On reachability and deadlock-freeness of Hybrid Adaptive Petri nets ⋆

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Abstract: Petri nets (PN) represent a well known family of formalisms for the modeling and analysis of Discrete Event Systems (DES). As most formalisms for DES, PNs suffer from the state explosion problem. A way to overcome this difficulty is to relax the original discrete model and deal with a *fully* or *partially* continuous model. In contrast to continuous Petri nets that consider a full continuous firing of transitions, what can lead to the loss some properties of the original discrete model, this paper deals with Hybrid Adaptive Petri nets (HAPNs), that consider partially continuous firings. In an HAPN, a threshold is associated with each transition: if the load of the transition is higher than its threshold, it behaves as continuous; if it is lower, it behaves as discrete. This way, transitions *adapt* dynamically to their load. The reachability space and the deadlock-freeness property of HAPNs are studied and compared to those of discrete and continuous Petri nets.

Keywords: Automata, Petri Nets and other tools; Discrete event systems modeling and control; Hybrid systems modeling and control.

1. INTRODUCTION

The *state explosion problem* is a crucial drawback in the analysis of discrete event systems. An interesting technique to overcome this difficulty is to relax the original discrete model and deal with a continuous approximation. Such a relaxation aims at computationally more efficient analysis methods, at the price of losing some precision.

Unfortunately, the transformation to a continuous model may not always preserve important properties of the original discrete model. For instance, in the context of Petri nets (PNs), the transformation from discrete to continuous [1, 2, 3] does not preserve, in general, deadlock-freeness, liveness, reversibility, etc [4].

This paper focuses on Hybrid Adaptive Petri nets (HAPNs) [5], a Petri net based formalism in which the firing of transitions is partially relaxed. The transitions of a HAPN can behave in two different modes: *continuous* and *discrete*. The continuous mode will be chosen when transition workload is higher than a given threshold. This makes sense because the higher the workload the better the continuous approximation. Consequently, it also makes sense to commute to a discrete mode when the load becomes low.

This way, a HAPN is able to *adapt* its behaviour to the net workload, offers the possibility to represent more faithfully the discrete model and simplifies analysis techniques by behaving as continuous when the load is high. In contrast to [5], HAPNs will be defined and studied in the untimed framework. Notice that the introduction of time in a given system would produce a particular system trajectory that is also achievable in the untimed one. Thus, the results for some properties as deadlock-freneness in the untimed framework can be almost straightforwardly applied on timed systems. In the following it is assumed that the reader is familiar with Petri nets (PNs) (see [6, 9] for a gentle introduction).

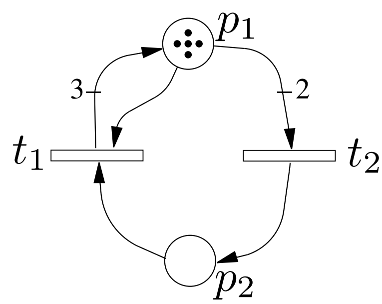

Fig. 1. A Petri net system that deadlocks as continuous but is deadlock-free as hybrid adaptive with appropriate thresholds.

Let us consider the PN system in Figure 1 [4] to introduce the behaviour of HAPNs. Let the initial marking of the system be $\boldsymbol{m_0} = (5, 0)$. If considered as a discrete system, it is deadlock-free: from the initial marking $\boldsymbol{m_0}$ only t_2 can fire, reaching $\boldsymbol{m_1} = (3, 1)$. From $\boldsymbol{m_1}$, both $\boldsymbol{m_0}$ and $\boldsymbol{m_2} = (1, 2)$ can be reached by firing t_1 and t_2 respectively. This behaviour is represented in the reachability graph and reachability space in Figure 2 (a). None of the reachable markings deadlocks the system, hence it is deadlock-free.

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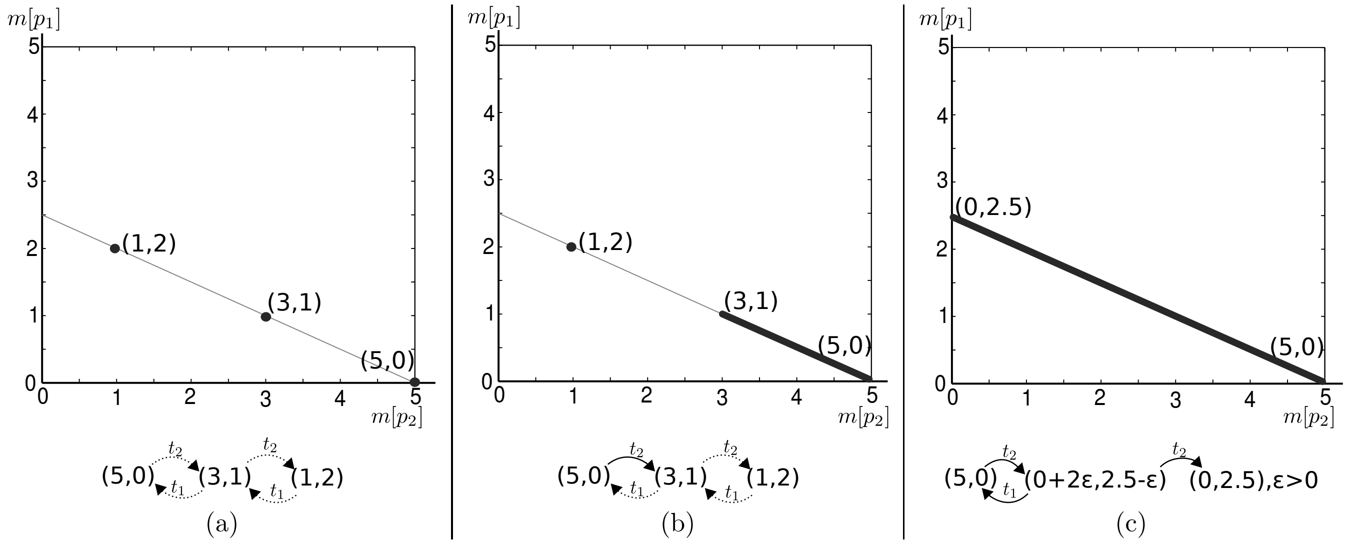

Fig. 2. Reachability spaces of the Petri net in Figure 1 when considered Discrete (a), Adaptive (b) and Continuous (c).

Consider now that the system is continuous [8] , i.e., each transition can be fired in any non-negative real amount less than or equal to its enabling degree. Given that at $\boldsymbol{m_0}$ the enabling degree of t_2 is 2.5, t_2 can fire in any amount in the interval $[0, 2.5]$. Figure 2 (c) shows the reachability space of the continuous PN. The firing of t_2 in an amount lower than 2.5 produces positive markings in both places and both transitions are enabled. However, the firing of t_2 in 2.5 from $\boldsymbol{m_0}$ leads to $(0, 2.5)$ where no transition is enabled and the system deadlocks. Consequently, deadlock-freeness is not preserved by the continuous PN.

Let us finally assume that the net system is adaptive. For these systems, a transition t_i can have two different firing modes: *continuous* and *discrete*. It behaves as continuous when its enabling degree is higher than a given threshold μ_i. Otherwise, t_i behaves as discrete.

When a discrete system is considered as adaptive, appropiate thresholds have to be defined. Let us define $\mu_1 = 1$ for t_1 and $\mu_2 = 1.5$ for t_2 for the system of Figure 1. At the initial marking $\boldsymbol{m_0} = (5, 0)$, t_1 is not enabled, and t_2 behaves as continuous, and it can fire in real amounts while it remains continuous. If t_2 is fired in an amount of 1, $\boldsymbol{m_1} = (3, 1)$ is reached. At $\boldsymbol{m_1}$, both t_1 and t_2 are enabled as discrete. The firing of $t_1(t_2)$ from $\boldsymbol{m_1}$ leads to $\boldsymbol{m_0}(\boldsymbol{m_2} = (1, 2))$. At $\boldsymbol{m_2}$ both transitions are discrete but only t_1 is enabled, whose firing leads to $\boldsymbol{m_1}$. Hence, although the adaptive system still keeps some continuous behaviour, it preserves the deadlock-freeness property of the discrete system. Figure 2 (b) shows the reachability space of the HAPN. The arrows of the *reachability graph* below the reachability space are solid for the continuous firings and dotted for the discrete ones.

In summary, deadlock-freeness property of a discrete system might not be preserved by the continuous aproximation; nevertheless, it could be preserved by the *hybrid adaptive* approximation.

The rest of the paper is organised as follows: In Section 2, HAPNs are formally defined. Section 3 studies the reachability space of HAPNs and relates it to those of the discrete and continous Petri nets. Section 4 presents some preliminary results about deadlock-freeness in HAPNs. Finally, conclusions and future work are presented in Section 5.

2. HYBRID ADAPTIVE PETRI NETS

This section defines the basic concepts related to HAPNs.

Definition 1. A HAPN is a tuple $\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post}, \boldsymbol{\mu} \rangle$ where:

- $P = \{p_1, p_2, ..., p_n\}$ and $T = \{t_1, t_2, ..., t_m\}$ are disjoint and finite sets of places and transitions.
- **Pre** and **Post** are $|P| \times |T|$ sized, natural valued, incidence matrices.
- $\boldsymbol{\mu} \in (\mathbb{R}_{\geq 0} \cup \infty)^{|T|}$ is the vector of thresholds.

Definition 2. A HAPN system is a tuple $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$, where $\boldsymbol{m_0} \in (\mathbb{N} \cup \{0\})^{|P|}$ is the initial marking.

Given a place (transition) $v \in P(T)$, its *preset*, ${}^\bullet v$, is defined as the set of its input transitions (places), and its *postset* $v^\bullet$ as the set of its output transitions (places).

As in continuous PNs, the enabling degree of t_i at m is defined as:

$$enab(t_i, m) = min_{p \in {}^\bullet t_i} \left\{ \frac{m[p]}{Pre[p, t_i]} \right\} \quad (1)$$

The threshold μ_i of a transition t_i determines the values of the enabling degree for which the transition behaves in continuous (C) or in discrete (D) mode:

$$mode(t_i, m) = \begin{cases} C & \text{if } enab(t_i, m) > \mu_i \\ D & \text{otherwise} \end{cases} \quad (2)$$

If a transition t_i is in *continuous* mode then $enab(t_i, m) > \mu_i$ what implies that t_i is enabled as continuous. On the other hand, if t_i is in *discrete* mode then it is enabled iff $enab(t_i, m) \geq 1$. This two conditions together imply that t_i is enabled (either as discrete or continuous) iff the following expression is true:

$$mode(t_i, m) = C \vee (mode(t_i, m) = D \wedge enab(t_i, m) \geq 1)$$

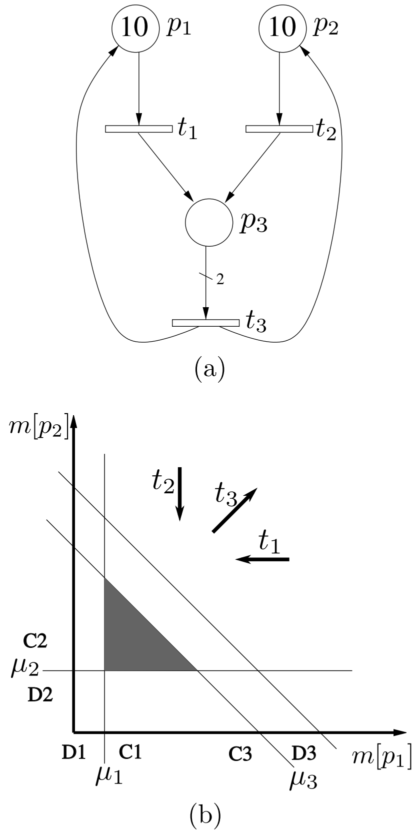

Fig. 3. Example of a Hybrid Adaptive Petri net (a) and the behaviour of its transitions (b).

This expression is equivalent to:

$$enab(t_i, m) > \mu_i \vee (enab(t_i, m) \leq \mu_i \wedge enab(t_i, m) \geq 1)$$

what simplifies to:

$$enab(t_i, m) > \mu_i \vee enab(t_i, m) \geq 1$$

Figure 3 (b) explains the behaviour of the transitions of the HAPN in Figure 3 (a), with any $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$. It shows the regions in which t_1, t_2 and t_3 behave as discrete (regions D_1, D_2, D_3) or continuous (C_1, C_2, C_3). In the triangular region of the center of Figure 3 (b), the PN behaves as continuous, and in the other regions, it has a partially discrete behaviour.

Notice that if $\boldsymbol{\mu} = \mathbf{0}$, all transitions will behave as continuous, and if $\boldsymbol{\mu} = \infty$ all transition will behave as discrete. Hence, the HAPN formalism includes both the continuous and discrete PN formalisms.

A transition t_i that is enabled can fire. The admissible firing amounts depend on its mode. If $mode(t_i, m) = C$, t_i can fire in any real amount $\alpha \in \mathbb{R}_{\geq 0}$ that does not make the enabling degree cross the threshold μ_i, i.e., $0 < \alpha \leq enab(t_i, m) - \mu_i$. If $mode(t_i, m) = D$, t_i can fire as a usual discrete transition in any natural amount $\alpha \in \mathbb{N}$ such that $0 < \alpha \leq enab(t_i, m)$.

The firing of t in a certain amount $\alpha \leq$ enab(t, $\boldsymbol{m}$) leads to a new marking $\boldsymbol{m'}$, and it is denoted as $\boldsymbol{m} \xrightarrow{\alpha t} \boldsymbol{m'}$. It holds $\boldsymbol{m'} = \boldsymbol{m} + \alpha \cdot \boldsymbol{C}[P, t]$, where $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the token flow matrix (incidence matrix if $\mathcal{N}$ is self-loop free). Hence, as in discrete systems, $\boldsymbol{m} = \boldsymbol{m_0} + \boldsymbol{C} \cdot \boldsymbol{\sigma}$, the state (or fundamental) equation summarizes the way the marking evolves, where $\boldsymbol{\sigma}$ is the firing count vector of the fired sequence. Right and left natural annullers of the token flow matrix are called T- and P-semiflows, respectively. As in discrete systems, when $\mathbf{y} \cdot \boldsymbol{C} = \mathbf{0}$, $\mathbf{y} > \mathbf{0}$ the net is said to be *conservative*, and when $\boldsymbol{C} \cdot \mathbf{x} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ the net is said to be *consistent*.

A discrete, continuos or hybrid adaptive Petri net is *choice free*[9] if each place has at most one output transition, i.e., $\forall p \quad |p^{\bullet}| \leq 1$. A Petri net is said to be *ordinary* iff: $\forall p \in P \quad \forall t \in T, Pre[p, t] \in \{0, 1\}$ and $Post[p, t] \in \{0, 1\}$.

The set of all the reachable markings of a given HAPN system $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ is denoted as reachability space, RS($\mathcal{N}$, $\boldsymbol{m_0}$):

Definition 3. RS($\mathcal{N}$, $\boldsymbol{m_0}$) $= \{\boldsymbol{m} \mid \exists\ \sigma = \alpha_1 t_{\gamma_1} \ldots \alpha_k t_{\gamma_k}$ *such that* $\boldsymbol{m_0} \xrightarrow{\alpha_1 t_{\gamma_1}} \boldsymbol{m_1} \xrightarrow{\alpha_2 t_{\gamma_2}} \boldsymbol{m_2} \cdots \xrightarrow{\alpha_k t_{\gamma_k}} \boldsymbol{m_k} = \boldsymbol{m}$ *where* $\alpha_i \in \mathbb{R}^+$ *if* $mode(t_{\gamma_i}, m_{i-1}) = C$, *and* $\alpha_i \in \mathbb{N}^+$ *if* $mode(t_{\gamma_i}, m_{i-1}) = D\}$

Liveness and deadlock-freeness properties are defined in a similar way to those of discrete systems.

Definition 4. Let $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ be a HAPN system.

- $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ deadlocks iff a marking $\boldsymbol{m} \in$ RS ($\mathcal{N}$, $\boldsymbol{m_0}$) exists such that $\forall t \in T$, t is not enabled.
- $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ is live iff for every transition t and for any marking $\boldsymbol{m} \in$ RS ($\mathcal{N}$, $\boldsymbol{m_0}$) there exists $\boldsymbol{m}$' $\in$ RS($\mathcal{N}$, $\boldsymbol{m}$) such that t is enabled at $\boldsymbol{m}$'.
- $\mathcal{N}$ is structurally live iff $\exists\ \boldsymbol{m_0}$ such that $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ is live.

3. REACHABILITY SPACE OF HAPNS

In this section, the reachability space (RS) of HAPN systems is studied and compared to the RS of discrete and continuous systems.

The following definitions will be used in the rest of the paper: $\mathcal{N}_D$ denotes a discrete Petri net with a given structure $\langle P, T, \mathbf{Pre}, \mathbf{Post} \rangle$, $\mathcal{N}_C$ denotes the continuous net with the same structure, and $\mathcal{N}_A$ denotes the hybrid adaptive Petri net with the same structure and an arbitrary $\boldsymbol{\mu}$. In order to compare the reachability spaces, the same initial marking $\boldsymbol{m_0} \in \mathbb{N}^{|P|}$ is considered for all three types of Petri nets (discrete, continuous or adaptive).

For the study of the RS we will focus on ordinary PNs. Notice that although ordinary PNs are a subclass of general PNs, any non-ordinary Petri net can be converted to an equivalent ordinary PN[10]. It will be proved that, under rather general conditions, the RS of a HAPN $\mathcal{N}_A$ contains the RS of $\mathcal{N}_D$, and that the RS of $\mathcal{N}_C$ contains the RS of $\mathcal{N}_A$. This is a straightforward consequence of the fact that, in contrast to continuous nets, HAPNs are a partial, non-full, relaxation of discrete nets.

Theorem 5. RS($\mathcal{N}_D$, $\mathbf{m_0}$) $\subseteq$ RS($\mathcal{N}_A$, $\mathbf{m_0}$) for any ordinary HAPN $\mathcal{N}_A$ with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$.

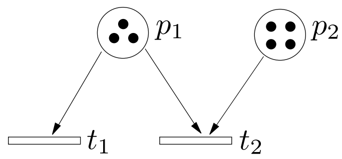

Fig. 4. A net whose reachability space as discrete is not contained in the reachability space as adaptive with with $\boldsymbol{\mu} = (1.5, 1.5)$, see Figure 5.

Proof. Let $\boldsymbol{m} \in \text{RS}(\mathcal{N}_D,\ \boldsymbol{m_0})$. Then, there exists $\sigma_d = t_{\gamma_1} \ldots t_{\gamma_k}$ such that $\boldsymbol{m_0} \xrightarrow{1t_{\gamma_1}} \boldsymbol{m_1} \xrightarrow{1t_{\gamma_2}} \boldsymbol{m_2} \cdots \xrightarrow{1t_{\gamma_k}} \boldsymbol{m_k} = \boldsymbol{m}$ in $\langle \mathcal{N}_D,\ \boldsymbol{m_0} \rangle$. We will prove that there exists a sequence $\sigma_a = \beta_1 t_{\gamma_1} \ldots \beta_k t_{\gamma_k}$ such that $\boldsymbol{m_0} \xrightarrow{\beta_1 t_{\gamma_1}} \boldsymbol{m_1} \xrightarrow{\beta_2 t_{\gamma_2}} \boldsymbol{m_2} \cdots \xrightarrow{\beta_k t_{\gamma_k}} \boldsymbol{m_k} = \boldsymbol{m}$ in $\langle \mathcal{N}_A,\ \boldsymbol{m_0} \rangle$.

Let us start with t_{γ_1}, and let us check if $\beta_1 = 1$ can be chosen. Two cases must be considered.

a) $enab(t_{\gamma_1}, \boldsymbol{m_0}) \leq \mu_{t_{\gamma_1}}$. From the definition of HAPN, t_{γ_1} behaves as discrete, i. e., $mode(t_{\gamma_1}, \boldsymbol{m_0}) = D$. Given that t_{γ_1} is enabled in $\langle N_D,\ \boldsymbol{m_0} \rangle$, it holds that $enab(t_{\gamma_1}, \boldsymbol{m_0}) = min_{p \in {}^\bullet t_{\gamma_1}} \{m_0[p]\} \geq 1$. Hence, it is also enabled in $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ in the same amount. Therefore, $\beta_1 = 1$ can be chosen, and the same $\boldsymbol{m_1}$ of the discrete system is reached.

b) $enab(t_{\gamma_1},\ \boldsymbol{m_0}) > \mu_{t_{\gamma_1}}$. From the definition of HAPN, t_{γ_1} behaves as continuous, i. e., $mode(t_{\gamma_1}, \boldsymbol{m_0}) = C$. Since $\mu_{t_{\gamma_1}} \in \mathbb{N}$ and $enab(t_{\gamma_1}, \boldsymbol{m_0}) > \mu_{t_{\gamma_1}}$, it holds that $enab(t_{\gamma_1},\ m_0) - \mu_{t_{\gamma_1}} \geq 1$. Therefore, $\beta_1 = 1 \leq$ $enab(t_{\gamma_1})$ - $\mu_{t_{\gamma_1}}$ can be chosen and $\boldsymbol{m_1}$ is reached.

The same reasoning can be applied to the rest of the transitions in the sequence $t_{\gamma_2} \ldots t_{\gamma_k}$. □

If non ordinary PNs or non natural thresholds are considered, $\text{RS}(\mathcal{N}_D, \boldsymbol{m_0})$ is in general not contained in $\text{RS}(\mathcal{N}_A, \boldsymbol{m_0})$. Let us show both cases through examples.

When non natural thresholds, $\boldsymbol{\mu} \notin \mathbb{N}^{|T|}$, are considered, $\text{RS}(\mathcal{N}_D, \boldsymbol{m_0})$ is in general not contained in $\text{RS}(\mathcal{N}_A, \boldsymbol{m_0})$ for ordinary HAPN. Let us show it with the following example. Consider the net of the Figure 4 as discrete, $\mathcal{N}_D$, with the initial marking $\boldsymbol{m_0}$ = (3,4). Both t_1 and t_2 can be fired until the place p_1 is empty (when enabling degree is 0). Its reachability space $\text{RS}(\mathcal{N}_D,\ \boldsymbol{m_0})$ is represented in Figure 5 (a). Let us consider now the net as adaptive, with $\boldsymbol{\mu} = (1.5, 1.5)$ Thus, t_1 can fire as continuous while $m[p_1] > 1.5$. And t_2 can fire as continuous while $m[p_1] > 1.5$ and $m[p_2] > 1.5$. When $m[p_1] = 1.5$, t_1 changes from continuos to discrete, and it can fire a discrete amount. Analogously, t_2 changes to discrete and can fire as discrete when $m[p_1] = 1.5$. Its reachability space is shown in Figure 5 (c). Notice that $\text{RS}(\mathcal{N}_D,\ \boldsymbol{m_0})$ contains some markings that are not reachable in $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$. For example, the marking $\boldsymbol{m_2}$ = (1, 4) $\in \text{RS}(\mathcal{N}_D,\ \boldsymbol{m_0})$, but $\boldsymbol{m_2} \notin \text{RS}(\mathcal{N}_A,$ $\boldsymbol{m_0})$.

If non-ordinary PN are considered, $\text{RS}(\mathcal{N}_D)$ is in general no contained in $\text{RS}(\mathcal{N}_A)$, with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$. This can be shown through an example. The reachability space of the HAPN in Figure 1 with $\boldsymbol{\mu} = (1, 1)$ is shown in Figure 6. Transition t_2 is enabled as continuous from marking $(5, 0)$ to $(2, 1.5)$, where it changes to discrete. If t_2 is fired as discrete (from $(2, 1.5)$), $(0, 2.5)$ is reached. In $(0, 2.5)$ none of the transitions are enabled (and the net deadlocks). Transition t_1 is enabled as continuous from $(2, 1.5)$ to $(3, 1)$, where it is enabled as discrete. When t_1 is fired as discrete from $(3, 1)$, $(5, 0)$ is reached and t_1 becomes not enabled.

The marking $\boldsymbol{m} = (1, 2)$ is reachable in the discrete Petri net, but not in the adaptive one with $\forall \mu, \mu = 1$. Therefore, $\text{RS}(\mathcal{N}_D, \boldsymbol{m_0})$ is not, in general, included in $\text{RS}(\mathcal{N}_A, \boldsymbol{m_0})$ with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$ for non ordinary HAPNs.

On the other hand, it is straightforward to prove that, given that HAPNs allow real-valued markings, the RS of $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ is not, in general, included in $\text{RS}(\mathcal{N}_A, \boldsymbol{m_0})$. Nonetheless, if $\boldsymbol{\mu} = \infty$, the HAPN always behaves as discrete and its RS is trivially identical to that of the discrete PN.

Let us finally compare the RS of the HAPN to the RS of its associated continuous PN.

Theorem 6. $\text{RS}(\mathcal{N}_A, \boldsymbol{m_0}) \subseteq \text{RS}(\mathcal{N}_C, \boldsymbol{m_0})$ with $\boldsymbol{\mu} \in \mathbb{R}^{|T|}_{\geq 0}$.

Proof. Let $\boldsymbol{m} \in \text{RS}(\mathcal{N}_A, \boldsymbol{m_0})$. Therefore, there exists $\sigma_a = \beta_1 t_{\gamma_1} \ldots \beta_k t_{\gamma_k}$ such that $\boldsymbol{m_0} \xrightarrow{\beta_1 t_{\gamma_1}} \boldsymbol{m_1} \xrightarrow{\beta_2 t_{\gamma_2}} \boldsymbol{m_2} \cdots \xrightarrow{\beta_3 t_{\gamma_k}} \boldsymbol{m_k} = \boldsymbol{m}$ *where* $\beta_i \in \mathbb{R}^+$ *if* $mode(t_{\gamma_i}, m_{i-1})$ $= C$ *and* $\beta_i \in \mathbb{N}^+$ *if* $mode(t_{\gamma_i}, m_{i-1}) = D$

For any of the β_i of σ_a, if $mode(t_{\gamma_i}, \boldsymbol{m_{i-1}}) = C$, then t_{γ_i} will be also enabled in $\langle \mathcal{N}, \boldsymbol{m_{i-1}}, \rangle$ and the same $\beta_i \in T$ can be chosen. If $mode(t_{\gamma_i}, m_{i-1}) = D$, then t_{γ_i} will be also enabled in $\langle \mathcal{N}, \boldsymbol{m_{i-1}}, \rangle$ and also the same $\beta_i \in \mathbb{N}^+$ can be chosen because $\beta_i \in \mathbb{R}$. Consequently, the same firing sequence σ_a of the HAPN system can be chosen in the continuous system and the same marking $\boldsymbol{m}$ is obtained.

The following Corollary is straightforwardly obtained from Theorems 5 and 6.

Corollary 7. $\text{RS}(\mathcal{N}_D, \boldsymbol{m_0}) \subseteq \text{RS}(\mathcal{N}_A, \boldsymbol{m_0}) \subseteq \text{RS}(\mathcal{N}_C, \boldsymbol{m_0})$ for ordinary nets with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$.

Furthermore, let us show through an example that the RS of the continuous system is, in general, not contained in the RS of the HAPN system, i.e., $\text{RS}(\mathcal{N}_C, \boldsymbol{m_0}) \nsubseteq \text{RS}(\mathcal{N}_A, \boldsymbol{m_0})$ with $\boldsymbol{\mu} \in \mathbb{R}^{|T|}$. In the PN system of Figure 1 (with $\boldsymbol{\mu}$ = (1.5, 1.5)), the marking $\boldsymbol{m}$ = (0.5, 2) is included in $\text{RS}(\mathcal{N}_C, \boldsymbol{m_0})$, but cannot be reached by the HAPN, i.e., it is not included in $\text{RS}(\mathcal{N}_A, \boldsymbol{m_0})$. Both spaces are trivially equal if all the transitions of the HAPN always behave as continuous, i.e., when $\boldsymbol{\mu} = 0$.

4. DEADLOCK-FREENESS IN HAPNS

This section studies the deadlock-freeness property of HAPNs, and relates it to deadlock-freeness of the equivalent discrete PNs. Although for arbitrary $\boldsymbol{\mu}$ deadlock-freeness of the discrete PN is, in general, not preserved by the HAPN, it is shown that the appropriate selection of $\boldsymbol{\mu}$ can preserve the property for a large class of nets.

Let us first show, by considering the net in Figure 1, that:

$$\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle \text{ is deadlock-free} \not\Rightarrow \langle \mathcal{N}_A, \boldsymbol{m_0} \rangle \text{ is deadlock-free.}$$

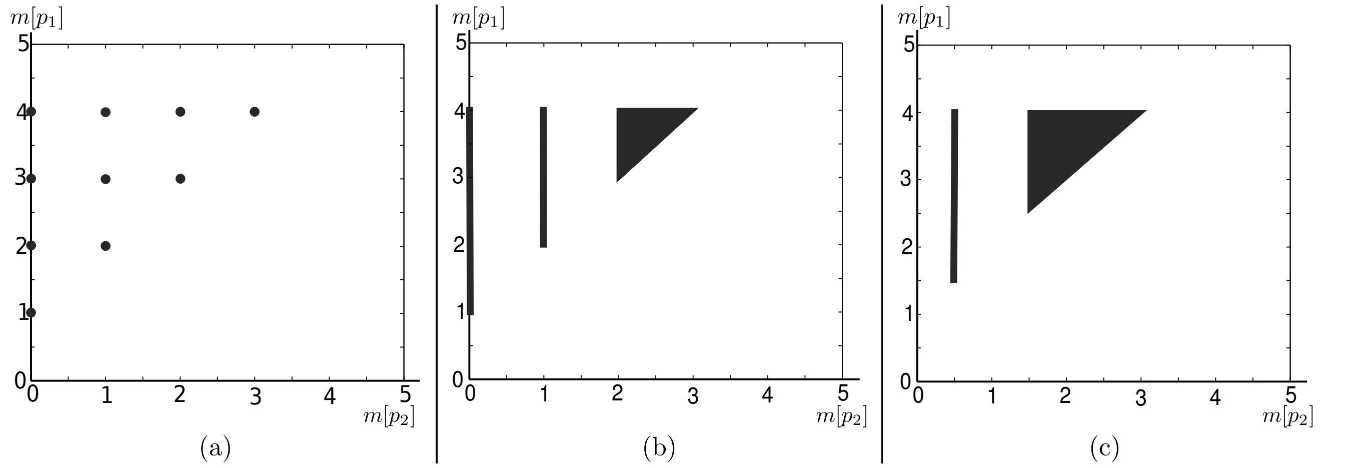

Fig. 5. Reachability space (RS) of the Petri Net of Figure 4 behaving as Discrete (a), HAPN with $\boldsymbol{\mu} = (2, 2)$ (b) or HAPN with $\boldsymbol{\mu} = (1.5, 1.5)$ (c).

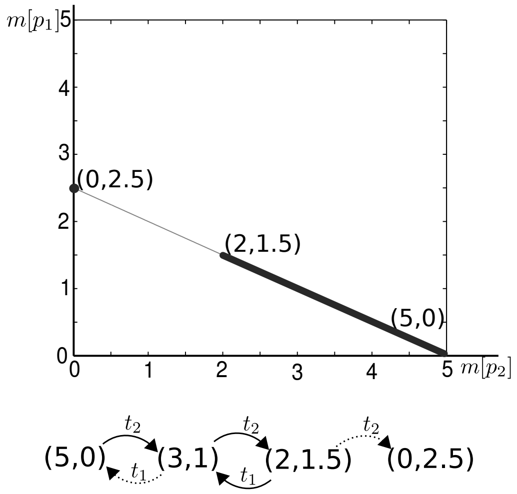

Fig. 6. Reachability space and reachability graph of the Petri Net of the Figure 1 behaving as HAPN with $\boldsymbol{\mu} = (1, 1)$.

The system in Figure 1 with $\boldsymbol{m_0} = (5, 0)$ is deadlock-free if considered as discrete. However, if considered as HAPN with $\boldsymbol{\mu} = (1, 1)$ it deadlocks after firing t_2 as continuous in an amount of 1.5, and again t_2 as discrete, see Fig. 6.

Furthermore, in general, deadlock-freeness of a HAPN system does not guarantee deadlock-freeness of the equivalent discrete system:

$$\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle \text{ is deadlock-free} \not\Rightarrow \langle \mathcal{N}_D, \boldsymbol{m_0} \rangle \text{ is deadlock-free.}$$

The system in the Figure 1 with $\boldsymbol{m_0} = (4, 0)$ deadlocks as discrete. If considered as HAPN, it is deadlock-free with $\boldsymbol{m_0} = (4, 0)$ and $\boldsymbol{\mu} = (1.5, 1.5)$ because t_2 commutes from continuous to discrete when $m[p_1] = 3$, and $m[p_1]$ never empties.

Although the deadlock-freness property of discrete systems is not preserved in general by HAPNs with arbitrary $\boldsymbol{\mu}$, it will be proved that for choice free nets with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$ deadlock-freness of the HAPN system is necessary and sufficient for deadlock-freeness of the discrete system. Let us first prove that it is a sufficient condition.

Theorem 8. Let $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ be an ordinary deadlock-free HAPN system with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$. Then, the discrete system $\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle$ is deadlock-free.

Proof. Let us assume that the discrete $\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle$ deadlocks at a marking $\boldsymbol{m}$. According to Theorem 5, marking $\boldsymbol{m}$ can be reached by $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$. Given that the net is ordinary, for every transition t, there exists $p \in {}^{\bullet}t$ such that $\boldsymbol{m}[p] = 0$, i.e., $\boldsymbol{m}$ is a deadlock for $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$. $\square$

For the necessary condition, two technical lemmas are introduced before stating the final result. The first one states that if a sequence σ is fireable in the adaptive system, its *ceil sequence* $\lceil \sigma \rceil$ is also fireable in the discrete one.

Definition 9. Let $\sigma = \alpha_1 t_{\gamma_1} \; \alpha_2 t_{\gamma_2} \ldots \alpha_k t_{\gamma_k}$ be a firing sequence of a given HAPN $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$. The ceil sequence, $\lceil \sigma \rceil$ of σ is defined as: $\lceil \sigma \rceil = \alpha'_1 t_{\gamma_1} \; \alpha'_2 t_{\gamma_2} \; \ldots \; \alpha'_k t_{\gamma_k}$ where

$$\alpha'_i = \left\lceil \sum_{1 \leq j \leq i | t_{\gamma_i} = t_{\gamma_j}} \alpha_j \right\rceil - \sum_{1 \leq j < i | t_{\gamma_i} = t_{\gamma_j}} \alpha'_j$$

For example, for the sequence $\sigma_1 = 0.1\, t_1\; 0.8\, t_2\; 0.1\, t_1\; 0.2\, t_1\; 0.8\, t_2$ in the HAPN of Figure 3 (a), the ceil sequence $\lceil \sigma_1 \rceil$ is defined as $\lceil \sigma_1 \rceil = 1\; t_1\; 1\; t_2\; 0\; t_1\; 0\; t_1\; 1\; t_2$.

Lemma 10. Let $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ be an ordinary choice-free HAPN system with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$. If σ is a fireable sequence in $\langle \mathcal{N}_A, \boldsymbol{m_0} \rangle$ then $\lceil \sigma \rceil$ is fireable in $\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle$.

Proof. Let us assume without loss of generality that $\sigma = \alpha_1 t_{\gamma_1} \ldots \alpha_k t_{\gamma_k}$ and $0 < \alpha_j \leq 1$ for every $j \in \{, 1 \ldots, k\}$. Induction on the length of σ: $|\sigma| = k$.

- Base case ($|\sigma| = 1$). Let $\sigma = \alpha_1 t_{\gamma_1}$, then $\forall p \in {}^{\bullet}t_{\gamma_1}, \boldsymbol{m_0}[p] \geq \alpha_1$ and given that $\boldsymbol{m_0}[p] \in \mathbb{N}$, it holds that $\boldsymbol{m_0}[p] \geq \lceil \alpha_1 \rceil$. Thus $\lceil \sigma \rceil = \lceil \alpha_1 \rceil t_{\gamma_1}$ can be fired in $\langle \mathcal{N}_D, \boldsymbol{m_0} \rangle$.
- Inductive step. Assume that the Lemma holds for $|\sigma| = k$. Let us consider the $k + 1$ firing, i.e., $t_{\gamma_{k+1}}$ fires in α_{k+1}. Two cases can occur:
  a) $\alpha'_{k+1} = 0$. In this case, the Lemma trivially holds.
  b) $\alpha'_{k+1} = 1$. Let $\boldsymbol{m}_\mathrm{i}$ and $\boldsymbol{\sigma}_i$ ($\boldsymbol{m}'_\mathrm{i}$ and $\boldsymbol{\sigma}'_i$) be the marking and firing count vector obtained just after

the firing of t_{γ_i} in an amount α_i (α'_i). If $t_{\gamma_{k+1}}$ fires in the HAPN system, it means that $\boldsymbol{m}_\mathbf{k}[p] > 0$ for every $p \in {}^\bullet t$. Notice that, by definition of ceil sequence, after the k^{th} firing the following inequalities are satisfied: $\boldsymbol{\sigma}'_k[t] \geq \boldsymbol{\sigma}_k[t]$ and $\boldsymbol{\sigma}'_k[t^q] \geq \boldsymbol{\sigma}_k[t^q]$ for every $t^q \in {}^\bullet({}^\bullet t)$. Given that the net is choice-free, for every place p it holds that $|{}^\bullet p| = 0$ or $|{}^\bullet p| \geq |p^\bullet| = 1$. If for $p \in {}^\bullet t$, it holds that $|{}^\bullet p| \geq |p^\bullet| = 1$, then the previous inequalities ensure $\boldsymbol{m}'_\mathbf{k}[p] \geq 1$. If p has no input transitions, then it must hold that $\boldsymbol{\sigma}'_{k+1}[t] \leq \boldsymbol{m}_\mathbf{0}[p]$. Therefore $t_{\gamma_{k+1}}$ can fire from $\boldsymbol{m}'_\mathbf{k}$ an amount of 1.

□

The second lemma states that if a certain sequence σ deadlocks a HAPN, then its firing count vector is in the naturals.

Lemma 11. Let $\langle \mathcal{N}_A, \boldsymbol{m}_\mathbf{0} \rangle$ be an ordinary choice-free HAPN system with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$. If σ is a fireable sequence $\boldsymbol{m}_\mathbf{0} \xrightarrow{\sigma} \boldsymbol{m}$, such that $\langle \mathcal{N}_A, \boldsymbol{m}_\mathbf{0} \rangle$ deadlocks at $\boldsymbol{m}$, then $\boldsymbol{\sigma} \in (\mathbb{N} \cup \{0\})^{|T|}$, where $\boldsymbol{\sigma}$ is the firing count vector of σ.

Proof. Let us first prove that if $\boldsymbol{m}$ is a deadlock marking then for every transition t there exists $p \in {}^\bullet t$ such that $\boldsymbol{m}[p] = 0$. Notice that just after the last firing of t in the sequence σ, which is necessarily discrete firing given that $\boldsymbol{\mu} \in \mathbb{N}^{|P|}$, at least one place $p \in {}^\bullet t$ becomes empty. Assume that after such a firing, a transition $t' \in {}^\bullet p$ fires. If the firing of t' is discrete then t would become enabled again; if it is continuous then t' is sufficiently enabled to fire also as discrete what would enable t. Hence, after the last firing of t, no transition $t' \in {}^\bullet p$ can fire and p remains empty.

Assume that $\boldsymbol{\sigma}[t] > 0$ is not a natural number and that $\boldsymbol{m}[p] = 0$ for a given $p \in {}^\bullet t$. Then, there exists $t' \in {}^\bullet p$ such that $\boldsymbol{\sigma}[t']$ is not a natural number and $\boldsymbol{\sigma}[t'] \leq \boldsymbol{\sigma}[t] - \boldsymbol{m}_\mathbf{0}[p]$. Notice that there also exists $p' \in {}^\bullet t'$ such that $\boldsymbol{m}[p'] = 0$, hence $t'' \in {}^\bullet p'$ exists such that $\boldsymbol{\sigma}[t'']$ is not a natural number and $\boldsymbol{\sigma}[t''] \leq \boldsymbol{\sigma}[t'] - \boldsymbol{m}_\mathbf{0}[p'] \leq \boldsymbol{\sigma}[t] - \boldsymbol{m}_\mathbf{0}[p] - \boldsymbol{m}_\mathbf{0}[p']$. This reasoning can be repeated until a transition t^* is found such that it deadlocked with $\boldsymbol{\sigma}[t^*] < 1$. Contradiction since natural thresholds do not allow $\boldsymbol{\sigma}[t^*]$ to be less than 1. □

Therefore, because of Lemmas 10 and 11, if a deadlock marking $\boldsymbol{m}$ is reachable in $\langle \mathcal{N}_A, \boldsymbol{m}_\mathbf{0} \rangle$ when σ is fired, the same deadlock marking $\boldsymbol{m}'$ is reachable in $\langle \mathcal{N}_A, \boldsymbol{m}_\mathbf{0} \rangle$, when $\lceil \sigma \rceil$ is fired. Thus, if $\langle \mathcal{N}_D, \boldsymbol{m}_\mathbf{0} \rangle$ is deadlock-free, then $\langle \mathcal{N}_A, \boldsymbol{m}_\mathbf{0} \rangle$ is deadlock-free too.

Theorem 12. Let $\langle \mathcal{N}_D, \boldsymbol{m}_\mathbf{0} \rangle$ be an ordinary choice-free and deadlock-free discrete system. Then, the HAPN system $\langle \mathcal{N}_A, \boldsymbol{m}_\mathbf{0} \rangle$ is deadlock-free for any $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$.

The following Corollary is straightforwardly obtained from Theorems 8 and 12.

Corollary 13. Let $\mathcal{N}$ be an ordinary choice-free net. $\langle \mathcal{N}_D, \boldsymbol{m}_\mathbf{0} \rangle$ is deadlock-free iff $\langle \mathcal{N}_A, \boldsymbol{m}_\mathbf{0} \rangle$ is deadlock-free with $\boldsymbol{\mu} \in \mathbb{N}^{|T|}$.

5. CONCLUSIONS

As most formalisms for discrete event systems, Petri nets suffer from the state explosion problem. Such a problem renders enumerative analysis techniques unfeasible for large systems. The *hybrid adaptive Petri nets* considered here aim at alleviating the state explosion problem by partially relaxing the firing of transitions. More precisely, a transition can fire in real amounts when its load is *higher* than a given threshold, and it is forced to fire in discrete amounts when its load is lower than that threshold. This partial relaxation offers the chance of preserving important properties of discrete event systems, as deadlock-freeness, that are not always retained by fully continuous approximations.

This paper focused on the reachability space and the deadlock-freeness property of hybrid adaptive nets. For a rather general class of nets, an inclusion relationship was proved for the reachability spaces of the discrete, hybrid adaptive and continuous nets. With respect to deadlock-freeness, although this property is not preserved in general for arbitrary real thresholds, it was shown that it is necessary and sufficient for deadlock-freeness of choice-free nets with arbitrary natural thresholds.

Future work will focus on the definition of $\boldsymbol{\mu}$ for deadlock-freeness preservation of more general net structures.

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