Deadlock-Freeness Analysis of Continuous Mono-T-Semiflow Petri Nets

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***Abstract*—Most verification techniques for highly populated discrete systems suffer from the *state explosion* problem. The "fluidification" of discrete systems is a classical relaxation technique that aims to avoid the state explosion problem. Continuous Petri nets are the result of fluidifying *traditional* discrete Petri nets. In continuous Petri nets the firing of a transition is not constrained to the naturals but to the non-negative reals. Unfortunately, some important properties, as liveness, may not be preserved when the discrete net model is fluidified. Therefore, a thorough study of the properties of continuous Petri nets is required. This paper focuses on the study of deadlock-freeness in the framework of mono-T-semiflow continuous Petri nets, i.e., conservative nets with a single repetitive sequence (T-semiflow). The study is developed both on untimed and timed systems. Topological necessary conditions are extracted for this property. Moreover, a bridge relating deadlock-freeness conditions for untimed and timed systems is established.**

***Index Terms*—Continuous Petri nets, deadlock-freeness.**

I. INTRODUCTION

STATE explosion problem is a crucial drawback in the analysis of discrete event systems. A way to try to overcome this difficulty is to make the system continuous (total or partially) and apply different analysis techniques that may provide an approximation to the original behavior.

Transforming a discrete model into a continuous one is, in general, a classical relaxation aiming at computationally more efficient analysis techniques, at the price of losing some precision. Nevertheless, it should be pointed out that the transformation from discrete to continuous of some systems may yield a non appropriate continuous model: some qualitative properties as deadlock-freeness may not be preserved [3]. Therefore, it becomes interesting to analyze continuous Petri nets in order to establish when they offer a reasonable relaxation of their discrete counterparts, i.e., when they preserve the properties to be analyzed. In the manufacturing systems domain, and using Petri net models, this idea has been applied, for instance, in [1]–[4].

This paper focuses on the analysis of deadlock-freeness in the framework of continuous *mono-T-semiflow* (MTS) *Petri nets* (PNs) (see [5] for an introduction to MTS nets in the classical discrete setting). Deadlock-freeness is an essential qualitative property that applies to those systems fulfilling that at any reachable marking there is at least one enabled transition. Quantitative properties can also be studied in the continuous Petri nets setting. For example, in [6] an approach to compute throughput bounds for MTS was developed. However, when analyzing a system, it is reasonable to carry out a qualitative study first (it usually requires less effort), and then a quantitative one. For instance, it makes no sense to compute the throughput bounds of a system that is known to be nondeadlock-free.

The main properties of MTS nets are purely structural: Consistency with a single T-semiflow (i.e., all transitions are covered by *the* unique minimal T-semiflow) and conservativeness (i.e., all places are covered by P-semiflows). Therefore the membership problem for MTS can be decided in polynomial time. The modelling power of MTS nets is considerable since they represent a generalization of *choice-free* nets [7]. Notice that a subclass of choice free nets are weighted-T-systems [8], a weighted generalization of the well-known subclass of marked graphs.

The study of deadlock-freeness will be developed for both untimed and timed continuous MTS Petri nets. For the untimed nets there exists an indeterminism in the firing of the transitions. For the time interpretation infinite servers semantics, that is introduced in Section II, will be used. The results obtained for timed MTS systems apply to any system that under infinite servers semantics evolves through a transient and eventually reaches a steady state in which the marking of the places and the flow through transitions remain constant. It is important to notice that most of the liveness conditions presented in this paper are structural, i.e., they are independent of the initial marking of the system.

The paper is structured as follows. In Section II, untimed and timed continuous Petri nets are introduced. It is shown that the property of deadlock-freeness is not always preserved when the model is made continuous. In Section III, some necessary deadlock-freeness conditions for untimed nets are obtained. Section IV is devoted to the study of the same property in timed systems. Special attention is paid to the influence that the firing speed of transitions has on deadlock-freeness. In Section V, the conclusions of the work are presented.

II. UNTIMED AND TIMED CONTINUOUS PETRI NETS

A. Basic Definitions

The reader is assumed to be familiar with PNs (see, for example, [9] and [10]). The usual PN system, $\langle \mathcal{N}, \mathbf{m}_o \rangle$ ($\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post} \rangle$), will be said to be *discrete* so as to distinguish it from the *continuous* relaxation [1], [2]. A first difference between continuous and discrete PNs is in the marking, which in a discrete PNs is restricted to be in the naturals, while in continuous PNs is released into the non-negative real numbers. This is a consequence of the firing, which is modified in the same way. In a continuous system, a transition t is enabled iff every

Manuscript received September 2, 2004; revised June 1, 2005, December 5, 2005, and January 12, 2006. Recommended by Associate Editor R. Boel. This work was supported by a Grant from D.G.A. ref B106/2001, and supported in part by Project CICYT and FEDER TIC2001-1819.

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Digital Object Identifier 10.1109/TAC.2006.880957

input place is marked, i.e., for every $p \in {}^{\bullet}t$, $\mathbf{m}[p] > 0$. As in discrete systems, the *enabling degree* at $\mathbf{m}$ of a transition measures the maximal amount in which the transition can be fired in one go, i.e., $\mathrm{enab}(t, \mathbf{m}) = \min_{p \in {}^{\bullet}t}\{\mathbf{m}[p]/\mathbf{Pre}[p,t]\}$. The firing of t in a certain amount $\alpha \leq \mathrm{enab}(t, \mathbf{m})$ leads to a new marking $\mathbf{m}' = \mathbf{m} + \alpha \cdot \mathbf{C}[P,t]$, where $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$ is the token flow matrix. Hence, as in discrete systems, the state (or fundamental) equation ($\mathbf{m} = \mathbf{m}_o + \mathbf{C} \cdot \sigma$) summarizes the marking evolution. Notice that for a continuous transition, the fact of being enabled or not does not depend on the arc weights, although they are important to compute the enabling degree and to obtain the new marking after the firing.

Every concept based on the representation of the net as a graph can be directly applied to continuous nets, in particular, the conflict relationships. Two transitions, t and t', are said to be in *structural conflict relation* if ${}^{\bullet}t \cap {}^{\bullet}t' \neq \emptyset$. The *coupled conflict* relation is defined as the transitive closure of the structural conflict relation. Each equivalence class is called a *coupled conflict set* denoted, for a given t, $\mathrm{CCS}(t)$. The set of all the equivalence classes is denoted as SCCS. When $\mathbf{Pre}[P,t] = \mathbf{Pre}[P,t'] \neq \mathbf{0}$, t and t' are in *equal conflict (EQ)* relation.

Right and left natural annullers of the token flow matrix $\mathbf{C}$ are called T- and P-semiflows, respectively. A semiflow $\mathbf{v}$ will be said to be *minimal* when its support, $\|\mathbf{v}\|$, is not a proper superset of the support of any other, and the g.c.d. of its elements is one. When $\mathbf{y} \cdot \mathbf{C} = \mathbf{0}$, $\mathbf{y} > \mathbf{0}$ the net is said to be *conservative*, and when $\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ the net is said to be *consistent*. In a consistent net, a vector $\mathbf{x} > \mathbf{0}$ such that $\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$ represents a repetitive sequence, or in other words, a potential steady-state behavior of the system in which all transitions are fired.

In a *continuous* PNs, the marking evolution along time can be expressed by the state equation $\mathbf{m}(\tau) = \mathbf{m}_o + \mathbf{C} \cdot \boldsymbol{\sigma}(\tau)$ where τ represents time. Taking the derivative with respect to time, $\dot{\mathbf{m}}(\tau) = \mathbf{C} \cdot \dot{\boldsymbol{\sigma}}(\tau)$ is obtained. Let us denote $\mathbf{f} = \dot{\boldsymbol{\sigma}}$, since it represents the *flow* through the transitions, i.e., the number of times transitions are fired per time unit.

The firing semantics defines how the flow of transitions is computed. For continuous PNs several semantics have been proposed, the most important being *infinite servers (or variable speed)* and *finite servers (or constant speed)* [1], [2]. Both represent a first order (or deterministic) approximation of the discrete case [2]. Infinite servers semantics will be considered here. Under infinite servers semantics, the flow through a transition t is the product of the *firing speed*, $\boldsymbol{\lambda}[t] > 0$, and the enabling degree of the transition, i.e., $\mathbf{f}[t] = \boldsymbol{\lambda}[t] \cdot \mathrm{enab}(t, \mathbf{m}) = \boldsymbol{\lambda}[t] \cdot \min_{p \in {}^{\bullet}t}\{\mathbf{m}[p]/\mathbf{Pre}[p,t]\}$, leading to nonlinear ordinary differential and deterministic systems. A continuous timed system will be represented as $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_o \rangle$.

For example, for the net system in Fig. 1: $\mathbf{f}[t_1] = \boldsymbol{\lambda}[t_1] \cdot \min\{\mathbf{m}[p_1]/2, \mathbf{m}[p_4]/2\}$, $\mathbf{f}[t_2] = \boldsymbol{\lambda}[t_2] \cdot \min\{\mathbf{m}[p_2], \mathbf{m}[p_4]\}$, and $\mathbf{f}[t_3] = \boldsymbol{\lambda}[t_3] \cdot \mathbf{m}[p_3]$.

Observe that in this case, the net system can be seen as a piecewise linear system [11]: At a given moment, the evolution of the net system is ruled by a linear differential system that depends on the net structure and the current marking. If the net is join free (i.e., each transition has at most one input place) a single set of linear differential equations represents the evolution of the marking: $\dot{\mathbf{m}}(\tau) = \mathbf{C} \cdot \boldsymbol{\Lambda} \cdot \mathbf{m}(\tau)$, $\mathbf{m}(0) = \mathbf{m}_o$ where

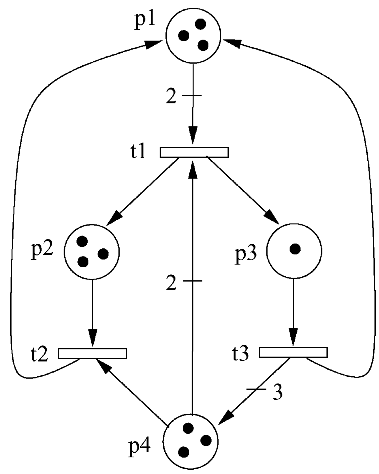

Fig. 1. Continuous PN system.

$\boldsymbol{\Lambda}[t,p] = \boldsymbol{\lambda}[t]/\mathbf{Pre}[p,t]$ if $p = {}^{\bullet}t$, and 0, otherwise. If the net system is conservative, $\mathbf{C}$ has redundant rows. Hence, some variables can be expressed in terms of the rest of the variables and the initial marking values.

The results reported in this paper refer to continuous MTS systems. The subclass of MTS nets is defined as follows:

Definition 1: A PN is a *mono-T-semiflow* (MTS) net iff it is conservative and has a unique minimal T-semiflow whose support contains all the transitions.

Since in PNs minimal T-semiflows represent all "basic" repetitive behaviors of the system, in MTS nets at most one repetitive behavior exists. Moreover, similarly to discrete systems, the unique T-semiflow stores the ratios of the flows of all transitions in such a repetitive behavior, i.e., in the steady state.

One important property of discrete (and continuous) MTS systems is that deadlock-freeness is equivalent to liveness [5], because all the infinite behaviors are "essentially conformed" by infinite repetition of sequences having (multiples of) the minimal T-semiflow as the firing count vector. Even more, for MTS systems, deadlock-freeness of the untimed model leads, for an arbitrary transition-time semantics (deterministic, exponential, coxian, ...), to non null throughput of the timed system. Thus there exists an interesting one-way bridge from logical or qualitative properties to performance or quantitative properties.

A classical concept in queueing network theory is the *visit ratio*. The visit ratio of transition t_j with respect to t_i, $\mathbf{v}^{(i)}[t_j]$, is the average number of times t_j is visited (fired) for each visit to (firing of) the reference transition t_i (thus $\mathbf{v}^{(i)}[t_i] = 1$) [12]. Observe that $\mathbf{v}^{(i)}$ is a "normalization" of the flow vector in the steady state, i.e., $\mathbf{v}^{(i)}[t_j] = \lim_{\tau \to \infty}(\mathbf{f}[t_j](\tau)/\mathbf{f}[t_i](\tau))$. Let $\mathbf{f}_{ss} = \lim_{\tau \to \infty} \mathbf{f}(\tau)$ denote the flow (or throughput) of the system in the steady state. Then, for any t_i, it holds $\mathbf{f}_{ss} = \mathbf{f}_{ss}[t_i] \cdot \mathbf{v}^{(i)}$. Clearly, the flow vector in the steady state is a right annuler of the incidence matrix $\mathbf{C}$, and therefore, in MTS systems, proportional to the unique T-semiflow.

B. No Deadlock-Freeness Preservation

This subsection shows that deadlock-freeness of the discrete net model is neither a necessary nor a sufficient condition for deadlock-freeness of the relaxed continuous approximation. In

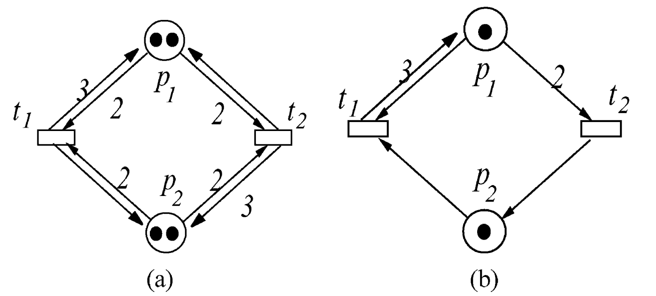

Fig. 2. Two untimed MTS systems which behave in very different ways if seen as discrete or as continuous.

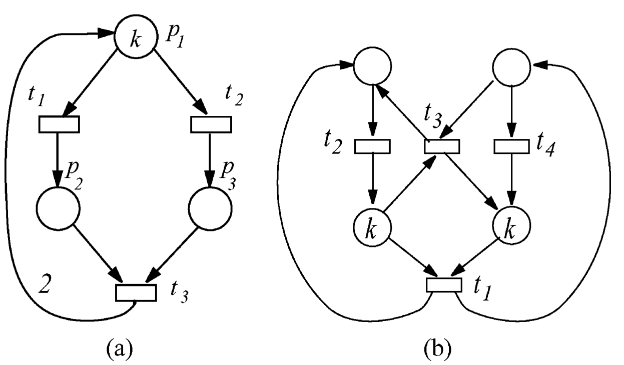

Fig. 3. Deadlock-freeness may also not be preserved by timed continuous systems.

other words, the transformation of a discrete system to a continuous one does not preserve in general the property of deadlock-freeness. This phenomenon, that may be surprising at first glance, can be easily accepted if one thinks, for instance, on the existence of non-linearizable differential equations systems (for example, due to the existence of a chaotic behavior).

Let us first consider the untimed case. Fig. 2(a) shows an untimed MTS system that is not live as discrete: A deadlock marking is reached if t_1 is fired. However, the net system never gets completely blocked as continuous unless an infinitely long sequence is considered. On the other hand, the system in Fig. 2(b) is live as discrete but reaches a deadlock as continuous if t_2 is fired in an amount of 0.5.

With respect to timed systems, the addition of the *infinite servers semantics* time interpretation may allow the timed continuous model to have infinite behavior (deadlock-freeness), while a "similar" timing in the discrete system leads to a deadlock. Let us consider the system in Fig. 3(a) as continuous under infinite servers semantics with $\boldsymbol{\lambda}[t_1] = \boldsymbol{\lambda}[t_2]$. It can be checked that it is live and the flow through transitions t_1 and t_2 is always the same. However, if the system is considered as discrete with a classical markovian time interpretation [13], i.e., all transitions are provided with an exponential probability distribution function law, the stochastic system will arrive to a deadlock marking with probability "1" (this is a particular case of the classical "gambler's ruin problem"). If $\boldsymbol{\lambda}[t_1] = \boldsymbol{\lambda}[t_2]$ the mean time for deadlock is a quadratic function of $k = \mathbf{m}_o[p_1]$ (see [2] for more details). The system in Fig. 2(b) never deadlocks as discrete under a markovian time interpretation with any $\boldsymbol{\lambda}[t_1]$, $\boldsymbol{\lambda}[t_2]$. However, if the system is considered as continuous under infinite servers semantics with $\boldsymbol{\lambda}[t_1] = 1$, $\boldsymbol{\lambda}[t_2] = 4$ it reaches a deadlock.

Liveness of a single transition can also be affected when considering a system as discrete or as continuous. Fig. 3(b) shows a non-MTS system (it has two T-semiflows), that considered as discrete is live (thus deadlock-free). However, for $k = 1$ a deterministic timing of transitions with t_4 faster than t_2 (i.e., $\theta_4 < \theta_2$ where θ_4 and θ_2 are the deterministic delays of t_4 and t_2) makes t_3 nonfireable, thus nonlive (in fact, it starves). Nevertheless, considering the model as continuous, it is live both for the untimed and the timed interpretations.

III. Deadlock-Freeness in Untimed Systems

In continuous systems, it may happen that a marking cannot be reached with a finite firing sequence, but there exists an infinite firing sequence that converges to that marking. As an example of this phenomenon, let us consider the system in Fig. 2(a). The marking $\mathbf{m}[p_1] = 0, \mathbf{m}[p_2] = 4$ cannot be reached with a finite firing sequence. However, if transition t_2 could be fired indefinitely in an amount equal to its enabling degree, the marking of p_1 and p_2 would converge to 0 and 4 respectively. Hence, it becomes reasonable to consider markings like $\mathbf{m}[p_1] = 0, \mathbf{m}[p_2] = 4$ as *reachable markings in the limit*. They are defined as follows:

Definition 2: [3] Let $\langle \mathcal{N}, \mathbf{m}_o \rangle$ be a continuous system. A marking $\mathbf{m} \in (\mathrm{IR}^+ \cup \{0\})^{|P|}$ is said to be lim-*reachable* iff a sequence of reachable markings $\{\mathbf{m}_i\}_{i \geq 1}$ exists verifying $\mathbf{m}_o \xrightarrow{\sigma_1} \mathbf{m}_1 \xrightarrow{\sigma_2} \mathbf{m}_2 \cdots \mathbf{m}_{i-1} \xrightarrow{\sigma_i} \mathbf{m}_i \cdots$ and $\lim_{i \to \infty} \mathbf{m}_i = \mathbf{m}$.

The set of lim-reachable markings is denoted $\lim - \mathrm{RS}(\mathcal{N}, \mathbf{m}_o)$. Liveness and deadlock-freeness definitions can immediately be obtained following the pattern of discrete systems.

Definition 3: [3] Let $\langle \mathcal{N}, \mathbf{m}_o \rangle$ be a continuous PN system.

- $\langle \mathcal{N}, \mathbf{m}_o \rangle$ lim-*deadlocks* iff a marking $\mathbf{m} \in \lim - \mathrm{RS}(\mathcal{N}, \mathbf{m}_o)$ exists such that $\mathrm{enab}(t, \mathbf{m}) = 0$ for every transition t.
- $\langle \mathcal{N}, \mathbf{m}_o \rangle$ is lim–*live* iff for every transition t and for any marking $\mathbf{m} \in \lim - \mathrm{RS}(\mathcal{N}, \mathbf{m}_o)$ a successor $\mathbf{m}'$ exists such that $\mathrm{enab}(t, \mathbf{m}') > 0$.
- $\mathcal{N}$ is *structurally lim-live* iff $\exists\ \mathbf{m}_o$ such that $\langle \mathcal{N}, \mathbf{m}_o \rangle$ is lim-live.

Although *lim-deadlocks* may only be reached in the limit, they represent an important system weakness: They enable the system to reach a marking in which all transitions have infinitely small enabling degrees. Furthermore, it must be pointed out that the concept of lim-reachability in continuous nets provides a better approximation to discrete nets, in the sense that lim-liveness of the continuous system is a sufficient condition for liveness of the discrete one [3].

In MTS systems any subset of transitions, $T' \subsetneq T$, can be disabled just by firing (indefinitely) every transition in T'.

Lemma 4: Let $\mathcal{N}$ be a MTS net. For every $\mathbf{m}_o$ and every $T' \subsetneq T$ a marking $\mathbf{m} \in \lim - \mathrm{RS}(\mathcal{N}, \mathbf{m}_o)$ exists such that for all $t \in T'$ $\mathrm{enab}(t, \mathbf{m}) = 0$. Moreover, this marking can be reached firing only transitions in T'.

Proof: Let $g_0 = \max_{t \in T'} \{\mathrm{enab}(t, \mathbf{m}_o)\}$. Let us pick a transition t_i in T' whose enabling degree is strictly positive (if such a transition does not exist the lemma trivially holds). Let

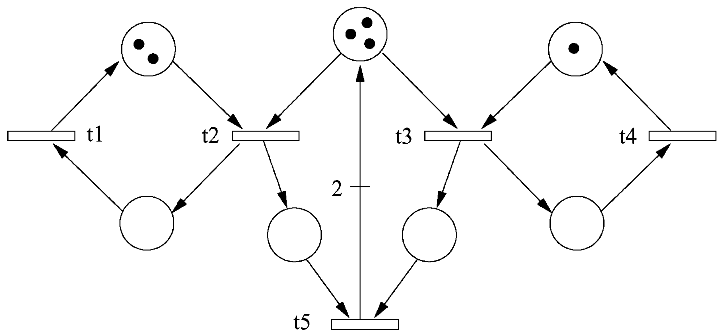

Fig. 4. Non-lim-live system according to Theorem 6.

us fire t_i in an amount equal to its enabling degree (maximum firing amount). This action yields a new marking $\mathbf{m}_1$. Let us recompute $g_1 = \max_{t \in T'}\{\text{enab}(t, \mathbf{m}_1)\}$. Let us pick another transition in T' with strictly positive enabling degree and let us repeat the described procedure. If after repeating these steps a finite number of times, k, it is obtained that every $t \in T'$ is disabled, i.e., $g_k = 0$, then the result is proved. Otherwise the procedure can be repeated indefinitely. Then, the enabling degree of every transition in T' must tend to zero, i.e., $\lim_{n \to \infty}(g_n) = 0$. Otherwise there would exist a repetitive sequence different from the T-semiflow. Contradiction. ■

Notice that disabling a subset of transitions T' is not equivalent to killing them: They could be enabled if the transitions not contained in T' were fired properly.

Lemma 4 leads to the equivalence between lim-deadlock-freeness and lim-liveness for continuous systems, a well-known result for the discrete case [5].

Property 5: A continuous MTS system is lim-live iff it is lim-deadlock-free.

Proof: Assume $\langle \mathcal{N}, \mathbf{m}_o \rangle$ is not lim-live. There is a transition t that cannot be fired from any successor of a certain reachable marking. The application of Lemma 4 on transitions $T - \{t\}$ leads to a deadlock. ■

Suppose that in a given system, $\langle \mathcal{N}, \mathbf{m}_o \rangle$, there is a transition, t, such that for any reachable marking t is never the only enabled transition. This means that if the rest of transitions, $T - \{t\}$, are disabled at a given marking $\mathbf{m}$, then t is also disabled at $\mathbf{m}$. Since every transition of the set $T - \{t\}$ can be disabled in the limit (Lemma 4), it can be inferred that $\langle \mathcal{N}, \mathbf{m}_o \rangle$ is not lim-live.

Theorem 6: Let $\langle \mathcal{N}, \mathbf{m}_o \rangle$ be a lim-live MTS system. For every transition t, $\exists\, \mathbf{m} \in \lim - \text{RS}(\mathcal{N}, \mathbf{m}_o)$ such that t is the only enabled transition at $\mathbf{m}$.

Theorem 6 establishes a necessary lim-liveness condition that is illustrated in Fig. 4. In that system, for every reachable marking in which t_2 is enabled either t_3 or t_4 is enabled. Hence, t_2 is never unavoidably "forced" to fire. Firing several times t_3 and t_4 a deadlock is reached.

Although the condition given by Theorem 6 is in general not easy to check, a simple structural condition (i.e., applicable independently of the initial marking) necessary for liveness can be extracted at net level.

Corollary 7: Let $\mathcal{N}$ be a MTS net. If $\mathcal{N}$ is structurally lim-live then for every $t \neq t'$, ${}^\bullet t \not\subseteq {}^\bullet t'$ (i.e., preconditions of transitions are non comparable)

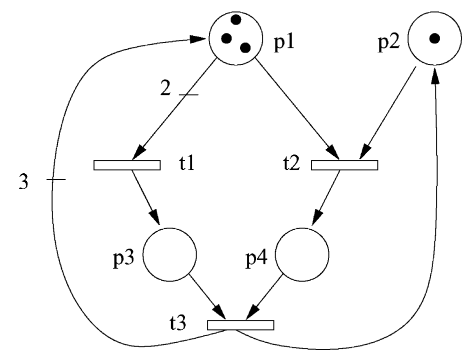

Fig. 5. System for which Corollary 7 detects non-lim-liveness.

Proof: If there exist $t \neq t'$ such that ${}^\bullet t \subseteq {}^\bullet t'$, for every marking in which t' is enabled, t is also enabled. Thus, Theorem 6 can be directly applied and non lim-liveness for an arbitrary initial marking is deduced. ■

Hence, topological conflicts in which the set of input places of one transition is contained in the set of input places of other transition must be forbidden if the system is wanted to be lim-live. For example, in the system in Fig. 5, for any reachable marking if t_2 is enabled then t_1 is also enabled. Thus t_2 is never the only enabled transition and therefore its firing can be always deliberately avoided. By firing successively t_1 and t_3 in amounts equal to their enabling degrees a deadlock is reached for any initial marking. (Notice however that this system is live if seen as discrete.)

In other words, Corollary 7 detects a kind of "structural contradiction" in the MTS net: On the one hand, all the transitions are included in the only repetitive sequence (the T-semiflow), and on the other hand there exist $t \neq t'$ such that ${}^\bullet t \subseteq {}^\bullet t'$, thus, the net gives the possibility of never firing transition t'. The result of this contradiction entails a deadlock.

IV. Deadlock-Freeness in Timed Systems

Deadlock-freeness and liveness definitions of untimed systems can be extended to timed systems:

Definition 8: Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_o \rangle$ be a timed continuous PN system.

- $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_o \rangle$ is *timed-deadlock-free* iff $\mathbf{f}_{ss}(\mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_o) \neq \mathbf{0}$;
- $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_o \rangle$ is *timed-live* iff $\mathbf{f}_{ss}(\mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_o) > \mathbf{0}$;
- $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$is *structurally timed-live* iff there exists $\mathbf{m}_o$ such that $\mathbf{f}_{ss}(\mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_o) > \mathbf{0}$.

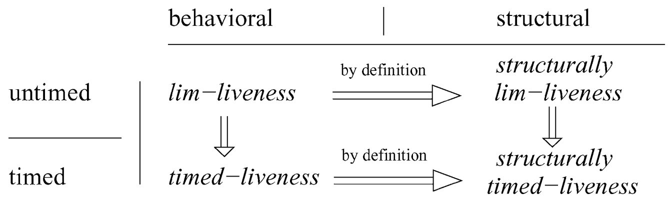

Fig. 6. Relationships among liveness definitions for continuous MTS models.

As in untimed nets structurally timed-liveness is a necessary condition for timed-liveness.

If a timed MTS system is not timed-live (timed-deadlock-free), it can be concluded that, seen as untimed, the system is non-lim-live (lim-deadlock-free) since the evolution of the timed system just gives a particular trajectory, i.e., a firing sequence, that can be fired in the untimed system. Therefore lim-liveness (lim-deadlock-freeness) is a sufficient condition for timed-liveness (timed-deadlock-freeness). The reverse is not true (Fig. 3(a) for $\boldsymbol{\lambda}[t_1] = \boldsymbol{\lambda}[t_2]$). Analogously, structurally lim-liveness is a sufficient condition for structurally timed-liveness. Relationships among liveness definitions are depicted in Fig. 6.

Recall that in the steady state the flow through transitions, $\mathbf{f}_{ss}$, is proportional to the vector of visit ratios, $\mathbf{v}^{(1)}$. Hence, the marking in the steady state, that will be denoted as $\mathbf{m}_{ss}$, verifies

$$\forall\, t\, \boldsymbol{\lambda}[t] \cdot \min_{p \in {}^\bullet t} \left\{ \frac{\mathbf{m}_{ss}[p]}{\mathbf{Pre}[p,t]} \right\} = \mathbf{f}_{ss} = k \cdot \mathbf{v}_t^{(1)}. \qquad (1)$$

A MTS system, $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_o \rangle$, verifies $\mathbf{v}_t^{(1)} > 0$ for every t, and hence it is timed-live iff $k > 0$.

A. Structural Timed-Liveness

The vector of firing speeds $\boldsymbol{\lambda}$ plays a crucial role in the evolution to the steady state. As the system in Fig. 3(a) shows, even structurally non-lim-live systems can be saved from deadlocking by choosing an adequate $\boldsymbol{\lambda}$. It can be seen that for any strictly positive initial marking it is always possible to find a $\boldsymbol{\lambda}$ that makes the system timed-live. One way to achieve this is to choose a $\boldsymbol{\lambda}$ that avoids any transient state, thus making the initial marking equal to the marking in the steady state and, therefore, avoiding a deadlock.

Proposition 9: Given a MTS net, $\mathcal{N}$, for every initial marking $\mathbf{m}_o > \mathbf{0}$, there exists $\boldsymbol{\lambda}$ such that $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_o \rangle$ is timed-live.

Proof: Let us define $\boldsymbol{\lambda}[t_i] = (\mathbf{v}^{(1)}[t_i])/(\mathrm{enab}(t_i, \mathbf{m}_o))$ where $\mathbf{v}^{(1)}$ is the vector of visit ratios normalized for t_1. Since $\mathbf{f}[t_i](\tau = 0) = \boldsymbol{\lambda}[t_i] \cdot \mathrm{enab}(t_i, \mathbf{m}_o)$ it is immediate to verify that $\mathbf{f}(\tau = 0) = \mathbf{v}^{(1)} = \mathbf{f}_{ss} > \mathbf{0}$ ■

For example, the continuous system in Fig. 7 is not lim-live as untimed. However, defining $\boldsymbol{\lambda} = (1\,1\,\lambda_3)$ where λ_3 is an arbitrary positive value, the timed version never deadlocks.

Another interesting problem consists in determining which continuous timed-nets are structurally timed-live (i.e., given $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$, $\exists\, \mathbf{m}_o$ such that $\mathbf{f}_{ss}(\mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_o) > \mathbf{0}$?).

Proposition 10: $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is structurally timed-live iff $\mathbf{m}$ defined as

$$\mathbf{m}[p] = \max_{t \in p^\bullet} \left\{ \frac{\mathbf{Pre}[p,t] \cdot \mathbf{v}^{(1)}[t]}{\boldsymbol{\lambda}[t]} \right\}$$

is a steady state marking for $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$.

Proof: ($\Leftarrow$) Let us assume that $\mathbf{m}$ is a steady-state marking. Given that $\mathbf{m} > \mathbf{0}$, its associated flow in the steady state fulfils $\mathbf{f}_{ss} > 0$ and therefore $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is structurally timed-live.

($\Rightarrow$) There exists $\mathbf{m}_o$ such that $\mathbf{f}_{ss}[t_1](\mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_o) = k > 0$. Let $\mathbf{m}_{ss}$ be the steady state marking associated to $\mathbf{m}_o$. Let us define $\boldsymbol{\mu} = (\mathbf{m}_{ss})/(k)$. Clearly $\mathbf{f}_{ss}(\mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \mathbf{v}^{(1)}$. Then, $\forall p\ \boldsymbol{\mu}[p] \geq \max_{t \in p^\bullet} \{\mathbf{Pre}[p,t] \cdot \mathbf{v}^{(1)}[t]/\boldsymbol{\lambda}[t]\}$. Those components of $\boldsymbol{\mu}$ being strictly greater, $\boldsymbol{\mu}[p] > \max_{t \in p^\bullet} \{\mathbf{Pre}[p,t] \cdot \mathbf{v}^{(1)}[t]/\boldsymbol{\lambda}[t]\}$, can be made equal (they are tokens stacked in synchronizations). This manipulation of $\boldsymbol{\mu}$ does not modify $\mathbf{f}_{ss}(\mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ and results in the marking $\mathbf{m}$ defined in the statement. ■

Let us apply Proposition 10 to the net in Fig. 5 with $\boldsymbol{\lambda} = (4\,1\,1)$. The vector of visit ratios of the net is $\mathbf{v}^{(1)} = (1\,1\,1)$, and so the marking defined by the statement of Proposition 10 is $\mathbf{m} = (1\,1\,1\,1)$. This marking is not a steady state marking since it does not verify (1). Therefore, the timed net with $\boldsymbol{\lambda} = (4\,1\,1)$ is not structurally timed-live.

B. Characterization of the $\boldsymbol{\Lambda}_{\mathcal{N}}$ Set

Given $\mathcal{N}$, an interesting problem lies in determining the set of firing speed vectors for which $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is structurally timed-live. In other words, the goal is to compute a set defined as follows.

Definition 11: $\boldsymbol{\Lambda}_{\mathcal{N}} = \{\boldsymbol{\lambda}\, |\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is structurally timed-live $\}$.

It has been seen that if $\mathcal{N}$ is structurally lim-live then for any $\boldsymbol{\lambda}$, $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is structurally timed-live (recall Fig. 6). Hence for structurally lim-live nets $\boldsymbol{\Lambda}_{\mathcal{N}}$ will be equal to all positive real vectors ($\boldsymbol{\Lambda}_{\mathcal{N}} = (\mathrm{IR}^+)^{|T|}$).

Let us show that the computation of $\boldsymbol{\Lambda}_{\mathcal{N}}$ can be simplified by considering separately each coupled conflict set.

Let us suppose that $\mathcal{N}$ has q coupled conflict sets, $\mathrm{CCS}_1 \ldots \mathrm{CCS}_q$ with $|\mathrm{CCS}_1| = n_1, \ldots, |\mathrm{CCS}_q| = n_q$, $|{}^\bullet\mathrm{CCS}_1| = s_1, \ldots, |{}^\bullet\mathrm{CCS}_q| = s_q$, and that transitions and places are sorted according to the coupled conflict they belong to: $t_{1,1} \ldots t_{1,n_1}, \ \ldots, t_{q,1} \ldots t_{q,n_q}$ and $p_{1,1} \ldots p_{1,s_1}, \ldots, p_{q,1} \ldots p_{q,s_q}$. Let us define $\boldsymbol{\Lambda}_{\mathrm{CCS}_i}$ as a set of vectors associated to the coupled conflict set CCS_i in the following way:

Difinition 12: $\boldsymbol{\Lambda}_{\mathrm{CCS}_i} = \{\boldsymbol{\lambda}_i \mid \boldsymbol{\lambda}_i \in (\mathrm{IR}^+)^{n_i}$ and $\exists\, \mathbf{m} \in (\mathrm{IR}^+)^{s_i}$ such that $\forall\, t \in \mathrm{CCS}_i\ \ \mathbf{v}^{(1)}[t] = \boldsymbol{\lambda}_i[t] \cdot \mathrm{enab}(t, \mathbf{m})\}$

From the previous definition it is obtained that for any $\boldsymbol{\lambda}_i \in \boldsymbol{\Lambda}_{\mathrm{CCS}_i}$ there exists a marking $\mathbf{m} \in (\mathrm{IR}^+)^{s_i}$ such that for every $t \in \mathrm{CCS}_i$ it holds $\boldsymbol{\lambda}_i[t] = \mathbf{v}^{(1)}[t]/\mathrm{enab}(t, \mathbf{m}) = \mathbf{v}[t]/\min_{p_j \in {}^\bullet t}\{\mathbf{m}[p_j]/\mathbf{Pre}[p_j, t]\} = \mathbf{v}[t] \cdot \max_{p_j \in {}^\bullet t}\{\mathbf{Pre}[p_j, t]/\mathbf{m}[p_j]\}$. Thus, by defining $\alpha_j = 1/\mathbf{m}[p_j]$ the set $\boldsymbol{\Lambda}_{\mathrm{CCS}_i}$ can be expressed in a more direct way.

Proposition 13: $\boldsymbol{\Lambda}_{\mathrm{CCS}_i} = \{\boldsymbol{\lambda}_i\, |\boldsymbol{\lambda}_i \in (\mathrm{IR}^+)^{n_i}$ and $\exists\, \boldsymbol{\alpha} \in (\mathrm{IR}^+)^{s_i}$ such that $\forall\, t \in \mathrm{CCS}_i\ \ \boldsymbol{\lambda}_i[t] = \mathbf{v}^{(1)}[t] \cdot \max_{p_j \in {}^\bullet t}\{\alpha_j \cdot \mathbf{Pre}[p_j, t]\}\}$

The following theorem states that $\boldsymbol{\Lambda}_{\mathcal{N}}$ can be computed as the cartesian product of all the $\boldsymbol{\Lambda}_{\mathrm{CCS}}$ of the net.

Theorem 14: $\boldsymbol{\Lambda}_{\mathcal{N}} = \{(\lambda_{1,1}, \ldots, \lambda_{q,n_q})\ |(\lambda_{i,1}, \ldots, \lambda_{i,n_i}) \in \boldsymbol{\Lambda}_{\mathrm{CCS}_i}\}$.

Proof:

- $\boldsymbol{\Lambda}_{\mathcal{N}} \subseteq \{(\lambda_{1,1}, \ldots, \lambda_{q,n_q})|(\lambda_{i,1}, \ldots, \lambda_{i,n_i}) \in \boldsymbol{\Lambda}_{\mathrm{CCS}_i}\}$ If $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_{\mathcal{N}}$, there exists $\mathbf{m}_o$ such that $\mathbf{f}_{ss}[t_1](\mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_o) =$

$l > \mathbf{0}$. Let $\mathbf{m}_{\mathrm{ss}}$ be its associated steady state marking. Then $\forall\, i = 1 \ldots q$, $\lambda_i = \{\lambda_{i,1} \ldots \lambda_{i,n_i}\} \in \mathbf{\Lambda}_{\mathrm{CCS}_i}$ with $\mathbf{m}_{i,j} = \mathbf{m}_{\mathrm{ss}_{i,j}}/l$.

- $\mathbf{\Lambda}_{\mathcal{N}} \supseteq \{(\boldsymbol{\lambda}_{1,1},\ldots,\boldsymbol{\lambda}_{q,n_q}) | (\lambda_{i,1},\ldots,\lambda_{i,n_i}) \in \mathbf{\Lambda}_{\mathrm{CCS}_i}\}$ If $\boldsymbol{\lambda} \in \{(\lambda_{1,1},\ldots,\lambda_{q,n_q}) | (\lambda_{i,1},\ldots,\lambda_{i,n_i}) \in \mathbf{\Lambda}_{\mathrm{CCS}_i}\}$ with the marking $m_{1,1}\ldots m_{1,s_1}$, $m_{2,1}\ldots m_{2,s_2}$, $m_{q,1}\ldots m_{q,s_q}$, then using this marking as $\mathbf{m}_o$, one has $\mathbf{f}_{ss}(\mathcal{N},\boldsymbol{\lambda},\mathbf{m}_o) = \mathbf{v}^{(1)} > \mathbf{0}$. Therefore $\boldsymbol{\lambda} \in \mathbf{\Lambda}_{\mathcal{N}}$.

■

Let us consider the net depicted in Fig. 5. Its vector of visit ratios is $\mathbf{v}^{(1)} = (1\,1\,1)$ and it has two CCSs: CCS_{t_1,t_2} and CCS_{t_3}. Applying Proposition 13 the following sets are computed: $\mathbf{\Lambda}_{\mathrm{CCS}_{t_1,t_2}} = \{(\lambda_1,\lambda_2) | \lambda_1 = \alpha_1, \lambda_2 = \max\{\alpha_1 \cdot 2, \alpha_2\}, \alpha_1 > 0, \alpha_2 > 0\} = \{(\lambda_1,\lambda_2) | \lambda_2 \geq 2 \cdot \lambda_1 > 0\}$ and $\mathbf{\Lambda}_{\mathrm{CCS}_{t_3}} = \{(\lambda_3) | \lambda_3 = \alpha_3, \alpha_3 > 0\}$. A direct application of Theorem 14 on these sets allows one to obtain the set $\mathbf{\Lambda}_{\mathcal{N}} = \{(\lambda_1,\lambda_2,\lambda_3) | \lambda_2 \geq 2 \cdot \lambda_1 > 0, \lambda_3 > 0\}$, i.e., the set of firing speeds that allows the system to reach a nondead steady state.

C. Restrictive Places

The expression in Proposition 13 describes the set of internal speeds of the transitions in a CCS that have to be considered in order to avoid the system to deadlock. That is, if the internal speeds of the transitions are in $\mathbf{\Lambda}_{\mathrm{CCS}}$, there exists a marking at which the transitions are fired proportionally to the vector of visit ratios. In other words, at that marking transitions are enabled in such a way that the flow vector is proportional to the vector of visit ratios.

Difinition 15: Given a steady state marking, $\mathbf{m}$, a place p is said to be a *restrictive* place for t, one of its output transitions, if the enabling degree of t at $\mathbf{m}$ is defined by the marking of p, i.e., $\mathrm{enab}(t,\mathbf{m}) = \mathbf{m}[p]/\mathbf{Pre}[p,t]$.

For a given $\langle \mathcal{N},\boldsymbol{\lambda}\rangle$, it turns out that in the steady state a place may be restrictive only for some of its output transitions. The set of transitions for which a place can be restrictive depends on the structure of the net and on the internal speeds of the transitions. This fact can be interpreted in the following way: in the steady state every transition is "demanding" a minimum quantity of marking to their input places in order to have a positive throughput. From another point of view, this means that places have to supply enough fluid to their output transitions. Hence, in the steady state, a place can be restrictive only for the output transition(s) that is (are) demanding the greatest amount of fluid.

Let us assume that the system $\langle \mathcal{N},\boldsymbol{\lambda},\mathbf{m}_o\rangle$ reaches a steady state at which the throughput is $\mathbf{f}_{ss}$. Thus, the marking of a given place p has to be big enough to allow its output transitions to fire according to $\mathbf{f}_{ss}$. Considering all the output transitions of a place p, its marking in the steady state, $\mathbf{m}_{\mathrm{ss}}[p]$, has to fulfill

$$\mathbf{m}_{\mathrm{ss}}[p] \geq \max_{t_i \in p^\bullet} \left\{ \frac{\mathbf{Pre}[p,t_i]}{\boldsymbol{\lambda}[t_i]} \cdot \mathbf{f}_{ss}[t_i] \right\} \tag{2}$$

This equation can be directly obtained from (1). Given a place p, (2) allows one to compute which of its output transitions is demanding the greatest marking in the steady state. And so, it is possible to deduce the transition(s) for which the place p can be restrictive. Since MTS are being considered, $\mathbf{f}_{ss}$ is proportional to the only T-semiflow of the system. Therefore, taking into account (2), the computation of the transition(s) for which a place p can be restrictive only depends on the T-semiflow of the system and on the vector of internal speeds $\boldsymbol{\lambda}$, i.e., it does not depend on the initial marking.

Proposition 16: The set of transitions for which a place p can be restrictive is given by

$$\left\{ t_j \in p^\bullet | \frac{\mathbf{Pre}[p,t_j]}{\boldsymbol{\lambda}[t_j]} \cdot \mathbf{v}^{(1)}[t_j] = \max_{t_i \in p^\bullet} \left\{ \frac{\mathbf{Pre}[p,t_i]}{\boldsymbol{\lambda}[t_i]} \cdot \mathbf{v}^{(1)}[t_i] \right\} \right\} \tag{3}$$

where $\mathbf{v}^{(1)}$ is the vector of visit ratios normalized for transition t_1.

According to Proposition 16, a place p can be restrictive for more than one transition only in the case that several of its output transitions are demanding exactly the same marking to place p. Notice that every transition has an input restrictive place. However, not every place has to be a restrictive place for one of its output transitions.

Let us assume that given a $\langle \mathcal{N},\boldsymbol{\lambda}\rangle$ the only possible restrictive place for a transition t is the place p. Thus, in a nondead steady state, place p has the *responsibility* of enabling transition t according to its visit ratio. The task of enabling transition t will be impossible for place p if it is an implicit place [14] (a place p is said to be *implicit* iff for any reachable marking $\mathbf{m}$ and for any transition t there exists $p_1 \in^\bullet t$, $p_1 \neq p$ such that $\mathbf{m}[p_1]/\mathbf{Pre}[p_1,t] = \mathrm{enab}(t,\mathbf{m})$, in other words p is never the only restrictive place of its output transitions). Therefore, if the system is desired to reach a non dead steady state, it must be avoided that the set of places that must enable transitions according to the vector of visit ratios is implicit.

In order to illustrate these reasonings, let us consider a CCS composed of two places and two transitions whose normalized visit ratios are v_1 and v_2; see Fig. 8. According to Proposition 13, the $\mathbf{\Lambda}_{\mathrm{CCS}}$ associated to that CCS is $\mathbf{\Lambda}_{\mathrm{CCS}} = \{(\lambda_1,\lambda_2) \mid \lambda_1 = v_1 \cdot \max\{\alpha_1 \cdot a, \alpha_2 \cdot c\}, \lambda_2 = v_2 \cdot \max\{\alpha_1 \cdot b, \alpha_2 \cdot d\}, \boldsymbol{\alpha} \in (\mathrm{IR}^+)^2\}$. Through some algebraic operations it can be seen that the region in the two dimensional plane associated to this $\mathbf{\Lambda}_{\mathrm{CCS}}$ can be written as: $\mathbf{\Lambda}_{\mathrm{CCS}} = \{(\lambda_1,\lambda_2) | (\lambda_1,\lambda_2) = \beta_1 \cdot (v_1 \cdot a, v_2 \cdot b) + \beta_2 \cdot (v_1 \cdot c, v_2 \cdot d), \boldsymbol{\beta} \in (\mathrm{IR}^+ \cup \{0\})^2\}$ (see Appendix I for an sketch of this equivalence). That is, the set $\mathbf{\Lambda}_{\mathrm{CCS}}$ can be seen as a two dimensional cone. Let us assume, without loss of generality, that $d/c \geq b/a$. Then, the slopes of the upper and lower edges of the cone are d/c and b/a, respectively. Fig. 9 depicts the region associated to $\mathbf{\Lambda}_{\mathrm{CCS}}$.

Only those speeds, $(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)$, in the cone allow the system to reach a nondead steady state. Assuming that this CCS is part of a MTS, $\langle \mathcal{N},\boldsymbol{\lambda},\mathbf{m}_o\rangle$, and that a non dead steady state is reached, the following results based on (3) hold.

- If (λ_1,λ_2) is not a border point of the cone $\mathbf{\Lambda}_{\mathrm{CCS}}$, then p_1 is the restrictive place of t_1 and p_2 is the restrictive place of t_2. Therefore, if any of the places is implicit the system will deadlock.
- If (λ_1,λ_2) is a point in the upper edge of the cone then p_1 is restrictive for t_1 and p_2 is restrictive for both transitions. If the system is wanted to reach a nondead steady state it is necessary that place p_2 is not implicit.

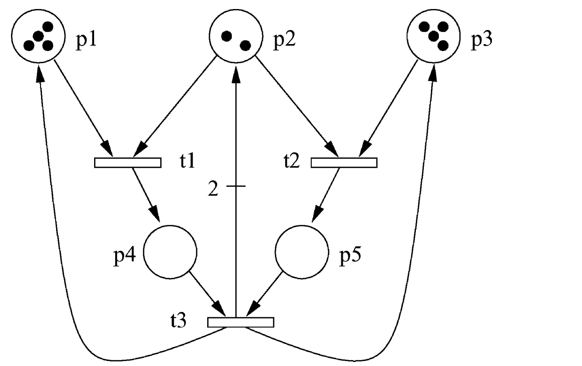

Fig. 7. Non-lim-live untimed system that never deadlocks as timed with $\boldsymbol{\lambda} = (1\ 1\ \lambda_3)$ for any $\lambda_3 > 0$. Every transition owns a CF place but the timed system deadlocks with $\boldsymbol{\lambda} = (2\ 1\ 1)$.

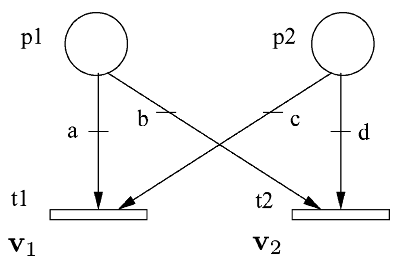

Fig. 8. Simple CCS.

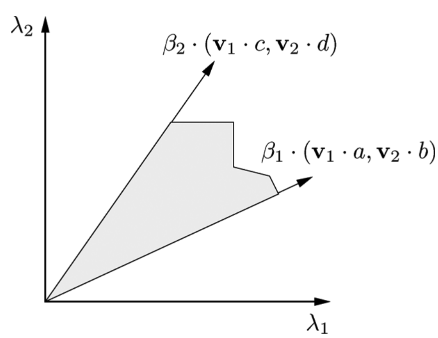

Fig. 9. $\boldsymbol{\Lambda}_{\mathrm{CCS}}$ set associated to the CCS in Fig. 8 assuming that $d/c \geq b/a$.

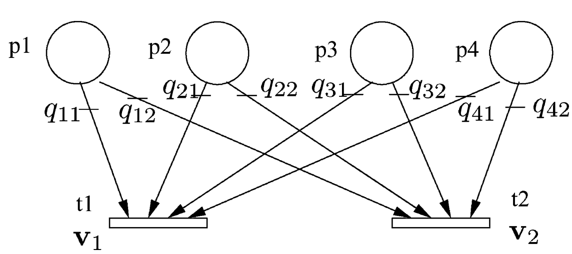

Fig. 10. CCS with four places and two transitions.

- If (λ_1, λ_2) is a point in the lower edge of the cone then p_1 is restrictive for both transitions and p_2 is restrictive for t_2. In this case if p_1 is implicit the system will deadlock.

It is not difficult to extend the previous results to a more complex CCS consisting of several input places and two transitions. The CCS in Fig. 10 has four input places and two transitions. The region containing the set $\boldsymbol{\Lambda}_{\mathrm{CCS}}$ has the shape of a cone. See Fig. 11 for the representation of $\boldsymbol{\Lambda}_{\mathrm{CCS}}$ assuming that $q_{42}/q_{41} \geq q_{32}/q_{31} \geq q_{22}/q_{21} \geq q_{12}/q_{11}$.

Three cones can be distinguished in the interior of $\boldsymbol{\Lambda}_{\mathrm{CCS}}$. The set of restrictive places depends on the cone to which $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$

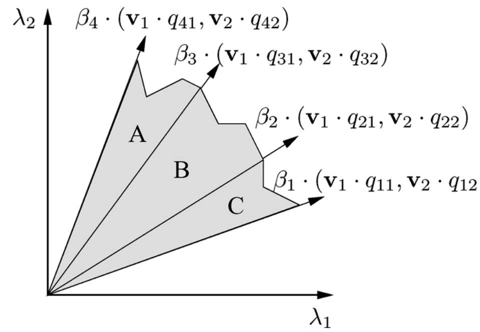

Fig. 11. $\boldsymbol{\Lambda}_{\mathrm{CCS}}$ set associated to the CCS in Fig. 10 assuming that $q_{42}/q_{41} \geq q_{32}/q_{31} \geq q_{22}/q_{21} \geq q_{12}/q_{11}$.

belongs. If (λ_1, λ_2) belongs to the cone A, the possible restrictive places for t_1 are p_1, p_2, p_3 and for t_2 the only possible restrictive place is p_4. Hence, in this case if p_4 is implicit the system will deadlock. A deadlock will also be reached if all three places p_1, p_2 and p_3 are implicit simultaneously. For a (λ_1, λ_2) in the cone B, the restrictive places for transition t_1 are p_1 and p_2, and for transition t_2 are restrictive places p_3 and p_4. If $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$ is cone C, the only restrictive place for transition t_1 is p_1 and the possible restrictive places for transition t_2 are p_2, p_3 and p_4. Finally, notice that in the edges of the cones it turns out that a single place can be restrictive for both transitions. For example for a (λ_1, λ_2) in the upper edge of the cone, place p_4 is restrictive for both transitions.

D. Critical Timed-Liveness

It has been seen that those $\boldsymbol{\lambda}$ vectors not included in $\boldsymbol{\Lambda}_{\mathcal{N}}$ do not allow the MTS system to reach a steady state with throughput greater than zero. Although $\boldsymbol{\Lambda}_{\mathcal{N}}$ is never an empty set (for every positive initial marking there exists $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_{\mathcal{N}}$), its "size" can be much smaller than desired. For example it is not desirable to use a vector of $\boldsymbol{\Lambda}_{\mathcal{N}}$ such that a minimum change in one of its components puts the vector out of $\boldsymbol{\Lambda}_{\mathcal{N}}$. It would mean that a small variation in the firing speed of one transition can kill the system. Hence, a new concept is needed to define wether a system can be "robust enough" to bear irregularities and variations happening in the real world.

Definition 17: $(\mathcal{N}, \boldsymbol{\lambda})$ is critically structurally timed-live iff $\boldsymbol{\lambda}$ is a border point of $\boldsymbol{\Lambda}_{\mathcal{N}}$

In some cases, the net structure can reduce dramatically the dimension of $\boldsymbol{\Lambda}_{\mathcal{N}}$. For every coupled conflict with *n* transitions $\boldsymbol{\Lambda}_{\mathrm{CCS}}$ is contained in $(\mathrm{IR}^+)^{|n|}$. Therefore the maximum dimension that a given region of the $\boldsymbol{\Lambda}_{\mathrm{CCS}}$ set can have is *n*. Apart from this constraint, the effective dimension of $\boldsymbol{\Lambda}_{\mathrm{CCS}}$ is also limited by the number of input places of the coupled conflict set, since $\boldsymbol{\Lambda}_{\mathrm{CCS}}$ is generated by as many independent variables as input places (see Proposition 13). Therefore, the following Proposition holds:

Proposition 18: Given a coupled conflict set CCS, the maximum dimension of any region of $\boldsymbol{\Lambda}_{\mathrm{CCS}}$ is bounded by the number of transitions in the CCS, $|CCS|$, and the number of input places of the CCS, $|^\bullet\mathrm{CCS}|$.

Since $\boldsymbol{\Lambda}_{\mathcal{N}}$ is the cartesian product of all the $\boldsymbol{\Lambda}_{\mathrm{CCS}}$ of the net (Theorem 14), if the net has a coupled conflict set CCS with less input places, $|^\bullet\mathrm{CCS}|$, than transitions, $|CCS|$, every region in $\mathbf{\Lambda}_{\mathcal{N}}$ will have a smaller dimension than the number of transitions of the net. That is, every point in $\mathbf{\Lambda}_{\mathcal{N}}$ will be a border point. From this reasoning, Proposition 19 is derived.

Proposition 19: Given a net $\mathcal{N}$, if there exists a CCS such that $|{}^\bullet\mathrm{CCS}| < |\mathrm{CCS}|$ then for every $\boldsymbol{\lambda} \in \mathbf{\Lambda}_{\mathcal{N}}$ $(\mathcal{N}, \boldsymbol{\lambda})$ is critically structurally timed-live.

For example, the CCS composed of $\{t_1, t_2\}$ in Fig. 3(a) has only one input place, p_1. Hence, if $\boldsymbol{\lambda} \notin \mathbf{\Lambda}_{\mathcal{N}}$ then for every initial marking the system will eventually deadlock, and if $\boldsymbol{\lambda} \in \mathbf{\Lambda}_{\mathcal{N}}$ then $(\mathcal{N}, \boldsymbol{\lambda})$ is critically structurally timed-live.

This means that in practice, all the coupled conflicts sets should have at least as many input places as transitions, otherwise the system will die or will remain in a critical timed-live state.

E. Robust Timed-Liveness

Going on with critical structurally timed-liveness, one could ask which features should be required to a system in order to be "safe," i.e., arbitrary variations in $\boldsymbol{\lambda} > 0$ do not cause $\boldsymbol{\lambda}$ to be out of $\mathbf{\Lambda}_{\mathcal{N}}$. In other words, the interest lies in looking for those net structures that allow the system to reach a nondead steady state for any $\boldsymbol{\lambda}$.

A place p is said to be *choice-free (CF)* iff $|p^\bullet| = 1$, i.e., p has a single output transition. It will be said that a transition *owns* its (input) CF places.

Theorem 20: Let $\mathcal{N}$ be a MTS net, $\forall \boldsymbol{\lambda} > 0$ $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is structurally timed-live iff every transition owns at least one CF place.

Proof: ($\Leftarrow$) Let $\boldsymbol{\lambda} \in (\mathrm{IR}^+)^{|T|}$. Let $P' = \{p_1, \ldots, p_k\}$ be a minimal set of CF places such that $(P')^\bullet = T$. For every $p_i \in P'$ let $t_i = p_i^\bullet$, and define $\mathbf{m}[p_i] = (\mathbf{Pre}[p_j, t_i] \cdot \mathbf{v}^{(1)}[t_i])/(\boldsymbol{\lambda}[t_i])$ if $p_i \in P'$, and $\mathbf{m}[p_i] = \max_{t_j \in p_i^\bullet}\{\mathbf{Pre}[p_i, t_j] \cdot \mathbf{v}^{(1)}[t_j]/\boldsymbol{\lambda}[t_j]\}$ otherwise. Then, it is obtained that for every $t_i \in T$ $\boldsymbol{\lambda}[t_i] \cdot \min_{p_j \in {}^\bullet t_i}\{\mathbf{m}[p_j]/\mathbf{Pre}[p_j, t_i]\} = \mathbf{v}^{(1)}[t_i]$. Therefore, $\mathbf{m}$ is a steady-state marking.

($\Rightarrow$) Let $\mathbf{\Lambda}_{\mathcal{N}} = (\mathrm{IR}^+)^{|T|}$ and let us assume that there exists a transition t_1 without CF places. Then, it holds that $\forall\, p \in^\bullet t_1 |p^\bullet| > 1$. Let us define $\{t_1, \ldots, t_k\}$ as $\{t_1, \ldots, t_k\} = ({}^\bullet t_1)^\bullet \subseteq CCS(t_1)$. According to Theorem 14, $\mathbf{\Lambda}_{\mathcal{N}}$ is the product of $\mathbf{\Lambda}_{CCS(t_i)}$, so $\mathbf{\Lambda}_{CCS(t_1)} = (\mathrm{IR}^+)^{|CCS(t_1)|}$. We will reach a contradiction to our initial assumption by finding an upper bound of $\boldsymbol{\lambda}[t_1]$, what clearly implies $\mathbf{\Lambda}_{\mathcal{N}} \neq (\mathrm{IR}^+)^{|T|}$. From Proposition 13, it is known that for any $\boldsymbol{\lambda} \in \mathbf{\Lambda}_{CCS(t_1)}$, a certain $\boldsymbol{\alpha} \in (\mathrm{IR}^+)^{|{}^\bullet CCS(t_1)|}$ exists verifying $\boldsymbol{\lambda}[t_i] = \mathbf{v}^{(1)}[t_i] \cdot \max_{\{p_j \in {}^\bullet t_i\}}\{\alpha_j \cdot \mathbf{Pre}[p_j, t_i]\}$ for every $1 \leq i \leq k$. Therefore, it holds that $\boldsymbol{\lambda}[t_1] = \mathbf{v}^{(1)}[t_1] \cdot \alpha_h \cdot \mathbf{Pre}[p_h, t_1]$ for a certain $p_h \in^\bullet t_1$. According to our assumption $|p_h^\bullet| > 1$, and thus another transition $t_j \in \{t_2, \ldots, t_k\}$ must exist such that $t_j \in p_h^\bullet$ and $\boldsymbol{\lambda}[t_j] \geq \mathbf{v}^{(1)}[t_j] \cdot \alpha_h \cdot \mathbf{Pre}[p_h, t_j]$. Hence, it is obtained that $(\boldsymbol{\lambda}[t_1])/(\mathbf{v}^{(1)}[t_1] \cdot \mathbf{Pre}[p_h, t_1]) = \alpha_h \leq (\boldsymbol{\lambda}[t_j])/(\mathbf{v}^{(1)}[t_j] \cdot \mathbf{Pre}[p_h, t_j]) \leq \max_{\{j \in \{2 \ldots k\}\}}\{(\boldsymbol{\lambda}[t_j])/(\mathbf{v}^{(1)}[t_j] \cdot \mathbf{Pre}[p_h, t_j])\}$. Therefore, it holds that $\boldsymbol{\lambda}[t_1] \leq \mathbf{v}^{(1)}[t_1] \cdot \mathbf{Pre}[p_h, t_1] \cdot \max_{\{j \in \{2 \ldots k\}\}}\{(\boldsymbol{\lambda}[t_j])/(\mathbf{v}^{(1)}[t_j] \cdot \mathbf{Pre}[p_h, t_j])\}$. That is, $\boldsymbol{\lambda}[t_1]$ is bounded by a value that depends on $\boldsymbol{\lambda}[t_2], \ldots, \boldsymbol{\lambda}[t_k]$. Contradiction. ■

From a different point of view, Theorem 20 states that the transitions can be enabled independently iff every transition owns a CF place. Observe that this condition does not guarantee that the system will always reach a non-dead steady state for every initial marking. For example in Fig. 7, every transition owns a CF place (and it is not a CF net). However, with that initial marking, choosing $\boldsymbol{\lambda} = (2\,1\,1)$ the system cannot reach a live steady state. This happens because the enabling degree of t_1 and t_2 is always the same, since p_1 and p_3 are implicit places [14], so they can be removed without changing the possible behaviors (trajectories for the timed case) of the system. From Theorem 20 it can be inferred that for those nets that have a transition without CF places there exists a $\boldsymbol{\lambda}$ for which the timed system deadlocks independently of the initial marking. Transitions $\{t_1, t_2, t_3\}$ in Fig. 12 do not own CF places. For $\boldsymbol{\lambda} = (1\,1\,2\,1\,1\,1)$, $\boldsymbol{\lambda} \notin \mathbf{\Lambda}_{\mathcal{N}}$, the system will evolve to a deadlock, no matter which the initial marking is.

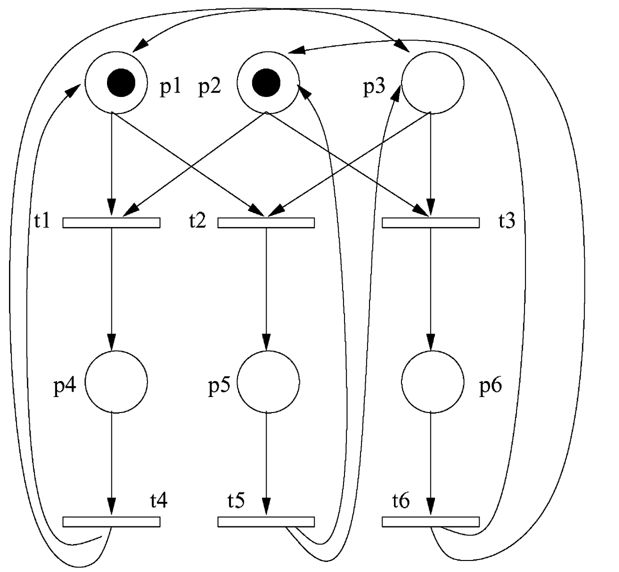

Fig. 12. Transitions t_1, t_2 and t_3 do not own CF places. For $\boldsymbol{\lambda} = (1\,1\,2\,1\,1\,1)$ no steady state with positive throughput is possible.

F. Coming Back to Structural Liveness in Untimed Systems

According to Theorem 20, if $\mathcal{N}$ has a transition without CF places, a $\boldsymbol{\lambda}$ exists such that $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is not structurally timed-live. Therefore $\mathcal{N}$ is not structurally lim-live, since structurally timed-liveness is a necessary condition for structurally lim-liveness (see Fig. 6).

Theorem 21: Let $\mathcal{N}$ be a MTS net. If $\mathcal{N}$ is structurally lim-live then every transition owns at least one CF place.

The system shown in Fig. 12 is not lim-live according to Theorem 21, since there are three transitions, t_1, t_2 and t_3 that do not own a CF place. In this case, the firing of a sequence that corresponds to vector $\sigma = (0\,1\,0\,0\,1\,0)$ leads to marking $\mathbf{m} = (0\,2\,0\,0\,0\,0)$, where the system lim-deadlocks. Notice that in consistent continuous systems in which every transition can be fired at least once, there do not exist spurious solutions of the state equation [3], hence a sequence can be fired that reaches marking $\mathbf{m}$.

Transition t does not own a CF place iff all the input places of t are contained in the set of input places of the rest of the transitions. Thus, Theorem 21 can be rewritten as: If $\mathcal{N}$ is structurally lim-live then for every t, ${}^\bullet t \not\subseteq {}^\bullet(T\setminus\{t\})$. Notice the similarity of this statement to that of Corollary 7. In fact, Theorem 21 and Corollary 7 express exactly the same condition if all the coupled conflict sets of the net have at most two transitions, but in general Theorem 21 has a greater decision power.

V. Conclusion

Continuous PNs may be used to overcome the state explosion problem of highly populated discrete systems. In continuous nets the firing of transitions is not discrete, but continuous. In this paper, the attention is focused on the study of continuous MTS systems, that is, systems that are consistent, conservative and have only one T-semiflow. Such systems are easy to characterize and offer an interesting modelling power.

Since MTS systems have only one repetitive sequence, lim-deadlock-freeness becomes equivalent to lim-liveness. From an untimed point of view, that is, without time interpretation, necessary structural lim-liveness conditions have been obtained. For MTS systems these conditions provide a better characterization than the one established by the rank Theorem [14]. For example, Theorem 21 allows one to decide on lim-liveness of the system in Fig. 12 while the rank Theorem does not. It is remarkable that all those conditions (Theorem 6, Corollary 7, and Theorem 21) are stated on topological features of the net, hence disregarding the arc weights. This is a logical consequence of the continuous firing of transitions: the arc weights have an influence on the amount in which one transition is enabled (enabling degree) but deciding whether a transition is enabled or not does not depend on the value of its input arc weights.

Lim-liveness and structural lim-liveness have been defined for timed systems [3]. For such systems a necessary and sufficient condition for structural timed liveness has been derived (Proposition 10). A a new concept, *critical timed-liveness*, has been introduced. A critical timed-live system can reach a live steady state, however any small variation in one of its firing speeds will cause the system to deadlock. The set of speeds that allow the system to reach non-dead steady states has been characterized. The existing relationships between lim-liveness of untimed systems and liveness of timed systems offers the possibility of applying some results obtained for timed systems (Theorem 20) to untimed systems (Theorem 21). This way, previous results for untimed systems (Corollary 7) are improved.

Appendix I

Outline of the equivalence of the expressions for $\boldsymbol{\Lambda}_{\text{CCS}}$ in Subsection IV-C: $\boldsymbol{\Lambda}_{\text{CCS}} = \{(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) | \boldsymbol{\lambda}_1 = \mathbf{v}_1 \cdot \max\{\alpha_1 \cdot a, \alpha_2 \cdot c\}, \boldsymbol{\lambda}_2 = \mathbf{v}_2 \cdot \max\{\alpha_1 \cdot b, \alpha_2 \cdot d\}, \boldsymbol{\alpha} \in (\mathrm{IR}^+)^2\}$ and $\boldsymbol{\Lambda}_{\text{CCS}} = \{(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) | (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \beta_1 \cdot (\mathbf{v}_1 \cdot a, \mathbf{v}_2 \cdot b) + \beta_2 \cdot (\mathbf{v}_1 \cdot c, \mathbf{v}_2 \cdot d), \boldsymbol{\beta} \in (\mathrm{IR}^+ \cup \{0\})^2\}$. It will be assumed, without loss of generality, that $d/c \geq b/a$.

"$\supseteq$" Given β_1, β_2, define $\alpha_1 = (\beta_1 \cdot a + \beta_2 \cdot c)/a$ and $\alpha_2 = (\beta_1 \cdot b + \beta_2 \cdot d)/d$.

"$\subseteq$" Given α_1, α_2.

- If $\alpha_1 \cdot a \geq \alpha_2 \cdot c$ and $\alpha_1 \cdot b < \alpha_2 \cdot d$ then define $\beta_1 = d \cdot (\alpha_1 \cdot a - \alpha_2 \cdot c)/(a \cdot d - b \cdot c)$ and $\beta_2 = a \cdot (\alpha_2 \cdot d - \alpha_1 \cdot b)/(a \cdot d - b \cdot c)$.
- If $\alpha_1 \cdot a \geq \alpha_2 \cdot c$ and $\alpha_1 \cdot b \geq \alpha_2 \cdot d$ then define $\beta_1 = \alpha_1$ and $\beta_2 = 0$.
- If $\alpha_1 \cdot a < \alpha_2 \cdot c$ and $\alpha_1 \cdot b \leq \alpha_2 \cdot d$ then define $\beta_1 = 0$ and $\beta_2 = \alpha_2$.
- If $\alpha_1 \cdot a < \alpha_2 \cdot c$ and $\alpha_1 \cdot b < \alpha_2 \cdot d$ then it holds $\alpha_1 \cdot a \cdot d < \alpha_2 \cdot c \cdot d < \alpha_1 \cdot b \cdot c$, then $a \cdot d < b \cdot c$. Thus, this case is not possible because it was assumed $d/c \geq b/a$.

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