounded, i.e., $\forall b \in \mathbb{N}\ \exists p \in P\ \exists \boldsymbol{m} \in \mathrm{RS}_C(\mathcal{N}, \boldsymbol{m}_0)$ s.t. $m[p] > b$. If $\boldsymbol{m}$ is not in $RS_Q(\mathcal{N}, \boldsymbol{m}_0)$, but $m[p] > b$, because of Lemma 16, $\forall \varepsilon > 0$, $\exists \boldsymbol{m}'$ s.t. $||\boldsymbol{m}' - \boldsymbol{m}|| < \varepsilon$, so we can find another $\boldsymbol{m}' \in RS_C(\mathcal{N}, \boldsymbol{m}_0)$ as near to $\boldsymbol{m}$ as desired, such that also $m'[p] > b$. And because of Lemma 15, if $\boldsymbol{m}' \in RS_Q(\mathcal{N}, \boldsymbol{m}_0)$, then $\exists k \in \mathbb{N}$ s. t. $k \cdot \boldsymbol{m}' \in \mathrm{RS}_D(\mathcal{N}, k \cdot \boldsymbol{m}_0)$. Hence, in the discrete PN, $\forall b \in \mathbb{N}$, $\exists k \cdot \boldsymbol{m} \in \mathrm{RS}_D(\mathcal{N}, k \cdot \boldsymbol{m}_0)$ s.t. $k \cdot m[p] > b$. Consequently, $\exists k \in \mathbb{N}$ s.t. the discrete system $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0 \rangle_D$ is unbounded.

($\Longleftarrow$) Let us suppose $\exists k \in \mathbb{N}$ s.t. the discrete system $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0 \rangle_D$ is not bounded. It means $\forall b \in \mathbb{N}$, $\exists p \in P$ $\exists \boldsymbol{m} \in \mathrm{RS}_D(\mathcal{N}, k \cdot \boldsymbol{m}_0)$ s.t. $m[p] > b$.

If $\boldsymbol{m} \in \mathrm{RS}_D(\mathcal{N}, k \cdot \boldsymbol{m}_0)$, then also $\boldsymbol{m} \in \mathrm{RS}_C(\mathcal{N}, k \cdot \boldsymbol{m}_0)$. Because of Property 7, for each marking $\boldsymbol{m} \in \mathrm{RS}_C(\mathcal{N}, k \cdot \boldsymbol{m}_0)$, the marking $\boldsymbol{m}' = \frac{\boldsymbol{m}}{k}$ is reachable in $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$.

Given that $m[p] > b$ then also $m'[p] = \frac{m[p]}{k} > \frac{b}{k}$, and it holds for all the reals, i.e., $\forall b \in \mathbb{R}^+$. Thus, for every $c = \frac{b}{k} \in \mathbb{R}^+$, the proposition holds. Consequently, $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is also not bounded. □

3.3 Deadlock-freeness

In Section 3.3 and Section 3.4, some results about the preservation of a homothetic property Π when the system is fluidified are presented. The results and its proofs are analogous in the case of three properties Π: deadlock-freeness, liveness and reversibility. For a didactic purpose, in this section the results are explained for DF property. In the following section, the results are extended to Π, and enunciated for liveness and reversibility.

As previously defined (Section 2.2), DF in continuous PN consider only the markings that are reachable with *finite* firing sequences (reachability); while lim-DF consider also *infinite* firing sequences (lim-reachability). Both concepts will be considered here.

Needed for Theorem 19, a technical result is presented below. It sets that, given a reachable deadlock marking, either its firing sequence is in $\mathbb{Q}$ (it is in $RS_Q(\mathcal{N}, \boldsymbol{m}_0)$) or there exists another deadlock marking that it is in the

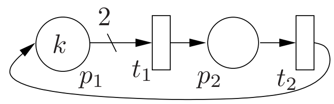

Fig. 4. PN which deadlocks as discrete for any k, and it lim-deadlocks as continuos

rationals ($\mathbb{Q}$). It is equivalent to say that, if $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is not DF, then also $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_Q$ is not DF.

Lemma 18. Let $\boldsymbol{m}_d \in RS_C(\mathcal{N}, \boldsymbol{m}_0)$ be a deadlock, then $\exists \boldsymbol{m}'_d \in RS_Q(\mathcal{N}, \boldsymbol{m}_0)$ s.t. it is a deadlock.

Proof. Assume $\boldsymbol{m}_d \in RS_C(\mathcal{N}, \boldsymbol{m}_0) \setminus RS_Q(\mathcal{N}, \boldsymbol{m}_0)$. Because of Lemma 16, $\forall \varepsilon > 0$ another $\boldsymbol{m}'_d \in RS_Q(\mathcal{N}, \boldsymbol{m}_0)$ exists such that $||\boldsymbol{m}'_d - \boldsymbol{m}_d|| < \varepsilon$ $\forall p$ s.t. $m_d[p] = 0$, also $m'_d[p] = 0$. Since $\forall t \in T$, t is not enabled in $\boldsymbol{m}_d$, then also $\forall t \in T$, t is not enabled in $\boldsymbol{m}'_d$. Hence, $\exists \boldsymbol{m}'_d \in RS_Q(\mathcal{N}, \boldsymbol{m}_0)$ that is a deadlock in the continuous system. $\square$

Let us proof that, if a discrete PN is homothetically deadlockfree (HDF), it will be also DF as continuous. However, the opposite implication is not true.

Theorem 19. $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$ is HDF $\Longrightarrow \langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is DF

Proof. Let us suppose $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ deadlocks. It means $\exists \boldsymbol{m} \in RS_C(\mathcal{N}, \boldsymbol{m}_0)$ that is a deadlock. Because of Lemma 18, if $\boldsymbol{m}$ is a deadlock, then there exists $\boldsymbol{m}' \in RS_Q(\mathcal{N}, \boldsymbol{m}_0)$ that is a deadlock. Because of Lemma 15, $\exists k \in \mathbb{N}$ s. t. $\boldsymbol{m}'' = k \cdot \boldsymbol{m}'$, where $\boldsymbol{m}'' \in RS(\mathcal{N}, k \cdot \boldsymbol{m}_0)$. Since $\forall t \in T$, $\exists p \in {}^\bullet t, \boldsymbol{m}'[p] = 0$, then also $\forall t \in T$, $\exists p \in {}^\bullet t, k \cdot \boldsymbol{m}''[p] = 0$, and consequently $\boldsymbol{m}''$ it is also deadlock: $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0 \rangle_D$ deadlocks. $\square$

Theorem 19 can be illustrated by the example in Fig. 2 (a). However, the opposite implication does not hold, i.e.,

$\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is DF $\not\Longrightarrow \langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$ is HDF

The PN in Fig. 4 is a counter example: the net system is DF when considered continuous, with $\boldsymbol{m}_0 = (1, 0)$; because it can always fire a small amount, and it will never reach a deadlock with a finite firing sequence [8], but when the net system is considered discrete, $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0 \rangle_D$ deadlocks for every k.

The previous results deals with DF. What happens if lim-DF is considered? A continuous system which is lim-DF, could be homothetic DF as discrete. When this is not the case, a minimun value of k can be considered for homotethic monotonicity. We will denote that a property Π is *homothetic from n* if $\exists n \in \mathbb{N}$, s.t. $\forall k \geq n$, with $k \in \mathbb{N}$, Π holds in $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0 \rangle_D$.

Now, the implication can be formulated.

Theorem 20. $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is lim-DF $\Longrightarrow \exists n \in \mathbb{N}$ s.t. $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$ is HDF from n

Proof. Let us suppose $\forall n \in \mathbb{N}$ $\exists k \geq n$, $k \in \mathbb{N}$, such that the discrete system $\langle N, k \cdot \boldsymbol{m}_0 \rangle_D$ deadlocks. It means there exists an infinite ordered set $A = \{a_1, a_2, a_3 \dots\}$, such that $\forall a_i \in A$, $a_i < a_{i+1}$ and $\langle \mathcal{N}, a_i \cdot \boldsymbol{m}_0 \rangle_D$ deadlocks.

For each a_i for which it deadlocks, $\exists \boldsymbol{m}_d \in \mathrm{RS}_D(\mathcal{N}, a_i \cdot \boldsymbol{m}_0)$ s.t. $\boldsymbol{m}_d$ is a deadlock. It holds that $\forall t \in T, \exists p \in {}^\bullet t, \boldsymbol{m}_d[p] < Pre[p, t]$. Because of the definitions of continuous firings (Property 7), marking $\frac{\boldsymbol{m}_d}{a_i}$ is reachable in $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$.

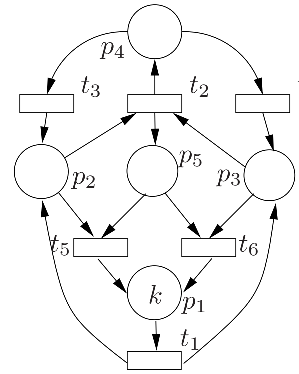

Fig. 5. Ordinary PN, it is live as discrete for any k (homothetic deadlock-freeness), but not lim-deadlockfree as continuos

Given that a_i tends to infinite, making $a_i \rightarrow \infty$, then $\frac{\boldsymbol{m}_d}{a_i}[p] \rightarrow 0$, and it will reach a deadlock in the limit. Consequently, the continuous $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ is not lim-deadlockfree. $\square$

However, the reverse is not true:

$\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$ is HDF $\not\Longrightarrow \langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is lim-DF

The PN system in Fig. 5 is a counter example. When discrete, the net system is homothetic DF from $\boldsymbol{m}_0 = (1, 0, 0, 0, 0)$. It can be easily proved that the discrete system is monotonic DF from $\boldsymbol{m}_0$ because every siphon contains a marked trap [4, 5]. It is straightforward that monotonic DF implies homothetic DF. However, when the net system is considered as continuous, the infinite firing sequence $t_1\ t_2\ \frac{1}{2}t_3\ \frac{1}{2}t_4\ \frac{1}{2}t_2\ \frac{1}{4}t_3\ \frac{1}{4}t_4\ \frac{1}{4}t_2\ \frac{1}{8}t_3\ \frac{1}{8}t_4 \dots$ can be fired, leading to the deadlock marking $\boldsymbol{m} = (0, 0, 0, 0, 2)$: the continuous system reaches a deadlock in the limit [8].

3.4 Liveness and reversibility

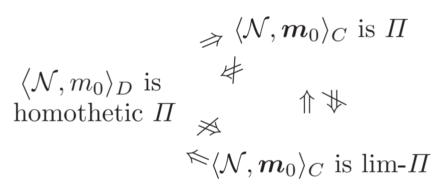

Fig. 6. Relations w.r.t. a property Π

The theorems and lemmas presented here are analogous to the ones presented for DF (Section 3.3); even the proofs are technically analogous. Figure 6 summarizes the relations among a certain property Π (DF, L or R), when considered homothetic in the discrete system (homothetic Π), when considered for the continuous system (Π) or considerig also reachability in the limit (lim-Π).

Analogously to Lemma 18:

Lemma 21. $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is not $\Pi \Longrightarrow \langle \mathcal{N}, \boldsymbol{m}_0 \rangle_Q$ is not Π

The proof of this lemma when $\Pi = \mathrm{L}$ is similar to the one of Lemma 18, but instead of consider a marking $\boldsymbol{m}_d$ which is a deadlock, a marking $\boldsymbol{m}_l$ from which $\exists t$ s.t. t is not enabled from $\boldsymbol{m}_l$ should be considered. In the lemma when $\Pi = \mathrm{R}$, a marking $\boldsymbol{m}_r$ from which the initial marking is no reachable should be considered.

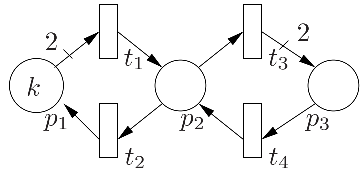

Fig. 7. Non reversible, non live discrete PN system. It is reversible and live as continuous

Given these lemmas, the general result about preservation of a homothetic property by fluidization can be formulated (equivalent to Theorem 19).

Theorem 22. $\langle\mathcal{N},\boldsymbol{m}_0\rangle_D$ is homoth. $\mathit{\Pi} \Longrightarrow \langle\mathcal{N},\boldsymbol{m}_0\rangle_C$ is $\mathit{\Pi}$

The proofs of the theorem for liveness ($\mathit{\Pi}$ = L) and for reversibility ($\mathit{\Pi}$ = R) would be similiar to the proof of Theorem 19. The lemmas obtained from Lemma 21 for $\mathit{\Pi}$ = L and $\mathit{\Pi}$ = R would be also used in these proofs.

As an illustrative example, consider the Petri net example on Fig. 2 (a). It is homothetic live and homothetic reversible. Thus it preserves these properties when fluidified.

If the opposite implication is considered, again, even in the case of considering not *every k* but a *big enough k*, the implication is not true:

$\langle\mathcal{N},\boldsymbol{m}_0\rangle_C$ is $\mathit{\Pi} \not\Longrightarrow \exists n \in \mathbb{N}$, $\langle\mathcal{N},\boldsymbol{m}_0\rangle_D$ is homothetic $\mathit{\Pi}$ from n.

Let us consider the example in Fig. 7. When considered a continuous system, it is live and reversible from $\boldsymbol{m}_0 =$ (1,0,0): the marking can decrease firing t_1 and t_2, but it can also increase in the same amount firing t_3 and t_4. However, if the net system is discrete, it is neither live or reversible for $\boldsymbol{m}_0 =$ (1,0,0), nor for any proportional initial marking $k\cdot\boldsymbol{m}_0$. It is because for any value of k, transitions t_1 and t_2 can be fired until $m[p_1] < 2$. Then, no transition is enabled, the system is deadlocked and the system is neither live nor reversible.

Let us now consider the properties in the limit, i.e., lim-$\mathit{\Pi}$. In this case, it holds:

Theorem 23. $\langle\mathcal{N},\boldsymbol{m}_0\rangle_C$ is $\mathit{\Pi} \Longrightarrow \exists n \in \mathbb{N}$ s.t.$\langle\mathcal{N},\boldsymbol{m}_0\rangle_D$ is homothetic $\mathit{\Pi}$ from n.

The proofs of Theorem 23 for $\mathit{\Pi}$ = L and for $\mathit{\Pi}$ = R would be analogous to that of Theorem 20, in which also Lemma 15 would be used.

Analogously to lim-deadlockfreenes, the reverse is not true:

$\langle\mathcal{N},\boldsymbol{m}_0\rangle_D$ is homothetic $\mathit{\Pi} \not\Longrightarrow \langle\mathcal{N},\boldsymbol{m}_0\rangle_C$ is lim-$\mathit{\Pi}$

Again, the PN system in Fig. 5 is live and reversible for $\boldsymbol{m}_0$=(1,0,0,0,0) if it is a discrete system. However, when considered continuous, the infinite firing sequence t_1 t_2 $\frac{1}{2}t_3$ $\frac{1}{2}t_4$ $\frac{1}{2}t_2$ $\frac{1}{4}t_3$ $\frac{1}{4}t_4$ $\frac{1}{4}t_2$ $\frac{1}{8}t_3$ $\frac{1}{8}t_4 \ldots$ would reach the deadlock marking $\boldsymbol{m}_d = (0,0,0,0,2)$ in the limit, so the system is not lim-live and not lim-reversible from $\boldsymbol{m}_d$.

4. HOMOTHETIC MONOTONICITY OF BOUNDEDNESS IN DISCRETE PN SYSTEMS

The aim of this brief section is to propose a condition to characterize (sufficient and necesary condition) homothetic boundedness for discrete PN systems.

By definition, if a discrete net system is structurally bounded (SB) (i.e., bounded for any initial marking $\boldsymbol{m}_0$), then it is homothetically bounded (HB): $\mathcal{N}$ is SB $\Rightarrow$ $\langle\mathcal{N},\boldsymbol{m}_0\rangle_D$ is HB.

However, the opposite is not true: $\langle\mathcal{N},\boldsymbol{m}_0\rangle_D$ is HB $\not\Rightarrow \mathcal{N}$ is SB. The reason is that if there is an empty siphon in $\langle\mathcal{N},\boldsymbol{m}_0\rangle_D$, some transitions can never be fired from $\boldsymbol{m}_0$; so the system can be B and HB for that $\boldsymbol{m}_0$, but $\mathcal{N}$ can be unbounded for a different initial marking.

Furthermore, if $\nexists$ empty siphon in $\langle\mathcal{N},\boldsymbol{m}_0\rangle_D$, a reasonable condition for real systems) then every transition would be fireable from $\boldsymbol{m}_0$ or from a given $k\cdot\boldsymbol{m}_0$, and then HB implies SB. Considering a result of boundedness in continuous systems from [8], a more general result can be obtained.

Theorem 24. Given a net system such that every siphon is marked, the following statements are equivalent:

(1) $\mathcal{N}$ is SB
(2) $\langle\mathcal{N},\boldsymbol{m}_0\rangle_D$ is HB
(3) $\langle\mathcal{N},\boldsymbol{m}_0\rangle_C$ is B
(4) $\langle\mathcal{N},\boldsymbol{m}_0\rangle_C$ is lim-B

Proof.
(1) $\Rightarrow$ (2) Trivially holds by definition.
(2) $\Rightarrow$ (3) Because of Theorem 17.
(3) $\Rightarrow$ (1) Proved in [8].
(4) $\Leftrightarrow$ (1) Proved in [8]. $\square$

The computation of structural boundedness of a PN system can be done in polynomial time [3], as well as checking the existence of empty siphons [9]. Consequently, boundedness of a continuous system can also be checked in polynomial time.

5. HOMOTHETIC DEADLOCK-FREENESS IN SB DISCRETE SYSTEMS

The aim of this section is to characterize homothetic deadlock-freeness for discrete PN systems. For this purpose, a technique for the characterization of DF in discrete PN systems considered in [3] is recalled. It will allow us to characterize not only DF of a given system $\langle\mathcal{N},\boldsymbol{m}_0\rangle_D$, but also HDF (for any scaled initial marking $k\cdot\boldsymbol{m}_0$).

5.1 *Classic method for the characterization of DF*

This general sufficient condition for DF, based on the state equation, exploits the definition: "a deadlock corresponds to a marking in which no transition is fireable".

Proposition 25. Let $\langle\mathcal{N},\boldsymbol{m}_0\rangle_D$ be a PN system. If there does not exist any solution $(\boldsymbol{m},\sigma)$ to the following system, then $\langle\mathcal{N},\boldsymbol{m}_0\rangle_D$ is deadlockfree.

$$\begin{array}{l}\boldsymbol{m} = \boldsymbol{m}_0 + C\cdot\sigma\\ \boldsymbol{m} \geq 0, \sigma \geq 0\\ \bigvee_{p\in{}^\bullet t}\boldsymbol{m}[p] < Pre[p,t],\ \forall t \in T\end{array} \qquad (1)$$

Notice that the system above contains $|T|$ "complex conditions", (one for each transition), which are non linear, due to the "$\vee$" connective. Thus, (1) can be handled by solving independently a set of $\prod_{t\in T}|{}^\bullet t|$ systems of linear inequalities. Notice the number of systems grows exponentially.

In [3], some transformations and rules are considered in order to reduce the number of systems generated by (1). Furthermore, in Theorem 34 of [3], it was proved that the system can be rewritten as a single system of linear inequalities for every structurally bounded PN system (to ease the task of reviewers, we included in this draft Appendix A).

Let us illustrate the key idea with the example in Fig. 2 (a). Initially, the system that characterizes the sufficient condition for deadlockfreeness is: If there not exists solution to the following system, then the net system is DF.

$$\begin{array}{ll} \boldsymbol{m} = \boldsymbol{m}_0 + C \cdot \sigma & \\ \boldsymbol{m} \geq 0; \sigma \geq 0 & \\ (\boldsymbol{m}[p_1] = 0 \vee \boldsymbol{m}[p_2] = 0) & \{t_1 \textit{ is not enabled}\} \\ (\boldsymbol{m}[p_1] = 0 \vee \boldsymbol{m}[p_3] = 0) & \{t_2 \textit{ is not enabled}\} \\ (\boldsymbol{m}[p_4] = 0 \vee \boldsymbol{m}[p_5] = 0) & \{t_3 \textit{ is not enabled}\} \end{array} \tag{2}$$

This system is not linear; however, by applying the transformation and the reduction rules in [3], it can be converted to a single system. Such rules force the PN system to be transformed to the one in Fig. 2 (b).

The resulting system is:

$$\begin{array}{ll} \boldsymbol{m} = \boldsymbol{m}_0 + C \cdot \sigma & \\ \boldsymbol{m} \geq 0; \sigma \geq 0 & \\ \boldsymbol{m}[p_b] + \boldsymbol{m}[p_2] \leq 1 & \{t_1 \textit{ is not enabled}\} \\ \boldsymbol{m}[p_b] + \boldsymbol{m}[p_3] \leq 1 & \{t_2 \textit{ is not enabled}\} \\ \boldsymbol{m}[p_d] + \boldsymbol{m}[p_4] \leq 1 & \{t_3 \textit{ is not enabled}\} \\ \boldsymbol{m}[p_1] + 2 \cdot \boldsymbol{m}[p_a] \leq 2 & \{t_p \textit{ is not enabled}\} \\ \boldsymbol{m}[p_c] + \boldsymbol{m}[p_5] \leq 1 & \{t_q \textit{ is not enabled}\} \end{array} \tag{3}$$

5.2 Characterization of homothetic deadlock-freeness

The system characterizes the presence of deadlocks in the state equation. If they are not reachable with a real sequence from $\boldsymbol{m}_0$, then they will be also not reachable from $k \cdot \boldsymbol{m}_0$, then the discrete system is homothetic DF. Applying Theorem 19, it is straightforward that the continuous net system is deadlockfree.

For example, the equation system obtained for the PN on Fig. 2, has no solution in the real domain. Consequently, $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$ is homotethic DF, and $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_C$ is also DF as continuous.

In the case that the equation system obtained from (1) has a solution, it means there is a solution of the state equation that is a deadlock. But it is not enough to decide about the HDF of the system. Some additional methods can be used to refine the characterization.

- Check if the obtained deadlock marking $\boldsymbol{m}$ is a spurious solution of the state equation, i.e., a solution of the state equation that is not reachable in the discrete system. It can be checked with some classical techniques [9].
- Check if $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$ is monotonic DF, which implies that it is also homothetic DF. It is monotonic DF if every siphon of $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle_D$ contains a marked trap [4, 5] (with some marking restrictions in the case of non-ordinary PN).
- Check if the firing count vector σ obtained as solution is fireable from a scaled initial marking $k \cdot \boldsymbol{m}_0$ in the discrete system $\langle \mathcal{N}, k \cdot \boldsymbol{m}_0 \rangle_D$.
- Consider a system of inequalities equivalent to the one recalled here, but scaled by k. The SB of $\langle N, k \cdot \boldsymbol{m}_0 \rangle$ would be calculated with a parametric Linear Programming Problem (see, for example, [6]), and the marking of the places would be scaled by k. The system could be solved (as a Integer Programming Problem) for different values of k.

6. CONCLUSIONS

This paper sets that, given a property which has a *homothetic* behaviour in a discrete PN, that property will be preserved in the corresponding continuous PN when it is fluidified.

The conclusion is that just studying if a given property is homothetically monotonic in a certain discrete PN system, the preservation of the property by the fluidified PN system can be concluded. It means to determine whether the fluidization makes sense with respect to a certain property.

Homotethic boundedness in discrete systems is characterized here. Moreover, a method to check if a given PN system is homothetically deadlockfree has been explored. This method requires the Petri net systems to be structurally bounded.

Future work will be to further study the characterization of HDF in discrete PNs; and also to obtain the characterization of more types of nets, apart from the ones proposed here. Furthermore, techniques for the characterization of homothetic liveness and homotethic reversibility are under development.

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Appendix A. DEADLOCKFREENESS CHARACTERIZATION IN DISCRETE PN

A transformation and two rules are used in [3] to reduce the set of equations of the characterization of discrete PN.

Rule 1 presents a reduction of the number of disjuntives $\vee$ when the structural bound (SB) of a set π of places in $^\bullet t$ satisfies SB[p] $\leq$ Pre[p,t].

Rule 2 is the generalization of Rule 1 when every place fulfills SB[p] $\leq$ Pre[p,t] but one. In order to force SB[p] $\leq$ Pre[p,t], a previous transformation should can be applied, where the projected language of the Petri net system is preserved.

Rule 1 *Let t be a transition such that for every $p \in \pi \subseteq {}^\bullet p$ the following holds: $SB[p] \leq Pre[p,t]$. Replacing in (1) for the disabledness condition corresponding to transition t the following (less complex) condition the set of integer solutions is preserved:*

$$\left(\sum_{p\in\pi} \boldsymbol{m}[p] < \sum_{p\in\pi} Pre[p,t]\right) \vee \left(\bigvee_{p\in{}^\bullet t\setminus\pi} \boldsymbol{m}[p] < Pre[p,t]\right)$$

Rule 2 *Let t be a transition such that $^\bullet t = \pi \cup \{p'\}$, where $SB[p] \leq Pre[p,t]$ for every $p \in \pi$. Replacing in (1) for the disabledness condition corresponding to transition t the following (less complex) condition the set of integer solutions is preserved:*

$$SB[p'] \sum_{p\in\pi} \boldsymbol{m}[p] + \boldsymbol{m}[p'] < SB[p'] \sum_{p\in\pi} Pre[p,t] + Pre[p',t]$$