Basic Qualitative Properties of Petri Nets with Multi-Guarded Transitions

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***Abstract*— In many discrete event dynamic systems, there exist tasks that can start performing before every input data is available. A common device exhibiting such a feature is the multiplexer: its output can be produced as soon as data is available in the selected channel without waiting for data in the other channels. The Petri net formalism can be easily extended to model this behavior by allowing a transition have several guards. Under this extension, a transition can fire as soon as the guard selected for the next firing is satisfied, what usually enhances the system performance with respect to the conventional "mono-guarded" system. This paper explores how some fundamental qualitative properties are preserved or lost when a conventional system is transformed to a system with several guards. The main properties studied are reachability, boundedness, and liveness.**

I. INTRODUCTION

Many formalisms for discrete event dynamic systems rely on the AND-causality paradigm, which is often associated with *rendez-vous* synchronizations. A well-known formalism based on AND-causality is the Petri net [6], [9] formalism. For many modeling purposes the AND-causality paradigm is appropriate. However, it can be excessively *rigid* when trying to model operations that can produce results when some input data are not available.

This paper deals with a class of Petri nets that associates several guards to every transition. This feature naturally models the fact that some operations can start performing even if some input data are not available.

Unfortunately, fundamental system properties as liveness may be lost when several guards are considered. The main goal of this work is to study how some basic qualitative properties change when a conventional Petri net is transformed to a Petri net with several guards. Namely, the paper focuses on reachability, boundedness, and specially on liveness.

A. Early Evaluation

In some discrete event dynamic systems an event cannot occur until all input data are available. Consider, for instance, a machine that assembles four legs and one board to make a table: The machine cannot produce a table until four legs and one board are made available to the machine. In contrast to this *strict* precondition, there exist tasks that can produce a result even if some input data are not available. A typical component in digital circuits that exhibits this behavior is the multiplexer.

The Petri net in Fig. 1 models a simple multiplexer with two input data (a and b), one input control signal (c) and one output data (z). Its behavior is given by the following expression:

$$\texttt{if } c \texttt{ then } z = a \texttt{ else } z = b \tag{1}$$

The availability of information is determined by the presence of tokens, e.g., a token in p_c means that the value of c is available. A simple strategy to compute the value of z is to wait for a, b, and c to be available, consume their values and apply (1). Although this strategy is correct, it turns out to be too conservative when aiming at improving the system performance. Let us assume that at a given instant, c and b are available (see Fig. 1), and that the value of c is *false*. Then, the result $z = b$ can be produced without waiting for a to be available. Analogously, if c is *true* there is no need to wait for b to yield the result.

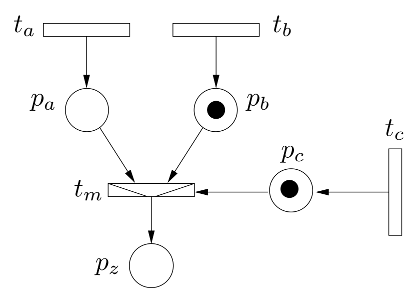

Fig. 1. A Petri net modeling a multiplexer with two input data and one input control signal.

The strategy of performing operations as soon as possible, which is commonly known as *early evaluation*, has already been applied to asynchronous design [1], [7], and can be used to enhance the system performance [5]. The synchronization mechanism of conventional Petri nets does not allow to model early evaluation in an easy way. This paper considers a class of Petri nets that models early evaluation by allowing a transition have several guards. The introduction of transitions with several guards enriches the formalism and allows one to describe many different behaviors, e.g., multiplexers, logic gates, conditional constructs, operations with OR-causality [13], etc.

A guard of a given transition is a set of input places that, depending on the system variables, represents a sufficient condition to fire the transition. For instance, early evaluation of transition t_m in Fig. 1 can be modeled by considering two guards for t_m, $\{p_a, p_c\}$ and $\{p_b, p_c\}$. For each firing of t_m, the value of c determines which guard must be satisfied to fire t_m, e.g., if c is *true* the guard $\{p_a, p_c\}$ must be satisfied, i.e., the availability of a and c suffices to fire t_m.

Once the guard for a given firing instance is selected, it must be satisfied, i.e., there must be enough tokens in the places contained in the guard, to enable the firing of the

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transition. The data in the rest of input places is not necessary and must be discarded, or equivalently consumed, in order to avoid the use of “out-of-date” data by subsequent firings. It will be shown that a natural way to discard unnecessary data is to allow places be negatively marked. Thus, in addition to transitions with several guards, the class of Petri nets considered here allows places have negative tokens to discard unnecessary data. In order to cover every potential evolution of the system, neither a time interpretation nor a policy for the selection of guards will be established.

The rest of the paper is organized as follows: Section II defines multi-guarded Petri nets. Section III shows that the reachability set is enlarged when several guards are considered. In Section IV the boundedness property is discussed. Section V is devoted to deadlock-freeness and liveness. The main results are summarized in Section VI.

II. MULTI-GUARDED PETRI NETS

A. Basic Definitions

In the following it is assumed that the reader is familiar with Petri nets (PN s) (see [6], [9] for instance).

Definition 1 (MPN): A *Multi-Guarded Petri Net* (MPN) is a tuple $\mathcal{N} = \langle P, T, Pre, Post, G\rangle$ where:

- P is a set of $|P|$ places
- T is a set of $|T|$ transitions
- $Pre : P \times T \to \mathbb{N} \cup \{0\}$ and $Post : P \times T \to \mathbb{N} \cup \{0\}$ are the *pre*- and *post*- incidence functions that specify the arc weights. The incidence matrix of the net is $\mathbb{C} = Post - Pre$. The *preset* and *postset* of a node $x \in P \cup T$ are denoted ${}^\bullet x$ and $x^\bullet$.
- $G : T \to 2^{2^P}$ assigns a set of guards $G(t)$ to every transition t. The following conditions must be fulfilled:
  - for every guard $h \in G(t)$ it holds $h \subseteq {}^\bullet t$
  - $\bigcup_{h \in G(t)} h = {}^\bullet t$

In a MPN, a transition t can also satisfy the condition $G(t) = \{\{{}^\bullet t\}\}$. Such transitions will be called *conventional* transitions; the rest of transitions will be called *multi-guarded* transitions. Conventional transitions will be represented graphically as empty rectangles, e.g., t_a in Fig 1, multi-guarded transitions will be represented as rectangles with oblique lines inside, e.g., t_m.

For a given MPN $\mathcal{N} = \langle P, T, Pre, Post, G\rangle$, its corresponding “conventional” PN is defined as $\mathcal{N}_c = \langle P, T, Pre, Post, G_c\rangle$ where $G_c(t) = \{\{{}^\bullet t\}\}$ for every $t \in T$. That is, $\mathcal{N}_c$ is a conventional PN, i.e., only one guard per transition, with the same structure as $\mathcal{N}$.

Definition 2 (MPN system): A *Multi-Guarded Petri Net System* is a tuple $\langle \mathcal{N}, M_0\rangle$ where $\mathcal{N}$ is a MPN, and $M_0 : P \to \mathbb{N} \cup \{0\}$ assigns an initial marking to each place p. The initial marking of place p is denoted as $M_0(p)$.

B. Enabling Condition and Firing Rule

Each guard of a multi-guarded transition t is a set of places from which tokens may be required to fire a given instance of t. In a real system, the guard for a given firing instance may depend on the particular values of the system variables, e.g., the guard of t_m in Fig. 1 is $\{p_a, p_c\}$ if c is *true*, and $\{p_b, p_c\}$ if c is *false*. In order to cope with every potential system behavior, the following assumption is made:

Assumption: The guard for a given firing instance of t is selected among the elements in $G(t)$. Such a selection is non-deterministic.

Notice that the potential behaviors that a system can exhibit under this assumption include the potential behaviors under any policy for the selection of guards, e.g., deterministic, probabilistic, etc. This way, if a given property, e.g., liveness, is satisfied under the assumed non-deterministic guard selection, it will be also satisfied under any other selection policy. The state of a MPN system is composed of a marking together with the guards selected for the next firings of the transitions:

Definition 3 (State): A *state* of a MPN system is a tuple $\langle M, g\rangle$ where $M : P \to \mathbb{Z}$ is the marking of the state, and $g : T \to 2^P$ are the guards selected for the next firings of the transitions. The following condition must be fulfilled: $g(t) \in G(t)$ for every $t \in T$.

The initial state of a MPN system is denoted as $s_0 = \langle M_0, g_0\rangle$, where g_0 is the initial selection of guards. A transition is said to be enabled at a given state if there are enough tokens in the set of input places selected as guard.

Definition 4 (Enabling condition): Transition t is enabled at state $\langle M, g\rangle$ if $M(p) \geq Pre(p, t)$ for every $p \in g(t)$.

In contrast to conventional transitions, a multi-guarded transition t can be enabled even if $p \in {}^\bullet t$ exists such that $M(p) < Pre(p, t)$. Hence, the enabling condition of MPNs can be seen as a relaxation with respect to the enabling condition of conventional PNs. Once transition t is enabled, it can fire yielding a new marking.

Definition 5 (Firing rule): Let transition t be enabled at state $\langle M, g\rangle$. The firing of t yields a new marking M' such that $M' = M + \mathbb{C}(P, t)$ where $\mathbb{C}(P, t)$ is the column of $\mathbb{C}$ that corresponds to transition t.

Thus, the firing of a multi-guarded transition t changes the system marking in the same way a transition in a conventional PN does: $Pre(p, t)$ tokens are removed/consumed from every $p \in {}^\bullet t$ (even if $p \notin g(t)$), and $Post(p, t)$ tokens are put/produced in every $p \in t^\bullet$. After the firing of t a new guard $g(t)$ is selected for the next firing of t, the guards for the rest of transitions are kept.

Unlike conventional transitions, the firing of a multi-guarded transition may produce negative markings (this is equivalent to antitokens in [10]). More precisely, every place $p \in {}^\bullet t$ such that $M(p) < Pre(p, t)$ will become negatively marked after the firing of t. If $M(p) < 0$ we say that place p has $|M(p)|$ negative tokens. Negative tokens will be depicted as white circles, see Fig. 2(b).

The described firing rule models satisfactorily the kind of behaviors we want to model. Consider the MPN system in Fig. 2(a) with $G(t_3) = \{\{p_3, p_5\}, \{p_4, p_5\}\}$. Let us assume that p_3 (p_4, p_5, p_6) contains a token iff a given variable a (b, c, z) is available, then the MPN system can be seen as a graphical representation of the following pseudocode:

```
loop
   compute a; compute b; compute c;
   if c then z = a else z = b end_if;
   deliver_output z;
end_loop
```

where operations are allowed to happen simultaneously.

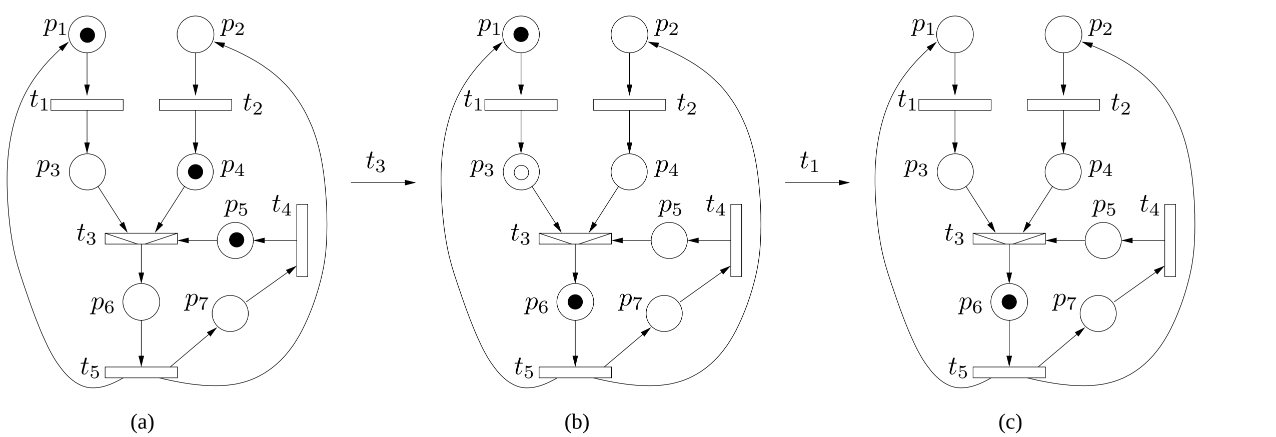


Fig. 2. (a) A MPN system with $G(t_3) = \{\{p_3, p_5\}, \{p_4, p_5\}\}$. If the initial guard for t_3 is $g(t_3) = \{p_4, p_5\}$, t_3 can fire (b). The negative token in p_3 cancels the positive token produced by the firing of t_1 (c).

Assume that the initial state is $\langle(1\ 0\ 0\ 1\ 1\ 0\ 0), (\{p_3, p_5\})\rangle$ (for clarity, only the guard of the multi-guarded transition t_3 is specified). At the initial state only t_1 is enabled, its firing drives the system to the state $\langle(0\ 0\ 1\ 1\ 1\ 0\ 0), (\{p_3, p_5\})$ at which t_3 is enabled. The firing of t_3 yields the marking $M = (0\ 0\ 0\ 0\ 0\ 1\ 0)$ at which a new guard for t_3 must be selected, i.e., the new state will be either $\langle M, (\{p_3, p_5\})\rangle$ or $\langle M, (\{p_4, p_5\})\rangle$.

Let us now assume that the initial state is $\langle(1\ 0\ 0\ 1\ 1\ 0\ 0), (\{p_4, p_5\})\rangle$. At such a state both t_1 and t_3 are enabled. Assume that t_3 fires first what yields the marking $(1\ 0\ -1\ 0\ 0\ 1\ 0)$, and produces a negative token in p_3, see Fig. 2(b). If t_1 fires from marking $(1\ 0\ -1\ 0\ 0\ 1\ 0)$ the incoming positive token will be cancelled by the negative token in p_3 leading to marking $(0\ 0\ 0\ 0\ 0\ 1\ 0)$, see Fig. 2(c). Thus, the negative token gets rid of the input data not contained in the selected guard. Such data is in fact "out-of-date" data that must not be consumed by t_3 in the next firing instance: Imagine that no negative token exists in Fig. 2(b), then if $\{p_3, p_5\}$ is the guard for the next firing, t_3 will consume data produced by t_1 for the previous firing instance of t_3.

III. Reachability

The fact that a transition t of a given MPN is enabled at state $s = \langle M, g\rangle$ and fires leading to state $s' = \langle M', g'\rangle$ is denoted by $s \xrightarrow{t} s'$. A firing sequence from s is a sequence $\sigma_{seq} = t_a t_b \dots t_i \ \in T^*$ such that $s \xrightarrow{t_a} s_a \xrightarrow{t_b} s_b \dots \xrightarrow{t_i} s_i$, this is denoted by $s \xrightarrow{\sigma_{seq}} s_i$.

Definition 6 (Reachability set): The reachability set of the MPN system $\langle \mathcal{N}, M_0\rangle$ is defined as $\mathcal{RS}(\mathcal{N}, M_0) = \{M|$ there exists a firing sequence σ_{seq} such that $\langle M_0, g_0\rangle \xrightarrow{\sigma_{seq}} \langle M, g\rangle\}$.

Since the firing of a multi-guarded transition t changes the marking in the same way a transition in a conventional PN does, the state equation $M = M_0 + \mathbb{C} \cdot \sigma$ provides a necessary condition for the reachability of M, where σ is the firing count vector of the transitions (notice that the marking of a MPN system can have negative values). Hence, as in conventional PNs, vectors $Y \geq 0$, $Y \cdot \mathbb{C} = 0$ ($X \geq 0$, $\mathbb{C} \cdot X = 0$) represent P-semiflows, also called conservative components (T-semiflows, also called consistent components). A semiflow V will be said to be *minimal* when its support, $\|V\|$[1], is not a proper superset of the support of any other, and the greatest common divisor of its elements is one. A MPN $\mathcal{N}$ is conservative (consistent) if there exists $Y > 0$ such that $Y \cdot \mathbb{C} = 0$ ($X > 0$ such that $\mathbb{C} \cdot X = 0$).

Given that a transition t is enabled at $\langle M, g\rangle$ iff $M(p) \geq Pre(p, t)$ for every $p \in g(t)$, the firing of t does not produce negative markings in the places contained in $g(t)$. In other words, for every reachable marking M, there exists at least one place $p \in P$ such that $M(p) \geq 0$.

Proposition 1: Let $\langle \mathcal{N}, M_0\rangle$ be a MPN system. If $M \in \mathcal{RS}(\mathcal{N}, M_0)$ then there exists $\sigma \in (\mathbb{N} \cup \{0\})^{|T|}$ such that $M = M_0 + \mathbb{C} \cdot \sigma$ and $p \in P$ exists such that $M(p) \geq 0$.

The following proposition states that the reachability set of a MPN system contains the reachability set of its corresponding conventional PN system.

Proposition 2: Let $\mathcal{N}$ be a MPN and $\mathcal{N}_c$ be its corresponding conventional PN. Then, $\mathcal{RS}(\mathcal{N}_c, M_0) \subseteq \mathcal{RS}(\mathcal{N}, M_0)$ for every $M_0 \geq 0$.

Proof: Let $M \in \mathcal{RS}(\mathcal{N}_c, M_0)$, then a firing sequence $\sigma_{seq} = t_a t_b \dots t_i$ exists such that $M_0 \xrightarrow{t_a} M_1 \dots \xrightarrow{t_i} M$ (given that every transition has only one guard, it is not necessary to specify them in the sequence). Since $\mathcal{N}_c$ is a conventional PN, if t_a is enabled at M_0 then $M_0(p) \geq Pre(p, t)$ for every $p \in {}^\bullet t$. Hence, t_a can also fire in the MPN $\mathcal{N}$ from $s_0 = \langle M_0, g_0\rangle$, whatever $g_0(t)$ is, leading to the same marking M_1. This reasoning can now be applied to the rest of transitions in σ_{seq}. $\square$

Given that MPN systems can reach negative markings and conventional PN systems cannot, in general $\mathcal{RS}(\mathcal{N}, M_0) \not\subseteq \mathcal{RS}(\mathcal{N}_c, M_0)$. The enlargement of the reachability set in a MPN system with respect to the conventional PN system has a direct impact on boundedness, deadlock-freeness and liveness, e.g., the set difference $\mathcal{RS}(\mathcal{N}, M_0) \backslash \mathcal{RS}(\mathcal{N}_c, M_0)$ might be infinite, and might contain markings that kill a given transition or make it live.

[1]The support, or set of non-null entries, of V is denoted by $\|V\|$.

IV. BOUNDEDNESS

A MPN system is said to be bounded if the number of positive and negative tokens in each place does not exceed a finite number k for any marking reachable from M_0.

Definition 7 (Boundedness):

- A MPN system $\langle \mathcal{N}, M_0\rangle$ is bounded if there exists $k \in \mathbb{N}$ such that for every $M \in \mathcal{RS}(\mathcal{N}, M_0)$ it holds that $-k \leq M(p) \leq k$ for every $p \in P$.
- A MPN $\mathcal{N}$ is structurally bounded (str. bounded) if for any initial marking $M_0 \in (\mathbb{N} \cup \{0\})^{|P|}$, the system $\langle \mathcal{N}, M_0\rangle$ is bounded.

Given that $\mathcal{RS}(\mathcal{N}_c, M_0) \subseteq \mathcal{RS}(\mathcal{N}, M_0)$ (Proposition 2), boundedness of $\langle \mathcal{N}, M_0\rangle$ is a sufficient condition for boundedness of $\langle \mathcal{N}_c, M_0\rangle$.

Proposition 3: Let $\mathcal{N}$ be a MPN and $\mathcal{N}_c$ be its corresponding conventional PN. If $\langle \mathcal{N}, M_0\rangle$ is bounded then $\langle \mathcal{N}_c, M_0\rangle$ is bounded.

The converse of Proposition 3 is not true given that $\mathcal{RS}(\langle \mathcal{N}, M_0\rangle)$ can be an infinite set and $\mathcal{RS}(\langle \mathcal{N}_c, M_0\rangle)$ finite: The system in Fig. 2(a) is bounded if all transitions are conventional. However, if t_3 is multi-guarded with $G(t_3) = \{\{p_3, p_5\}, \{p_4, p_5\}\}$, then p_3 (and p_4) is not lower-bounded, i.e., there does not exist $k \in \mathbb{N}$ such that for every $M \in \mathcal{RS}(\mathcal{N}, M_0)$ it holds that $-k \leq M(p_3)$. This can be easily checked by selecting repeatedly the guard $\{p_4, p_5\}$ and firing indefinitely the sequence $t_3t_5t_2t_4t_3t_5t_2t_4\ldots$. Thus, in contrast to conventional PNs, a conservative MPN (like the one in Fig. 2(a)) is not necessarily structurally bounded.

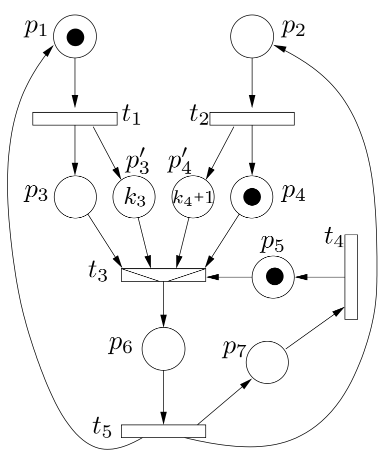


Fig. 3. Bounded version of the system in Fig. 2(a) with $G(t_3) = \{\{p_3, p_5, p_3', p_4'\}, \{p_4, p_5, p_3', p_4'\}\}$. For every reachable marking M it holds $M(p_3) \geq -k_3$ and $M(p_4) \geq -k_4$.

Similarly to conventional PNs, an easy way to *lower-bound* the input places of a multi-guarded transition is by adding some places. Assume that a place p is desired to store at most k negative tokens in any reachable marking. This can be achieved by adding a place p' to the net such that $Pre(p', T) = Pre(p, T)$, $Post(p', T) = Post(p, T)$, $M_0(p') = M_0(p)+k$ and including p' in every guard of every output transition of p. Given that p' is contained in every guard of its output transitions, $M(p') \geq 0$ for any reachable M. Moreover, the addition of p' entails the existence of the invariant $M(p') = M(p) + k$ that holds for any reachable M. Then, $M(p') \geq 0$ implies $M(p) \geq -k$.

If the described addition of places to lower-bound the input places of multi-guarded transitions is applied to a conservative MPN system, the system becomes bounded: Fig. 3 shows the net resulting of applying this approach on p_3 and p_4 of Fig. 2(a) (there is no need to lower-bound p_5 because it is contained in every guard). The new guards of t_3 are $G(t_3) = \{\{p_3, p_5, p_3', p_4'\}, \{p_4, p_5, p_3', p_4'\}\}$. Given, that every place is now lower-bounded and the net is conservative, every place is also upper-bounded, i.e., the system is bounded.

V. DEADLOCK-FREENESS AND LIVENESS

A. Definitions and Preliminary Results

A natural way to define deadlock-freeness and liveness is to adapt the existing definitions [9] for conventional PNs to the enabling condition in MPNs.

Definition 8 (Deadlock-freeness and Liveness):

- A MPN system $\langle \mathcal{N}, M_0\rangle$ is deadlock-free if for every $M \in \mathcal{RS}(\mathcal{N}, M_0)$ and for any selection of guards g there exists $t \in T$ such that t is enabled at $\langle M, g\rangle$.
- Transition t is live in $\langle \mathcal{N}, M_0\rangle$ if for every $M \in \mathcal{RS}(\mathcal{N}, M_0)$ and for any selection of guards g there exists $M' \in \mathcal{RS}(\mathcal{N}, M)$ such that t is enabled at $\langle M', g\rangle$.
- A MPN system $\langle \mathcal{N}, M_0\rangle$ is live if every $t \in T$ is live.
- A MPN $\mathcal{N}$ is structurally live (str. live) if an initial marking $M_0 \in (\mathbb{N} \cup \{0\})^{|P|}$ exists such that $\langle \mathcal{N}, M_0\rangle$ is live.

Deadlock-freeness of a MPN system is a sufficient condition for deadlock-freeness of its corresponding PN system.

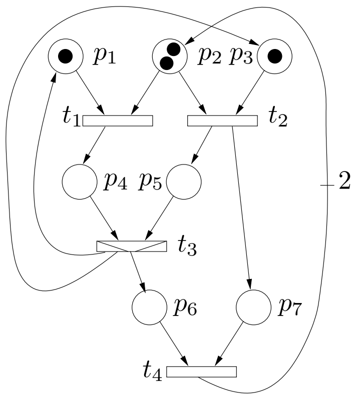
Proposition 4: Let $\mathcal{N}$ be a MPN and $\mathcal{N}_c$ be its corresponding conventional PN. If $\langle \mathcal{N}, M_0\rangle$ is deadlock-free then $\langle \mathcal{N}_c, M_0\rangle$ is deadlock-free.

Proof: Assume that $M \in \mathcal{RS}(\mathcal{N}_c, M_0)$ exists such that no transition is enabled at M, i.e., for every $t \in T$ there exists $p \in {}^\bullet t$ such that $M(p) < Pre(p, t)$. By Proposition 2, it holds that $M \in \mathcal{RS}(\mathcal{N}, M_0)$. By definition $\bigcup_{h \in G(t)} h = {}^\bullet t$, hence, a selection of guards g exists such that for every $t \in T$ there exists $p \in g(t)$ such that $M(p) < Pre(p, t)$, i.e., no transition is enabled at $\langle M, g\rangle$. $\square$

The fact that $\mathcal{RS}(\mathcal{N}_c, M_0) \subseteq \mathcal{RS}(\mathcal{N}, M_0)$ can produce two different phenomena related to liveness: a) The set $\mathcal{RS}(\mathcal{N}_c, M_0)$ does not contain deadlock markings, but $\mathcal{RS}(\mathcal{N}, M_0)$ does (thus, the converse of Proposition 4 does not hold); b) A given transition is not live in $\mathcal{RS}(\mathcal{N}_c, M_0)$ but is live in $\mathcal{RS}(\mathcal{N}, M_0)$. In other words, liveness of $\langle \mathcal{N}, M_0\rangle$ is neither necessary nor sufficient for liveness of $\langle \mathcal{N}_c, M_0\rangle$.

Fig. 4(a) illustrates case a). The system is live (and deadlock-free) if considered as a conventional PN. Assume now that t_3 is multi-guarded with $G(t_3) = \{\{p_4\}, \{p_5\}\}$, and that the guards selected for its first and second firings are $\{p_4\}$ and $\{p_5\}$. Then the firing sequence t_1, t_3, t_1 is feasible leading to marking $(0\ 0\ 2\ 1\ -1\ 1\ 0)$ at which no transition is enabled. The MPN system is not live (and not deadlock-free).

Case b) is illustrated in Fig. 4(b). Transitions t_3 and t_4 are not live if all transitions are conventional. If transition t_3 is multi-guarded with $G(t_3) = \{\{p_1\}, \{p_2\}\}$ every transition becomes live.



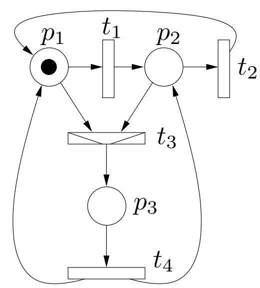


Fig. 4. (a) Live as conventional PN system, not live as MPN system. (b) Not live as PN system, live as MPN system.

B. Liveness and Boundedness

Live and bounded systems are often called *well-behaved* systems. It is well-known that a str. bounded and str. live conventional PN is conservative and consistent [9]. Such fundamental necessary conditions also hold for MPN systems.

Lemma 5: Let $\mathcal{N}$ be a MPN. If $\mathcal{N}$ is str. bounded and str. live, then $\mathcal{N}$ is consistent.

Proof: The proof is similar to the proof for conventional PNs [8]. If $\langle \mathcal{N}, M_0 \rangle$ is bounded and live, the set of reachable markings is finite and any strongly connected component of the reachability graph contains all transitions. Then, from any marking M in a given strongly connected component, all transitions can be fired reaching again M. Then, there exists a T-semiflow that contains all transitions. □

Lemma 6: Let $\mathcal{N}$ be a MPN. If $\mathcal{N}$ is str. bounded and consistent, then $\mathcal{N}$ it is conservative.

Proof: If $\mathcal{N}$ is str. bounded, then its corresponding conventional PN $\mathcal{N}_c$ is also str. bounded (Propostion 3). It is well-known that any str. bounded and consistent $\mathcal{N}_c$ is conservative [8]. Since the incidence matrices of $\mathcal{N}$ and $\mathcal{N}_c$ are identical, $\mathcal{N}$ is also conservative. □

From Lemma 5 and Lemma 6 the following proposition immediately follows:

Proposition 7: Let $\mathcal{N}$ be a MPN. If $\mathcal{N}$ is str. bounded and str. live, then $\mathcal{N}$ is consistent and conservative.

C. Equal Conflict Nets

As shown in Subsection V-A, liveness of a MPN system $\langle \mathcal{N}, M_0 \rangle$ is, in general, neither necessary nor sufficient for liveness of the corresponding conventional PN system $\langle \mathcal{N}_c, M_0 \rangle$. There is, however, an important net subclass, namely conservative and strongly connected Equal Conflict (EC) nets [12], for which liveness of $\langle \mathcal{N}, M_0 \rangle$ is necessary and sufficient for liveness of $\langle \mathcal{N}_c, M_0 \rangle$. This liveness preservation allows us to directly apply the existing liveness conditions for conventional EC PNs [12], [2] to EC MPNs.

Two transitions, t and t', are in Equal Conflict Relation (denoted by $(t, t') \in$ ECR) iff $t = t'$ or $Pre(P, t) = Pre(P, t') \neq 0$. This is an equivalence relation on the set of transitions. Each equivalence class is called Equal Conflict Set (ECS).

Definition 9 (EC MPN): A MPN is Equal Conflict (EC MPN) if for every $t, t' \in T$ such that ${}^\bullet t \cap {}^\bullet t' \neq \emptyset$ it holds $(t, t') \in$ ECR.

From a structural perspective, the class of EC nets subsumes several classes of ordinary Petri nets: Marked Graphs [3] ($\forall\ p\ |{}^\bullet p| = |p^\bullet| = 1$), Choice Free nets [11] ($\forall\ p\ |p^\bullet| = 1$), and Free Choice nets [4] ($\forall\ t, t'$, if ${}^\bullet t \cap {}^\bullet t' \neq \emptyset$ then ${}^\bullet t = {}^\bullet t'$).

In an ECS of conventional transitions, all transitions become simultaneously enabled when there are enough tokens in their input places, if there are not enough tokens then no transition in the ECS is enabled. Interestingly, an ECS of multi-guarded transitions can behave in the same way if the guards are selected appropriately: Let T_{EC} be an ECS, and let $p_l \in {}^\bullet T_{EC}$ be the last place that receives enough tokens to enable transitions, i.e., $M(p_l) \geq Pre(p_l, t)$ for any $t \in T_{EC}$ implies that $M(p) \geq Pre(p, t)$ for every $p \in {}^\bullet T_{EC}$ and any $t \in T_{EC}$. If every transition $t \in T_{EC}$ selects a guard that contains p_l, then all transitions in the ECS become enabled simultaneously. Such guard selection is feasible given that by definition $\bigcup\limits_{h \in G(t)} h = {}^\bullet t$ for every $t \in T$. Notice that p_l may differ along the system execution, hence, the guards for each firing instance must be updated to contain the new p_l.

A strongly connected and bounded EC PN system is live iff it is deadlock-free [12]. We establish a similar statement for strongly connected and conservative (not necessarily bounded, see Fig. 2(a)) EC MPN systems.

Proposition 8: A conservative and strongly connected EC MPN system is live iff it is deadlock-free.

Proof: Only the direction 'deadlock-freeness$\Rightarrow$ liveness' must be proved. Assume a conservative and strongly connected EC MPN system $\langle \mathcal{N}, M_0 \rangle$ exists such that it is deadlock-free and not live. Then, there exists a reachable marking M at which every transition in a non-empty set $T_d \subset T$ is dead, i.e., the transitions in T_d are not enabled at any marking reachable from M. Let us select the guards for transitions in $T \setminus T_d$ in such a way that they behave as conventional transitions. Hence, if t and t' are in ECR then $t \in T_d$ iff $t' \in T_d$. Thus, the input places of dead transitions keep every incoming token. Given that the system is conservative the input transitions of such places cannot fire indefinitely producing an ever increasing number of tokens and will eventually deadlock. Since the net is strongly connected every transition will finally deadlock. □

In order to prove the liveness equivalence between a conservative and strongly connected EC MPN system and its corresponding conventional EC PN system, a technical lemma will be used. Such a lemma states that, if the EC PN system is live, then for any marking M that is solution of the state equation, a marking M' exists such that M' can be reached from M by the EC MPN system, and from M_0 by the EC PN system. A similar lemma is proved in [12] for non-negative solutions of the state equation. Here, the scope is extended to handle the negative markings that a MPN system can reach.

Lemma 9: Let $\mathcal{N}$ be an EC MPN, and $\mathcal{N}_c$ be its corresponding conventional PN. If $\langle \mathcal{N}_c, M_0 \rangle$ is live and $M = M_0 + \mathbb{C} \cdot \sigma$ for a given $\sigma \in (\mathbb{N} \cup \{0\})^{|T|}$, then M' exists such that $M' \in \mathcal{RS}(\mathcal{N}, M) \cap \mathcal{RS}(\mathcal{N}_c, M_0)$.

Proof: Let $M = M_0 + \mathbb{C} \cdot \sigma$ for a given $\sigma \in (\mathbb{N} \cup \{0\})^{|T|}$. It will be shown that M' exists such that it is reachable by $\langle \mathcal{N}, M \rangle$ and by $\langle \mathcal{N}_c, M_0 \rangle$.

Let us decompose the firing count vector σ as $\sigma = \sigma^0 + \sigma^1$, where the firing count vector $\sigma_0 \geq 0$ can be fired from M_0 in $\mathcal{N}_c$ leading to M^0_{PN} at which no transition in $\|\sigma^1\|$ is enabled (such a decomposition can be obtained by starting with a null σ^0 and $\sigma^1 = \sigma$, and then, iteratively firing transitions in the successive $\|\sigma^1\|$ until no transition in $\|\sigma^1\|$ is enabled). Then, $M^0_{PN} = M_0 + \mathbb{C} \cdot \sigma^0$, $M = M^0_{PN} + \mathbb{C} \cdot \sigma^1$ and no transition in $\|\sigma^1\|$ is enabled at M^0_{PN}.

Since $\langle \mathcal{N}_c, M_0 \rangle$ is live, so is $\langle \mathcal{N}_c, M^0_{PN} \rangle$. Hence, $t \in T$ exists such that t is enabled at M^0_{PN}, i.e., $M^0_{PN}(p) \geq Pre(p,t)$ for every $p \in {}^\bullet t$. Given that the net is equal conflict, there is no transition in $\|\sigma^1\|$ that is in ECR with t (otherwise such transition in $\|\sigma^1\|$ would be enabled at M^0_{PN}). In other words, there is no transition in $\|\sigma^1\|$ that takes tokens from input places of t, and therefore $M(p) \geq Pre(p,t)$ for every $p \in {}^\bullet t$. Thus, t is also enabled at M in the MPN system, whatever the guard for t is.

Let us fire t from M^0_{PN} in the PN system leading to marking M^1_{PN}, and t from M in the MPN system leading to marking M^1_{MPN}. Then, $M^1_{MPN} = M^1_{PN} + \mathbb{C} \cdot \sigma^1$, i.e., the firing of t in both systems makes them advance in parallel. The procedure described for M_0 and M can now be repeated for M^1_{PN} and M^1_{MPN}: If there exists a transition in $\|\sigma^1\|$ that is enabled at M^1_{PN}, it is fired in the PN system leading to a marking that approaches M^1_{MPN}; If no transition in $\|\sigma^1\|$ is enabled at M^1_{PN}, then a transition $t \in T$ exists such that t is enabled at M^1_{PN} in the PN system, and at M^1_{MPN} in the MPN system, the firing of t in both systems leads to new markings M^2_{PN} and M^2_{MPN} from which the procedure is repeated.

Given that $\langle \mathcal{N}_c, M_0 \rangle$ is live, the firing count vector σ^1 can be eventually fired. Since the procedure makes the PN and MPN systems either approach or advance in parallel, when σ^1 is fired, a common successor M' is reached. A graphical sketch of the proof is shown in Figure 5. $\square$

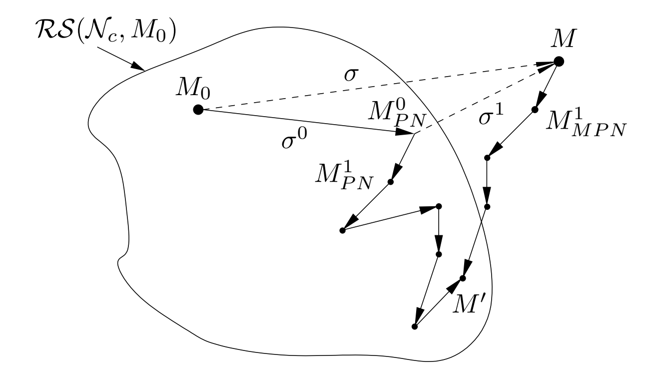


Fig. 5. Graphical sketch of the proof of Lemma 9. A common successor is reached by making the systems advance in parallel and approaching their markings when possible.

Proposition 10: Let $\mathcal{N}$ be a conservative and strongly connected EC MPN, and $\mathcal{N}_c$ be its corresponding conventional PN. $\langle \mathcal{N}, M_0 \rangle$ is live iff $\langle \mathcal{N}_c, M_0 \rangle$ is live.

Proof: ($\Rightarrow$) It follows from these facts: a) liveness and deadlock-freeness are equivalent in EC MPN systems (Proposition 8) and in EC PN systems [12]; b) deadlock-freeness of $\langle \mathcal{N}, M_0 \rangle$ implies deadlock-freeness of $\langle \mathcal{N}_c, M_0 \rangle$ (Proposition 4).

($\Leftarrow$) Assume that $\langle \mathcal{N}_c, M_0 \rangle$ is live. We will prove that every transition t is live in $\langle \mathcal{N}, M_0 \rangle$. Let M be a marking reached by the MPN system from M_0, then there exists $\sigma \geq 0$ such that $M = M_0 + \mathbb{C} \cdot \sigma$. By Lemma 9, a marking M' exists such that $M' \in \mathcal{RS}(\mathcal{N}, M) \cap \mathcal{RS}(\mathcal{N}_c, M_0)$. Since $\langle \mathcal{N}_c, M_0 \rangle$ is live, so is $\langle \mathcal{N}_c, M' \rangle$. Then, if the selected guards make the MPN system behave as a conventional one, any transition of the MPN system can eventually fire from M'. $\square$

VI. Conclusions

Multi-guarded transitions represent a natural approach to model early evaluation of operations. We have investigated how some qualitative properties change when a conventional PN system, $\langle \mathcal{N}_c, M_0 \rangle$, is transformed to a MPN system, $\langle \mathcal{N}, M_0 \rangle$. The following table summarizes the main results.

$\mathcal{RS}(\mathcal{N}, M_0)$	$\supseteq$	$\mathcal{RS}(\mathcal{N}_c, M_0)$
$\langle \mathcal{N}, M_0 \rangle$ bounded	$\Rightarrow$	$\langle \mathcal{N}_c, M_0 \rangle$ bounded
$\langle \mathcal{N}, M_0 \rangle$ deadlock-free	$\Rightarrow$	$\langle \mathcal{N}_c, M_0 \rangle$ deadlock-free
$\langle \mathcal{N}, M_0 \rangle$ live	$\not\Leftarrow$ $\not\Rightarrow$	$\langle \mathcal{N}_c, M_0 \rangle$ live
$\mathcal{N}$ str. bounded & str. live	$\Rightarrow$	$\mathcal{N}$ consistent & conservative
$\mathcal{N}$ is conservative and strongly connected EC:		
$\langle \mathcal{N}, M_0 \rangle$ live	$\Leftrightarrow$	$\langle \mathcal{N}_c, M_0 \rangle$ live

Acknowledgments

The author is very grateful for suggestions made by Jordi Cortadella of the Universitat Politècnica de Catalunya.

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