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Minimum-time Control for Structurally Persistent Continuous Petri Nets

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Abstract— **This paper addresses the minimum-time control problem of structurally persistent timed continuous Petri Net systems ($ContPN$). In particular, an *ON-OFF* controller is proposed to drive the system from a given initial marking (state) to the desired (final) marking in minimum-time. The controller is developed first for the discrete-time system ensuring that all transitions are fired as fast as possible in each sampling period until the required total firing counts are reached. After that, they are stopped suddenly. By taking the limit of the sampling period, the controller for continuous-time systems is obtained. Simplicity and the fact that it ensures minimum-time are the main advantages of the controller.**

I. INTRODUCTION

Petri Nets (PN) is a well known paradigm used for modeling, analysis, and synthesis of *discrete event systems* (DES). With strong facility to depict the sequence, concurrency, conflict and other synchronous relationships, it is widely applied in the industry, for the analysis of manufacturing [5], traffic, and software systems, etc. Similar to other modeling formalisms for DES, it also suffers from the *state explosion* problem. To overcome it, a classical relaxation technique called *fluidification* can be used.

Continuous Petri nets [4], [13] are fluid approximations of classical *discrete Petri nets* obtained by removing the integrality constrains, which means the firing counter vector and consequently the marking is no longer restricted to be in the naturals but relaxed into the non-negative real numbers. An important advantage of this relaxation is that more efficient algorithms are available for their analysis, e.g., reachability and controllability [10], [7] problems.

Different approaches have been proposed in the literature for the control of different classes of $ContPNs$, e.g., First-Order Hybrid Petri nets [3] or finite server semantics [1] etc. In this work, the minimum-time control problem of *timed continuous Petri nets* under *infinite server semantics* is considered. For this class of systems several control approaches have been considered. In [10], the steady state control and optimal steady state control are studied. Model Predictive Control (MPC) is used for optimal control problem in [9] assuming a discrete-time model. In [16], a Lyapunov-function-based dynamic control algorithm is studied while in [2] an efficient heuristics for minimum-time control is proposed.

Here, we design an *ON-OFF* controller for structurally persistent $ContPN$ systems. With this controller, we will prove that the system is driven from an initial marking to a final one in minimum-time. The basic idea of the proposed control strategy is to fire every transition as fast as possible until the required total firing count is achieved (ON), and then it is stopped (OFF). This kind of controller has been studied in the case of linear systems [12], [14] and it is proved to be minimum-time in some cases. Unfortunately our system is only piecewise linear and the classical results can not be applied.

This paper is organized as follows: Section II briefly recalls some basic concepts on $ContPN$. Section III states the control problem, as it is here addressed. In Section IV, the *ON-OFF* controller is proposed to reach the final state in minimum-time. Section V gives an example of the application of the *ON-OFF* controller to the $ContPN$ model of a manufacturing system. The conclusions are in Section VI.

II. CONTINUOUS PETRI NETS

The reader is assumed to be familiar with basic Petri net concepts (see [4], [13] for a gentle introduction).

Definition *2.1:* A continuous Petri net system is a pair $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ where $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post} \rangle$ is a net structure where:

- P and T are the sets of places and transitions respectively.
- $\boldsymbol{Pre}, \boldsymbol{Post} \in \mathbb{R}_{\geq 0}^{|P|\times|T|}$ are the pre and post matrices.
- $\boldsymbol{m}_0 \in \mathbb{R}_{\geq 0}^{|P|}$ is the initial marking (state).

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For $v \in P \cup T$, the sets of its input and output nodes are denoted as ${}^{\bullet}v$ and $v^{\bullet}$, respectively. Let p_i, $i = 1, \ldots, |P|$ and $t_j, j = 1, \ldots, |T|$ denote the places and transitions. Each place can contain a non-negative real number of tokens, this number represents the marking of the place. The distribution of tokens in places is denoted by $\boldsymbol{m}$. A transition $t_j \in T$ is enabled at $\boldsymbol{m}$ iff $\forall p_i \in^{\bullet} t_j$, $m(p_i) > 0$ and its enabling degree is given by

$$enab(t_j, \boldsymbol{m}) = \min_{p_i \in {}^{\bullet}t_j} \left\{ \frac{m(p_i)}{Pre(p_i, t_j)} \right\}$$

which represents the maximum amount in which t_j can fire. Transition t_j is called *k-enabled* under marking $\boldsymbol{m}$, if $enab(t, \boldsymbol{m}) = k$. An enabled transition t_j can fire in any real amount α, with $0 < \alpha \leq enab(t_j, \boldsymbol{m})$ leading to a new state $\boldsymbol{m'} = \boldsymbol{m} + \alpha \cdot \boldsymbol{C}(\cdot, t_j)$ where $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the *token flow matrix* and $\boldsymbol{C}(\cdot, j)$ is its j^{th} column.

If $\boldsymbol{m}$ is reachable from $\boldsymbol{m}_0$ through a finite sequence σ, the state (or fundamental) equation is satisfied: $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \vec{\sigma}$, where $\vec{\sigma} \in \mathbb{R}^{|T|}_{\geq 0}$ is the *firing count vector*, i.e., $\vec{\sigma}(t_j)$ is the cumulative amount of firings of t_j in the sequence σ.

If for all $p \in P$, $|p^{\bullet}| = 1$ then $\mathcal{N}$ is called *structurally persistent* PN, in the sense that independently by the initial marking, the net has no conflict.

***Property** 2.2:* [8] Let $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ be a structurally persistent PN system. If t_j is k-enabled at $\boldsymbol{m}$, it will remain k-enabled until t_j is fired.

In timed continuous Petri net($ContPN$) the state equation has an explicit dependence on time: $\boldsymbol{m}(\tau) = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \vec{\sigma}(\tau)$ which through time differentiation becomes $\dot{\boldsymbol{m}}(\tau) = \boldsymbol{C} \cdot \dot{\vec{\sigma}}(\tau)$. The derivative of the firing sequence $\boldsymbol{f}(\tau) = \dot{\vec{\sigma}}(\tau)$ is called the *firing flow*. Depending on how the flow is defined, many firing semantics appear, being the most used ones *infinite* and *finite* server semantics [13]. For a broad class of Petri nets it is shown that infinite server semantics offers better approximation than finite server semantics [11]. This paper deals with infinite server semantics for which the flow of a transition t_j at time τ is the product of the firing rate, λ_j, and the enabling degree of the transition at $\boldsymbol{m}(\tau)$

$$f(t_j, \tau) = \lambda_j \cdot enab(t_j, \boldsymbol{m}(\tau)) = \lambda_j \cdot \min_{p_i \in {}^{\bullet}t_j} \left\{ \frac{m(p_i, \tau)}{Pre(p_i, t_j)} \right\} \tag{1}$$

For the sake of clarity, τ will be omitted in the rest of the paper when there is no confusion: $f(t_j)$, $\boldsymbol{m}$ and $m(p_i)$ will be used instead of $f(t_j, \tau)$, $\boldsymbol{m}(\tau)$ and $m(p_i, \tau)$.

III. Problem Statement

We now consider net systems subject to external control actions, and assume that the only admissible control law consists in *slowing down* the firing speed of transitions [13]. Under this assumption, the controlled flow of a $ContPN$ system is denoted as: $\boldsymbol{w}(\tau) = \boldsymbol{f}(\tau) - \boldsymbol{u}(\tau)$, with $0 \leq \boldsymbol{u}(\tau) \leq \boldsymbol{f}(\tau)$. The overall behavior of the system is ruled by: $\dot{\boldsymbol{m}} = \boldsymbol{C} \cdot (\boldsymbol{f}(\tau) - \boldsymbol{u}(\tau))$. In this paper, we assume that every transition is *controllable* (t_j is uncontrollable if the only control that can be applied is $u(t_j) = 0$).

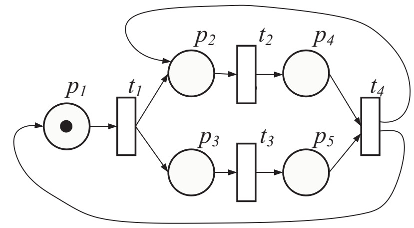

Fig. 1. Structurally Persistent Petri Net System

The problem we deal with in this work is: how to design a control action $\boldsymbol{u}$ that drives the system from the initial marking $\boldsymbol{m}_0$ to the desired final marking $\boldsymbol{m}_f$ in minimum-time?

***Example** 3.1:* Fig. 1 shows a structurally persistent and unbounded $ContPN$. Assume $\boldsymbol{m}_0 = [1\ 0\ 0\ 0\ 0]^T$, $\boldsymbol{\lambda} = [1\ 1\ 1\ 1]^T$, and the desired final marking $\boldsymbol{m}_f = [0.3\ 0.4\ 0.3\ 0.4\ 0.4]^T$.

Considering the model as untimed, the following firing sequence ensures the reachability of the final marking: $\sigma = t_1(0.8)t_2(0.5)t_3(0.5)t_4(0.1)$. Looking at the system as timed and considering σ, the final marking can be reached in the following way:

(i) fire first as fast as possible t_1 and stop the other transitions; since $\int_0^{1.61} f(t_1)d\tau = 0.8$, this firing takes 1.61 time units;
(ii) open t_2 until the integral of its flow is equal to 0.5 and stop the other transitions; this firing takes 0.98 time unit because $\int_0^{0.98} f(t_2)d\tau = 0.5$;
(iii) stop all transitions and fire only t_3 until its flow integral is 0.5; this will take 0.98 time unit because $\int_0^{0.98} f(t_3)d\tau = 0.5$;
(iv) finally, open only t_4 for 0.22 time unit because $\int_0^{0.22} f(t_4)d\tau = 0.1$.

The previous strategy on the time system corresponds to the following control actions $\boldsymbol{u}(\tau)$:

(i) $\boldsymbol{u}(\tau) = [0\ f(t_2)\ f(t_3)\ f(t_4)]^T$ for $0 \leq \tau \leq 1.61$;
(ii) $\boldsymbol{u}(\tau) = [f(t_1)\ 0\ f(t_3)\ f(t_4)]^T$ for $1.61 < \tau \leq 2.59$;
(iii) $\boldsymbol{u}(\tau) = [f(t_1)\ f(t_2)\ 0\ f(t_4)]^T$ for $2.59 < \tau \leq 3.57$;
(iv) $\boldsymbol{u}(\tau) = [f(t_1)\ f(t_2)\ f(t_3)\ 0]^T$ for $3.57 < \tau \leq 3.79$;
(v) if $\tau > 3.79$, $\boldsymbol{u}(\tau) = \boldsymbol{f}(\tau)$, i.e., all transitions are

stopped.

With these control actions, the system can reach the final marking in 3.79 time units, but as it will be shown, this is not a minimum-time controller because actions are unnecessarily sequentialized.

IV. Minimum-time Controller

In this section, an *ON-OFF* controller is proposed for structurally persistent $ContPN$ systems and it will be shown that it is a minimum-time controller. We will first present some assumptions, then the controller is designed for both discrete-time and continuous-time $ContPN$.

A. Minimal Firing Count Vector

In general, a marking $\boldsymbol{m}$ can be reached from $\boldsymbol{m}_0$ by using different firing sequences. For example, if the net is consistent and $\boldsymbol{m}$ is reached with σ, it is also reached when firing a T-*semiflow* $\alpha \geq 0$ times before (or interleaved with) σ. Here we introduce the notion of *minimal firing count vector*, and prove that it is unique under some assumptions for persistent nets.

***Definition** 4.1:* Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m}_0 \rangle$ be a $ContPN$ system and $\boldsymbol{m}_f$ be a reachable marking through a sequence σ, i.e., $\boldsymbol{m}_f = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \vec{\sigma}$. The firing count vector $\vec{\sigma}$ is said to be *minimal* if for any T-*semiflow* $\boldsymbol{x}$, $||\boldsymbol{x}|| \not\subseteq ||\vec{\sigma}||$, where $||\cdot||$ stands for the support of a vector, i.e., the index of the elements different than zero.

***Example** 4.2:* Let us consider the net system in Fig. 2 that is not structurally persistent because $p_1^\bullet = \{t_1, t_2\}$, and $p_2^\bullet = \{t_3, t_6\}$. Assume $\boldsymbol{m}_0 = [1\ 0\ 0\ 0]^T$ and $\boldsymbol{m}_f = [0\ 0\ 0\ 1]^T$. Firing the sequence $\sigma_1 = t_1(1)t_3(1)$ ($\vec{\sigma}_1 = [1\ 0\ 1\ 0\ 0\ 0]^T$) from $\boldsymbol{m}_0$ the obtained marking is $\boldsymbol{m}_f$. The same marking is obtained by firing $\sigma_2 = t_1(1)t_6(1)t_1(1)t_3(1)$ ($\vec{\sigma}_2 = \vec{\sigma}_1 + [1\ 0\ 0\ 0\ 0\ 1]^T$) since $[1\ 0\ 0\ 0\ 0\ 1]^T$ is a T-*semiflow*. Therefore, $\vec{\sigma}_1$ is a minimal firing count vector, while $\vec{\sigma}_2$ is not. Normally, the minimal firing count vector is not unique. For this net, $\vec{\sigma_3} = [0\ 1\ 0\ 1\ 0\ 0]^T$ ($\sigma_3 = t_2(1)t_4(1)$) is another minimal firing count vector leading to $\boldsymbol{m}_f$.

***Proposition** 4.3:* Let $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ be a structurally persistent PN system and $\boldsymbol{m}_f$ be a reachable marking. If one of the following assumptions is satisfied, there exits a unique minimal firing count vector $\vec{\sigma}$.

(A1) The matrix $\boldsymbol{C}$ has full rank;

(A2) The $ContPN$ is strongly connected and consistent.

Proof: Suppose there exist two minimal firing count vectors $\vec{\sigma}_1$ and $\vec{\sigma}_2$, then (1) $\boldsymbol{m}_f = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \vec{\sigma}_1$, (2) $\boldsymbol{m}_f = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \vec{\sigma}_2$. Subtracting (2) from (1), we obtain:

$$\boldsymbol{C} \cdot (\vec{\sigma}_1 - \vec{\sigma}_2) = \boldsymbol{C} \cdot \vec{\sigma}_{12} = 0$$

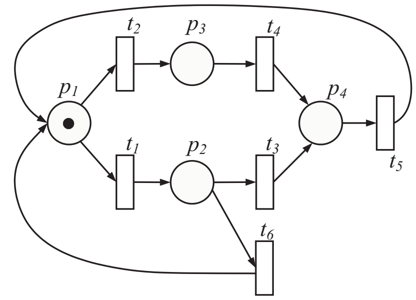

Fig. 2. A non Structurally Persistent Petri Net System

If (A1) is satisfied, we must have $\vec{\sigma}_{12} = 0$, so $\vec{\sigma}_1 = \vec{\sigma}_2 (\neq 0$, if $m_f \neq m_0)$.

If (A2) is satisfied, there is only one minimal T-*semiflow* [15], denoted by $\boldsymbol{x} > 0$. $\vec{\sigma}_{12}$ may have negative elements, but we can always find an $\alpha \geq 0$, such that $\vec{\sigma}_{12} + \alpha \cdot \boldsymbol{x} \geq 0$. Since $\boldsymbol{C} \cdot (\vec{\sigma}_{12} + \alpha \cdot \boldsymbol{x}) = 0$ and $\vec{\sigma}_{12} + \alpha \cdot \boldsymbol{x} \geq 0$, it is a T-*semiflow*. Therefore, there exists $\beta > 0$ such that $\vec{\sigma}_{12} + \alpha \cdot \boldsymbol{x} = \beta \cdot \boldsymbol{x}$, implying $\vec{\sigma}_{12} = (\beta - \alpha) \cdot \boldsymbol{x}$. If $\beta - \alpha = 0$ then $\vec{\sigma}_1 = \vec{\sigma}_2$ which is impossible by assumption. If $\beta - \alpha > 0$ then $\vec{\sigma}_1 = \vec{\sigma}_2 + (\beta - \alpha) \cdot \boldsymbol{x} > (\beta - \alpha) \cdot \boldsymbol{x}$. Therefore, $\vec{\sigma}_1$ is not a minimal firing count vector. Similarly, if $\beta - \alpha < 0$ then $\vec{\sigma}_2$ is not a minimal firing count vector. ■

Hence, for a structurally persistent system under assumption (A1) or (A2), any controller driving the system to $\boldsymbol{m}_f$ must follow *the* minimal firing count vector plus eventually a T-*semiflow*. If we are interested in the minimum-time controller then it should follow *only* the minimal firing count vector since a T-*semiflow* can be fired independently, before the minimal firing sequence and this firing takes time. In the following we will show that among all possible controllers having the integral of firing flow equal to the minimal firing count vector, the one corresponding to the *ON-OFF* strategy provides the minimum-time controller.

B. Minimum-time controller: Discrete-time Case

Sampling the continuous-time $ContPN$ system with a *sampling period* Θ, we obtain the discrete-time $ContPN$ [9] given by:

$$\begin{aligned} &\boldsymbol{m}(k+1) = \boldsymbol{m}(k) + \boldsymbol{C} \cdot \boldsymbol{w}(k) \cdot \Theta \\ &0 \leq \boldsymbol{w}(k) \leq \boldsymbol{f}(k) \end{aligned} \tag{2}$$

Here $\boldsymbol{m}(k)$ and $\boldsymbol{w}(k)$ are the marking and controlled flow at sampling instant k, i.e., at $\tau = k \cdot \Theta$. Let $u(t_j, k)$, $f(t_j, k)$ and $w(t_j, k)$ denote the control action, flow and controlled flow of transition t_j. The firing count of t_j in k^{th} sampling period is denoted by $s_k(t_j) = w(t_j, k) \cdot \Theta$.

Property 2.2 shows that for structurally persistent systems if two transitions t_1 and t_2 are enabled at the same time, the order of firing is not important (i.e., both sequences $t_1 t_2$ and $t_2 t_1$ are fireable).

***Example** 4.4:* Let us consider again the net system in Fig. 1 but now as discrete-time with $\Theta = 0.2$. Assume $\boldsymbol{m}_0 = \boldsymbol{m}(0) = [0\ 1\ 1\ 1\ 1]^T$, $\boldsymbol{\lambda} = [1\ 1\ 1\ 1]^T$ and $\boldsymbol{m}_f = [0.2\ 1.1\ 0.9\ 0.9\ 0.9]^T$. The minimal firing count vector in this case is $\vec{\sigma} = [0\ 0.1\ 0.1\ 0.2]^T$. The following controlled flow ensures the reaching of $\boldsymbol{m}_f$ in two sampling periods:

- At $k = 0$: $w(t_1, 0) = w(t_4, 0) = 0$ and $w(t_2, 0) = w(t_3, 0) = 0.5$. Then t_2, t_3 are fired in an amount $0.5 \cdot \Theta = 0.1$ and t_1 and t_4 are stopped. The system reaches $\boldsymbol{m}(1) = [0\ 0.9\ 0.9\ 1.1\ 1.1]^T$.
- At $k = 1$: $w(t_1, 1) = w(t_2, 1) = w(t_3, 1) = 0$ and $w(t_4, 1) = 1$. Then t_1, t_2 and t_3 are stopped while t_4 is fired in an amount $1 \cdot \Theta = 0.2$. After this sampling period, $\boldsymbol{m}_f$ is reached.

Under this control law, t_2 and t_3 are fired before t_4. Since t_4 is 1-enabled at $\boldsymbol{m}_0$ it can be fired first and $\boldsymbol{m}_f$ is still reached. Therefore, another control law corresponding to the same minimal firing count vector is:

- At $k = 0$: $w(t_1, 0) = w(t_2, 0) = w(t_3, 0) = 0$ and $w(t_4, 0) = 1$. Hence, t_1, t_2 and t_3 are stopped, and t_4 is fired in an amount $1 \cdot \Theta = 0.2$. Now, $\boldsymbol{m}(1) = [0.2\ 1.2\ 1\ 0.8\ 0.8]^T$.
- At $k = 1$: $w(t_1, 1) = w(t_4, 1) = 0$ and $w(t_2, 1) = w(t_3, 1) = 0.5$. Hence, t_2 and t_3 are fired in an amount 0.1 while t_1 and t_4 are stopped.

***Definition** 4.5:* Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \Theta, \boldsymbol{m}_0 \rangle$ be a discrete-time $ContPN$ system and $\boldsymbol{m}_f$ be a reachable final marking with a firing count vector $\vec{\sigma}$. Then, transition t_j is said to be *sufficiently fired* in the k^{th} sampling period if one of the following conditions holds:

- $s_k(t_j) \overset{\text{def}}{=} w(t_j, k) \cdot \Theta = f(t_j, k) \cdot \Theta$, i.e., $u(t_j, k) = 0$,
- $0 < s_k(t_j) \leq f(t_j, k) \cdot \Theta$ and $s_k(t_j) + \sum_{i=0}^{k-1} s_i(t_j) = \vec{\sigma}(t_j)$.

In the first case t_j is fired in the maximal amount, while in the second case it is the last firing of t_j in the corresponding sequence.

For instance, let us examine the first control law in Ex. 4.4 for $k = 0$. Transition t_4 is not sufficiently fired, because $\vec{\sigma}(t_4) = 0.2$ (should fire in an amount of 0.2) but $s_0(t_4) = w(t_4, 0) \cdot \Theta = 0$ (it is not fired). On the other hand t_2 and t_3 are sufficiently fired at $k = 0$ because $s_0(t_2) = s_0(t_3) = 0.1 = \vec{\sigma}(t_2) = \vec{\sigma}(t_3)$.

In control theory, an *ON-OFF* controller is a controller that switches abruptly between two states. It is frequently used in minimum-time problems and actually optimal control in many cases [12], [6]. Here we design an *ON-OFF* controller for structurally persistent Petri nets and prove that it is the minimum-time controller for such systems.

***Definition** 4.6:* Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \Theta, \boldsymbol{m}_0 \rangle$ be a structurally persistent discrete-time $ContPN$ system and $\boldsymbol{m}_f$ be a reachable final marking with the corresponding minimal firing count vector $\vec{\sigma}$. An *ON-OFF* controller is defined as: $u(t_j, k) =$

$$\begin{cases} 0 & \text{if } \sum_{i=0}^{k-1} s_i(t_j) + f(t_j,k)\cdot\Theta \leq \vec{\sigma}(t_j) \quad \text{(a)} \\ f(t_j,k) & \text{if } \sum_{i=0}^{k-1} s_i(t_j) = \vec{\sigma}(t_j) \quad \text{(b)} \\ f(t_j,k) - \frac{\vec{\sigma}(t_j) - \sum_{i=0}^{k-1} s_i(t_j)}{\Theta} & \text{if } \sum_{i=0}^{k-1} s_i(t_j) < \vec{\sigma}(t_j) \text{ and} \quad \text{(c)} \\ & \sum_{i=0}^{k-1} s_i(t_j) + f(t_j,k)\cdot\Theta > \vec{\sigma}(t_j) \end{cases} \tag{3}$$

Assuming at $k = 0$, $\sum_{i=0}^{k-1} s_i(t_j) = 0$.

(a) says that before reaching the required total firing count $\vec{\sigma}(t_j)$, we simply let transition t_j to fire *free*, i.e. $u(t_j, k) = 0$; (b) means once $\vec{\sigma}(t_j)$ is reached, the transition is completely stopped, i.e. $u(t_j, k) = f(t_j, k)$; (c) represents the last firing of t_j but can not fire at maximum speed to not overpass $\vec{\sigma}(t_j)$. After all the transitions are stopped, the system will stay in the final marking.

The basic idea of the *ON-OFF* controller is that in each sampling period k, every transition is sufficiently fired.

***Proposition** 4.7:* Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \Theta, \boldsymbol{m}_0 \rangle$ be a structurally persistent discrete-time $ContPN$ system and $\boldsymbol{m}_f$ be a reachable final marking with the corresponding minimal firing count vector $\vec{\sigma}$. The *ON-OFF* controller is a minimum-time controller driving the system to $\boldsymbol{m}_f$.

Proof: We will prove that whenever there exists a controller $\boldsymbol{G}$ driving the system to $\boldsymbol{m}_f$, it consumes at least the time of the *ON-OFF* controller. This will imply that the *ON-OFF* controller is the minimum-time controller.

Assume a non *ON-OFF* controller $\boldsymbol{G}$. Hence, there exists a transition t_j that is not sufficiently fired in a sampling period k. In other words, t_j has to be fired later in a sampling period l, $l > k$. Let us assume, without loss of generality, that t_j is not fired between the k^{th} and the l^{th} sampling period. It is always possible to "move" some amount of firing from the l^{th} sampling period to the k^{th} one until t_j becomes sufficiently fired in k. According to Property 2.2 this move does not affect the fireability of the other transitions. Iterating the procedure, all transitions can be made sufficiently fired in all sampling periods and the obtained controller is an *ON-OFF* one.

Obviously, the number of discrete-time periods necessary to reach the final marking after moving firings from a sampling period l to another one k with $k \le l$ is at least the same. Hence the number of sampling steps is not higher with the *ON-OFF* controller. ■

C. Minimum-time controller: Continuous-time Case

***Definition** 4.8:* Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m}_0 \rangle$ be a structurally persistent continuous-time $ContPN$ system, and $\boldsymbol{m}_f$ be a reachable final marking with the corresponding minimal firing count vector $\vec{\sigma}$. An *ON-OFF* controller is defined as:

$$u(t_j) = \begin{cases} 0 & \text{if } \int_0^{\tau^-} w(t_j)\mathrm{d}\tau < \vec{\sigma}(t_j) \quad (a) \\ f(t_j) & \text{if } \int_0^{\tau^-} w(t_j)\mathrm{d}\tau = \vec{\sigma}(t_j) \quad (b) \end{cases} \tag{4}$$

(a) means that if $\vec{\sigma}(t_j)$ is not reached then t_j is completely *ON*, i.e., $u(t_j) = 0$; else (b), t_j is completely *OFF*, i.e., $u(t_j) = f(t_j)$.

***Corollary** 4.9:* Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m}_0 \rangle$ be a structurally persistent continuous-time $ContPN$ system, and $\boldsymbol{m}_f$ be a reachable final marking with the corresponding minimal firing count vector $\vec{\sigma}$. The *ON-OFF* controller given by (4) is a minimum-time one driving the system to $\boldsymbol{m}_f$.

Proof: If we take sampling period $\Theta \to 0$ in Def. 4.6, the *ON-OFF* controller in Def. 4.8 is obtained. According to Proposition 4.7, this is the minimum-time controller. ■

Let us notice that once a place of a continuous-time $ContPN$ is marked, it will take infinite time to be emptied (like the discharging of a capacitor in an electrical RC-circuit). The *ON-OFF* controller of a structurally persistent net is a minimum-time controller if no place is emptied during the trajectory from $\boldsymbol{m}_0$ to $\boldsymbol{m}_f$. Otherwise, the final marking is reached at the limit, in infinite time. For example, to reach $\boldsymbol{m}_f = [0\ 0\ 1]^T$ in the net system in Fig. 3, p_1 has to be emptied while p_2 should be marked first and then emptied. Hence, $\boldsymbol{m}_f$ is reached at the limit. Nevertheless, if $\boldsymbol{m}_f = [0.5\ 0.5\ 0]^T$ (that is not strictly positive), it is reached in finite time since there exists a trajectory no emptying any place, i.e., firing t_1 in an amount 0.5.

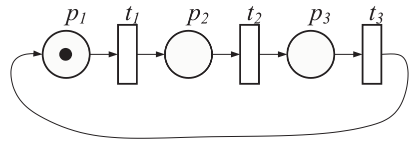

Fig. 3. $ContPN$ with $\boldsymbol{m}_0 = [1\ 0\ 0]$. Assume $\boldsymbol{m}_f = [0\ 0\ 1]$.

***Example** 4.10:* Let's consider the same problem in Ex. 3.1 (Fig. 1) but with an *ON-OFF* controller. The marking $\boldsymbol{m}_f$ is reached in 1.65 time units comparing with 3.79 time units in Ex. 3.1 where a different (pure sequentialized) controller with the same minimal firing count vector is applied. The marking trajectory is in Fig. 4.

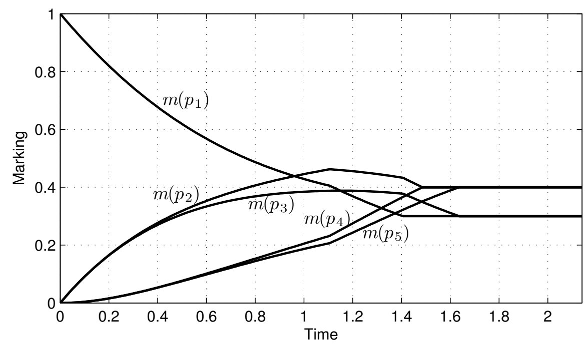

Fig. 4. Marking trajectories when applying *ON-OFF* in Ex. 3.1

V. Case Study

Let's consider the net system in Fig. 5, which models a table factory system (taken from [15]).

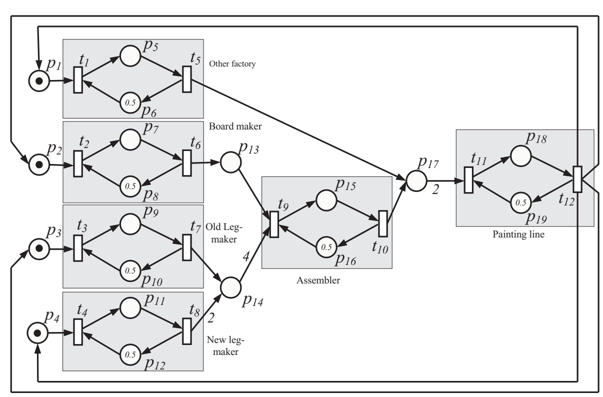

Fig. 5. Persistent PN model of a table factory system. Assume the firing rate of every transition to be equal to 1.

The system consists of several parts, including board maker, leg maker, assembler, painting line and is modeled by a weighted structurally persistent $ContPN$. Suppose in the initial marking $m_0(p_1) = m_0(p_2) = m_0(p_3) = m_0(p_4) = 1$, $m_0(p_6) = m_0(p_8) = m_0(p_{10}) = m_0(p_{12}) = m_0(p_{16}) =$

$m_0(p_{19}) = 0.5$, and the other places are empty. Assume $\boldsymbol{m}_f$ be $m_f(p_3) = m_f(p_{17}) = 0.1$, $m_f(p_4) = m_f(p_5) = 0.2$, $m_f(p_{13}) = 0.15$, and all the other places with markings equal to 0.25. The corresponding minimal firing count vector $\vec{\sigma} = [0.85\ 0.85\ 1.0\ 0.9\ 0.6\ 0.6\ 0.75\ 0.65\ 0.45\ 0.2\ 0.35\ 0.10]$.

Applying *ON-OFF* on the system under continuous time, Fig. 6 shows the stopping time instants of transitions. After t_9 is stopped at 4.28 time units, all the places are at the final state values.

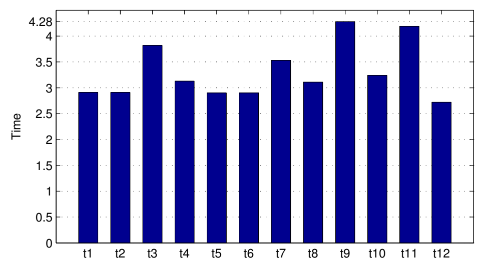

Fig. 6. Stopping time instants

Fig. 7 shows the marking trajectory of place p_3, p_{13}, p_{14} and p_{17}. Taking p_{17} as a example, it reaches the final state at 4.19 time units. That makes sense, because the marking of p_{17} is dependent on transitions t_5, t_{10} and t_{11}, which are stopped at 2.9, 3.24 and 4.19 time units, respectively. When t_{11} is stopped, p_{17} has reached the final state.

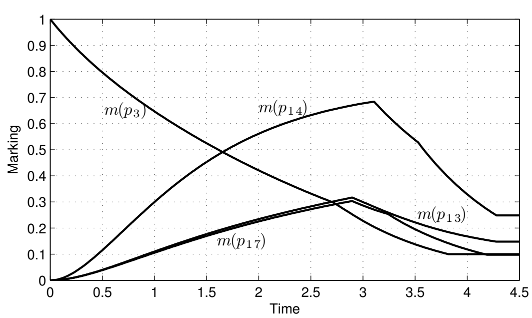

Fig. 7. Marking trajectories

VI. Conclusions

In this paper, an *ON-OFF* controller is designed for structurally persistent $ContPN$ systems which can drive the system from an initial marking to a final one in minimum-time. The idea behind is extremely simple and efficient: the system reaches its final marking in minimum-time if it follows the minimal firing count vector. In the framework of discrete-time systems, we design the *ON-OFF* controller, such that all transitions are fired as fast as possible, and suddenly stopped when the total firing counts are reached. Special attention should be paid to the last sampling period before stopping in order to prevent the total firing count from being exceeded. It is proved that with this controller the final marking is reached in minimum-time. By considering the limit (going to zero) of the sampling period, the results are extended to continuous time systems.

As a future work we plan to compare this procedure with the ones already developed [2], [9], [16].

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