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Distributed relative localization using the multi-dimensional weighted centroid

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***Abstract*—A key problem in multi-agent systems is the distributed estimation of the localization of agents in a common reference from relative measurements. Estimations can be referred to an anchor node or, as we do here, referred to the weighted centroid of the multi-agent system. We propose a Jacobi Over–Relaxation method for distributed estimation of the weighted centroid of the multi-agent system from noisy relative measurements. Contrary to previous approaches, we consider relative multi-dimensional measurements with general covariance matrices not necessarily fully diagonal. We analyze the method convergence and provide mathematical constraints that ensure avoiding ringing phenomena. We also prove our weighted centroid method converges faster than anchor-based solutions.**

***Index Terms*—Distributed Sensor Networks; Noisy Relative Measurements; Multi–agent Localization; Jacobi Over–Relaxation; Weighted Centroid**

I. Introduction

Localization is a central task in multi-agent systems. For example, in order to cooperatively manipulate a load, agents need to know their positions in a common frame [1]. Agents usually start at unknown locations, and they can only perceive nearby agents (*neighbors*). Each agent combines bearing and range measurements [2] of the position of its neighbors and it builds a 2D or 3D representation of the relative positions of the neighbors in its own local frame (*multi–dimensional* relative *full-position* measurements). This multi–dimensional sensed data is corrupted with noise, with associated covariance matrices which are not necessarily fully diagonal, as it is often assumed by several approaches [3]–[11]. The distributed localization problem, which often considers stationary agents, consists of combining these relative measurements to build an estimate of the agents' positions in a common frame.

It is well known [3]–[13] that the localization problem can be solved only up to an additive constant, which is equivalent to representing the agent locations relative to different reference frames. In order to remove this ambiguity, several solutions [3], [4], [12], [13] fix an *anchor*, e.g., the first agent, which is placed at the origin of the reference frame, and make the other agents to compute their locations relative to the position of the anchor. Here, instead, we prefer not fixing any anchor, which is also the approach followed in [5], [6], [8]–[10]. We propose using the *weighted centroid* of the multi–agent system as the origin of the reference frame since, as we will show, this speeds up the process.

We propose using a Jacobi Over–Relaxation (JOR) scheme, that includes a tuning parameter h. We discuss how to establish values for the parameter h to ensure that the agent positions estimated by the distributed localization method converge asymptotically and smoothly (without *ringing*) to the same values as if a centralized unit was used. Remarkably, the proposed constraints on h do not depend on any global information on the network topology or on the noise, as opposed to [5]–[7], [9]–[11].

The main contributions of this paper are: (i) The use of noisy multi-dimensional relative measurements with covariance matrices not necessarily fully diagonal. (ii) The establishment of mathematical constraints on the JOR parameter h to avoid ringing. (iii) The proof of the weighted centroid method speeding up the convergence, with respect to the anchor-based case. Compared to our previous work [12] on distributed localization, here we avoid using any anchors and we use a JOR scheme instead of the Jacobi methods used in [12]. Thus, all the analysis and results in Sections IV, V, VI, VII and the intermediary results in the appendix are novel.

The remaining of this paper is organized as follows. Section II reviews the related work. In Section III we define the notation that will be used in the paper and we define the localization problem. In Section IV we provide the definition of the weighted centroid. Section V presents the distributed algorithm executed by the agents to compute their estimated positions relative to the weighted centroid. We give the main result of the paper, that proves the convergence of the distributed weighted centroid localization method. In Section VI we discuss the selection of the parameter h to ensure not only convergence, but also to prevent the estimates from exhibiting ringing. Section VII formally compares the performance of the proposed distributed weighted centroid localization method against anchor based localization methods. In Section VIII we show numerical examples of the behavior of the method and in Section IX we provide the conclusions. The Appendix contains several auxiliary results used in the paper.

II. Related Work

A key problem in multi-agent systems is the distributed estimation of the localization of agents in a common reference from relative measurements. Agents usually start at unknown locations, and they can only take noisy measurements of the relative positions of nearby agents in their own local frames. The localization problem, which often considers stationary

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agents, consists of combining these relative measurements to build an estimate of the agents' positions in a common frame.

As agents move and their states change, the relative positions or relative states of nearby agents change accordingly [14], and it is necessary to take new measurements and update the current location estimates of the agents. This problem, known as cooperative localization [15], usually requires the existence of a good quality initial solution to the localization problem, i.e., the one associated to the agents in their initial positions. In this paper, we consider the localization problem for stationary agents, which can be used to generate these initial localization solutions to be used by higher level methods, such as cooperative localization algorithms, among others. Here, agents are *stationary*, they take a single set of relative measurements of nearby agents at the beginning, and they combine them to estimate their initial positions.

Several localization algorithms rely on range-only [16]–[19], or bearing-only [20], [21] relative measurements of positions. Alternatively, each agent can locally combine its observations and build an estimate of the relative full-position of its neighbors. In [2], several methods to combine bearing and range data to build the relative 2D and 3D full-position representations are presented. We focus on this last case, in which agents measure the relative $p-$dimensional *full-position* of their neighbors, e.g., 2D or 3D relative positions, being the measurements corrupted with noise. This localization problem can be solved by using linear optimization methods [3]–[13].

Some localization works [22], [23] consider not only the agents positions, but also their orientations, i.e., they consider full *poses*. This problem has also connections with multi-robot graph–SLAM scenarios where privacy is a concern [24]–[26]. A robot team has explored an environment, each robot has acquired a local graph-based map that includes the robot trajectory, and the goal is that robots fuse their maps in a distributed way. Robots can measure each other and exchange data when they are nearby. In order to avoid exchanging sensitive information (e.g., paths followed by each robot) that may be sniffed by other agents, robots update their estimated trajectories using the relative measurements of the pose of nearby agents. These methods [24]–[26] decouple the management of poses into two stages. In the first one, only orientations are taken into account. In the second stage, these orientations are used to represent the measurements using a common orientation, and then, position data is used to solve the full problem. In each stage, linear optimization methods as the ones considered in [3]–[13] are used.

Thus, the solution presented in this paper can be applied to scenarios that consider robot poses (orientations and positions) to e.g., solve a specific stage such as the position computation. They can also be applied to such cases provided the agents perform a synchronization [27], [28] to align their orientations, or they estimate a common orientation for their reference frames in a first stage, as proposed in [29]. In addition, the assumption on the relative measurements being expressed in a common alignment frame can be addressed by equiping the robots with sensors measuring for instance the north (compasses or magentometers) [30].

Formation control [28], [31], [32] and localization are related problems. Although some works discuss the effect of noise in the final result [28], formation algorithms usually assume noise–free measurements [31], [32].

From now on, we focus on the distributed localization from relative full-positions problem considered in this paper: a distributed localization scenario, with stationary agents that take a single set of measurements of nearby agents. The relative measurements are noisy, and they represent the $p-$dimensional full–position of the neighbors, e.g., 2D or 3D relative positions, being these measurements corrupted with noises. The problem addressed in this paper is close to the approaches in [3]–[13], which often rely on linear methods, such as [33] the Jacobi [3], [4], [12], [13], the Jacobi Over–Relaxation (JOR) [28], the Successive Over–Relaxation (SOR) [24]–[26], the Gauss–Seidel (GS), and the Richardson's method or gradient–descent strategies [5]–[7], [9]–[11], and which are also connected to distributed consensus ideas [5]–[11] and are thus resilient to delays and link failures [3], [5], [34].

In addition to the linear method used, the solutions can be classified according to several ideas that we discuss next. Several methods fix an anchor in the network [3], [4], [12], [13] and compute the localization taking this anchor as the origin of the reference frame. The intuition behind anchors is that it is well known that the relative localization problem can only be solved up to an additive constant [3]–[13]. This ambiguity can thus be removed by fixing the position of one of the agents (the anchor). The placement of the anchor influences the accuracy of the final results and it is common to analyze the estimation errors at the agents as a function of their distances to the anchor [35]. However, it is common to assume that the first agent is the anchor placed at the origin of the common reference frame and make the other agents compute their positions relative to the anchor [3], [13]. Thus, other works prefer not fixing any anchor [5], [6], [8]–[10] and compute the agents positions relative to a different coordinate frame, for instance, the centroid. We propose in this paper a method that uses the weighted centroid as the reference frame, and we do not use any anchor since, as we show, this slows down the full process.

Localization methods can also be classified depending on their requirements on additionally synchronization strategies, or on the required knowledge of global data for their adjustment. The inconvenience of SOR [24]–[26] and GS compared to JOR, Jacobi, Richardson's or gradient–descent methods is that they force a specific state update ordering [33], and thus, they require more sophisticated network synchronization policies. Thus, it is more interesting to use methods that do not impose this requirement such as the JOR, Jacobi, Richardson's or gradient–descent methods. Solutions based on the Jacobi [3], [4], [12], [13] require fixing an anchor, but a benefit of these solutions is that they do not require additional information for adjusting the algorithm. On the other hand, the Richardson's and gradient–descent methods include a parameter h which, in order to ensure convergence, must be adjusted using global information on the network topology and on the noise [5]–[7], [9]–[11], which is a limitation. Instead, some versions based on the JOR [28] or on gradient–descent

[8] establish values for the parameter h without knowing any global information, e.g., $0 < h \leq 1$, which is a strong advantage, and which is the approach we follow in this paper.

We can further classify the localization methods depending on whether they consider measurements and states that are scalar values or multi–dimensional variables. Most of the works on distributed localization assume scalar states, or fully diagonal covariance matrices [3]–[11], which is a simplification of the problem. In this paper, we consider more realistic scenarios, where the covariance matrices associated to multi–dimensional relative measurements are full (instead of fully diagonal), since each relative measurement may be the result of fusing different sensory data [2]. As far as we know, only [12], [13] have addressed multi–dimensional measurements with full covariance matrices, although in both cases they involved the use of an anchor agent and the Jacobi method in some part of the process.

Finally, all the previous works on distributed localization discuss the asymptotic convergence of the algorithms, but they do not pay attention to the way in which the solution is achieved. In some cases, the phenomena of ringing appears in discrete-time systems [36]. This makes the estimates at each time step change drastically, making it hard to use these oscillating estimates within a higher level task. Here, we propose mathematical constraints on h to avoid ringing.

To sum up, in this paper we propose a relative distributed localization that does not fix any anchor, and that considers relative measurements with full covariance matrices. To reach these goals (no anchor, full covariance matrices), we cope here with system matrices (Section V, eq. (13)) which do not satisfy the properties (row–stochastic, non–negative, primitive, a single eigenvalue equal to one) used in classical scalar consensus [28], [37] to establish the convergence for connected graphs. Thus, we adapt here in a non trivial way several properties and results that were established for classical scalar consensus. Here, agents compute their locations relative to a common reference frame that depends on the *weighted centroid* of their initial unknown estimates. We use here a Jacobi Over-Relaxation (JOR) scheme, which has milder requirements on the network synchronization policies than, e.g., SOR, but that requires a nontrivial analysis of its convergence [24], [25]. We compute the parameter h of the JOR to ensure not only convergence, but also to prevent the estimates from exhibiting ringing. Moreover, we provide conditions on h that do not depend on global information, e.g., on the network topology or on the noise.

III. PRELIMINARIES

We let $\mathbf{I}_n$ be the $n \times n$ identity matrix, $\mathbf{0}_{n_1 \times n_2}$ be a $n_1 \times n_2$ matrix with all entries equal to 0, and $\mathbf{1}_n$ and $\mathbf{0}_n$ be column vectors with its n entries equal to 1 and to 0. The dimensions are omitted when they can be easily inferred. The Kronecker product is denoted by $\otimes$.

Consider $n \in \mathbb{N}$ stationary agents. Each agent $i \in \{1, \ldots, n\}$ has a $p-$dimensional state $\mathbf{x}_i \in \mathbb{R}^p$ and it observes the states of a subset of the agents relative to its own state. This information is represented by the directed sensing graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, n\}$ are the agents, and $\mathcal{E}$ contains the m relative measurements, $|\mathcal{E}| = m$. There is an edge $e = (i, j) \in \mathcal{E}$ from i to j if node i has a relative measurement $\mathbf{z}_e \in \mathbb{R}^p$ of the state of agent j,

$$\mathbf{z}_e = \mathbf{x}_j - \mathbf{x}_i + \mathbf{v}_e, \qquad \mathbf{v}_e \sim N\left(\mathbf{0}_{p \times p}, \Sigma_{\mathbf{z}_e}\right),$$

where $\mathbf{v}_e$ is a Gaussian additive noise. We assume that if agents i, j measure each other, both measurements are combined into a single one from, e.g., by averaging the measurements expressed from i to j.

We let $\mathbf{z}, \mathbf{v} \in \mathbb{R}^{mp}$ and $\Sigma_{\mathbf{z}} \in \mathbb{R}^{mp \times mp}$ contain the information of all the measurements $\mathbf{z}_e$, $\mathbf{v}_e$, $\Sigma_{\mathbf{z}_e}$, following some order $\{1, \ldots, m\}$, i.e.,

$$\begin{aligned} \mathbf{z} &= (\mathbf{z}_1^T, \ldots, \mathbf{z}_m^T)^T, \\ \mathbf{v} &= (\mathbf{v}_1^T, \ldots, \mathbf{v}_m^T)^T, \\ \Sigma_{\mathbf{z}} &= \text{blkDiag}(\Sigma_{\mathbf{z}_1}, \ldots, \Sigma_{\mathbf{z}_m}). \end{aligned}$$

We assume that the measurements are independent since they were acquired individually by the agents, and thus the covariance matrix $\Sigma_{\mathbf{z}}$ is *block* diagonal. Note that matrix $\Sigma_{\mathbf{z}}$ would be fully diagonal for fully uncorrelated noises, and $\Sigma_{\mathbf{z}} = \mathbf{I}_{mp}$ for noise–free data.

Each agent i communicates with both its in and out neighbors in the sensing graph $\mathcal{G}$, and we assume $\mathcal{G}$ is weakly connected (i.e., its undirected version is connected). The out–edges and in–edges of agent $i \in \mathcal{V}$ are

$$\begin{aligned} \underline{\mathcal{E}}_i &= \{e \in \mathcal{E} | e = (i, j), j \in \mathcal{V}\} \\ \overline{\mathcal{E}}_i &= \{e \in \mathcal{E} | e = (j, i), j \in \mathcal{V}\}. \end{aligned}$$

The incidence matrix $\mathcal{A} \in \{0, 1, -1\}^{n \times m}$ of $\mathcal{G}$ is

$$\mathcal{A}_{i,e} = \begin{cases} 1, & \text{if } e \in \overline{\mathcal{E}}_i \\ -1, & \text{if } e \in \underline{\mathcal{E}}_i \\ 0, & \text{otherwise} \end{cases}, \forall i \in \mathcal{V}, e \in \mathcal{E}. \tag{1}$$

The estimation from relative measurements problem consists of estimating the states of the n agents using $\mathbf{z}$. Note that agents are stationary, i.e., $\mathbf{x}_i \in \mathbb{R}^p$ is constant, and agents compute their states using the measurements $\mathbf{z}$ they collected initially. Cooperative localization methods [15] instead recompute the states of the agents as they move, acquiring new measurements as agents move. Cooperative localization methods require knowing the initial states of the n agents. Thus, the estimation from relative measurements problem can be used for instance to give this initial solution.

It is well known [3]–[5], [8]–[13] that in estimation from relative measurements problems, solutions can be determined only up to an additive constant. Usually, one agent $a \in \mathcal{V}$, e.g., the first one, is taken as an anchor with fixed state, e.g., $\hat{\mathbf{x}}_a^a = \mathbf{0}$, and the states $\hat{\mathbf{x}}_i^a$ of all the other agents relative to the anchor are computed. We call such approaches *anchor-based* and we add the superscript a to their associated variables. We let $\mathcal{V}^a = \mathcal{V} \setminus \{a\}$ contain the non-anchor nodes, and $\mathcal{A}^a \in \mathbb{R}^{(n-1) \times m}$ be as $\mathcal{A}$ in (1) but without the row associated to the anchor. The Best Linear Unbiased Estimator [3] for $\mathbf{x}^a_{\mathcal{V}^a}$ is

$$\begin{aligned}\hat{\mathbf{x}}^a_{\mathcal{V}^a} =& \Sigma_{\hat{\mathbf{x}}^a_{\mathcal{V}^a}} \eta^a, \text{ where } \Sigma_{\hat{\mathbf{x}}^a_{\mathcal{V}^a}} = (\Upsilon^a)^{-1}, \\ \eta^a &= (\mathcal{A}^a \otimes \mathbf{I}_p) \Sigma_{\mathbf{z}}^{-1} \mathbf{z}, \\ \Upsilon^a &= (\mathcal{A}^a \otimes \mathbf{I}_p) \Sigma_{\mathbf{z}}^{-1} (\mathcal{A}^a \otimes \mathbf{I}_p)^T.\end{aligned} \quad (2)$$

We let $\hat{\mathbf{x}}^a_{\mathcal{V}} \in \mathbb{R}^{np}$ and $\Sigma_{\hat{\mathbf{x}}^a_{\mathcal{V}}} \in \mathbb{R}^{np \times np}$ include the anchor state,

$$\hat{\mathbf{x}}^a_{\mathcal{V}} = (\mathbf{0}, (\hat{\mathbf{x}}^a_{\mathcal{V}^a})^T)^T, \ \Sigma_{\hat{\mathbf{x}}^a_{\mathcal{V}}} = \text{blkDiag}(\mathbf{0}, \Sigma_{\hat{\mathbf{x}}^a_{\mathcal{V}^a}}). \quad (3)$$

Anchor–based methods make agents compute in a distributed way a $\hat{\mathbf{x}}^a_{\mathcal{V}^a}$ as in (2) satisfying

$$\Upsilon^a \ \hat{\mathbf{x}}^a_{\mathcal{V}^a} = \eta^a. \quad (4)$$

The anchor–free expression is similar, but using the original $\mathcal{A} \in \mathbb{R}^{n \times m}$ instead of $\mathcal{A}^a \in \mathbb{R}^{(n-1)\times m}$:

$$\begin{aligned}&\Upsilon \ \mathbf{x}^\star = \eta, \text{ where } \eta = (\mathcal{A} \otimes \mathbf{I}_p) \Sigma_{\mathbf{z}}^{-1} \mathbf{z}, \text{ and} \\ &\Upsilon = (\mathcal{A} \otimes \mathbf{I}_p) \Sigma_{\mathbf{z}}^{-1} (\mathcal{A} \otimes \mathbf{I}_p)^T, \text{ where} \\ &\Upsilon = \begin{bmatrix} \Upsilon_{11} & \dots & \Upsilon_{1n} \\ \vdots & \ddots & \vdots \\ \Upsilon_{n1} & \dots & \Upsilon_{nn} \end{bmatrix}, \text{ where, } \forall i,j \in \mathcal{V}: \\ &\Upsilon_{ii} = \sum_{e \in (\underline{\mathcal{E}}_i \cup \overline{\mathcal{E}}_i)} \Sigma_{\mathbf{z}_e}^{-1}, \\ &\Upsilon_{ij} = -\Sigma_{\mathbf{z}_e}^{-1}, \text{ if } e = (i,j) \in \mathcal{E} \text{ or } e = (j,i) \in \mathcal{E}, \\ &\Upsilon_{ij} = \mathbf{0}_{p \times p}, \text{ otherwise.}\end{aligned} \quad (5)$$

This anchor–free expression (5) is more general than (4). As discussed later in Lemma A.2, vectors $\mathbf{x}^\star$ satisfying (5) include all anchor–based vectors $\hat{\mathbf{x}}^a_{\mathcal{V}}$ as in (2)–(4), plus an additive term, which is equivalent to expressing $\hat{\mathbf{x}}^a_{\mathcal{V}}$ relative to a different coordinate frame. The goal is that the agents compute in a distributed and fast way a vector $\mathbf{x}^\star$ satisfying (5), as explained next.

IV. Weighted Centroid Reference Frame

We begin with the definition of the weighted centroid representation of the states of the agents.

***Definition** 4.1 (Weighted Centroid):* Given matrix Υ in (5), we define the *weighting matrix* $\mathbf{w} \in \mathbb{R}^{np \times p}$ and *weighted centroid matrix* $M^c_\star \in \mathbb{R}^{np \times np}$, as follows. Note that $\mathbf{w}$ is not a single vector but several.

$$\begin{aligned}\mathbf{w} &= D(\mathbf{1}_n \otimes \mathbf{I}_p) = [\Upsilon_{11}, \Upsilon_{22}, \dots, \Upsilon_{nn}]^T, \\ M^c_\star &= (\mathbf{1}_n \otimes \mathbf{I}_p)(\mathbf{w}^T(\mathbf{1}_n \otimes \mathbf{I}_p))^{-1} \mathbf{w}^T.\end{aligned} \quad (6)$$

We define the *weighted goal estimates* $\hat{\mathbf{x}}^c_{\mathcal{V}}$ as the goal centralized estimates in (2)–(3), with the positions expressed relative to their weighted centroid:

$$\hat{\mathbf{x}}^c_{\mathcal{V}} = \Pi \hat{\mathbf{x}}^a_{\mathcal{V}}, \text{ with } \quad \Pi = \mathbf{I}_{np} - M^c_\star. \quad (7)$$

■

As the following result shows, the weighted centroid representation of the goal centralized estimates is unique.

***Lemma** 4.1 (Weighted goal estimates):* Given all possible anchor agents a, a',..., and their associated goal centralized estimates in (2)–(3), $\hat{\mathbf{x}}^a_{\mathcal{V}}$, $\hat{\mathbf{x}}^{a'}_{\mathcal{V}}$,... obtained by the selection the anchor a, a',..., the *weighted goal estimates* $\hat{\mathbf{x}}^c_{\mathcal{V}}$, $\hat{\mathbf{x}}^{c'}_{\mathcal{V}}$,... obtained with (7) are the same:

$$\hat{\mathbf{x}}^c_{\mathcal{V}} = \Pi \hat{\mathbf{x}}^a_{\mathcal{V}} = \hat{\mathbf{x}}^{c'}_{\mathcal{V}} = \Pi \hat{\mathbf{x}}^{a'}_{\mathcal{V}}. \quad (8)$$

Proof: From Lemma A.2, vectors $\mathbf{x}^\star$ satisfying (5) include all anchor–based vectors $\hat{\mathbf{x}}^a_{\mathcal{V}}$ as in (2)–(4), plus an additive term, which is equivalent to expressing $\hat{\mathbf{x}}^a_{\mathcal{V}}$ relative to a different coordinate frame, so that

$$\hat{\mathbf{x}}^a_{\mathcal{V}} = \hat{\mathbf{x}}^{a'}_{\mathcal{V}} + (\mathbf{1}_n \otimes \mathbf{I}_p) \hat{\mathbf{x}}^a_{a'}. \quad (9)$$

Substituting this in (7), we get

$$\begin{aligned}\hat{\mathbf{x}}^c_{\mathcal{V}} &= \Pi \hat{\mathbf{x}}^a_{\mathcal{V}} = \Pi(\hat{\mathbf{x}}^{a'}_{\mathcal{V}} + (\mathbf{1}_n \otimes \mathbf{I}_p)\hat{\mathbf{x}}^a_{a'}), \\ &= \Pi \hat{\mathbf{x}}^{a'}_{\mathcal{V}} + (\mathbf{I}_{np} - M^c_\star)(\mathbf{1}_n \otimes \mathbf{I}_p)\hat{\mathbf{x}}^a_{a'}.\end{aligned} \quad (10)$$

Now we use (6), that gives $M^c_\star(\mathbf{1}_n \otimes \mathbf{I}_p) = (\mathbf{1}_n \otimes \mathbf{I}_p)$, which combined with (10) gives

$$\hat{\mathbf{x}}^c_{\mathcal{V}} = \Pi \hat{\mathbf{x}}^{a'}_{\mathcal{V}} + \mathbf{0} = \hat{\mathbf{x}}^{c'}_{\mathcal{V}}, \quad (11)$$

concluding the proof. ■

Figure 3 in Section VIII shows some examples of *weighted goal estimates* $\hat{\mathbf{x}}^c_{\mathcal{V}}$ (Def. 4.1) and of goal centralized estimates in (2)–(3) (red circles). In a centralized setup, the localization problem would be solved by compiling the relative measurements, and computing the goal centralized estimates ((2)–(3)), or the *weighted goal estimates* $\hat{\mathbf{x}}^c_{\mathcal{V}}$, using (7).

In a distributed scenario, as the one considered in this paper, each agent $i \in \mathcal{V}$ only knows the relative measurements $\mathbf{z}_e$, $\Sigma_{\mathbf{z}_e}$ associated to its in and out neighbors, $e \in (\underline{\mathcal{E}}_i \cup \overline{\mathcal{E}}_i)$, and the estimate of its own position. In addition, each agent i can only exchange its estimated position with its neighbors. The goal is that agents obtain an estimated position that converges asymptotically to the same goal centralized estimates or *weighted goal estimates* as the ones discussed so far.

In the remaining of the paper, we propose a distributed algorithm for computing the goal centralized estimates in (2)–(3), with the positions expressed relative to their weighted centroid (Def. 4.1), and we discuss its properties in Sections V, VI and VII.

V. Weighted Centroid Localization

In this section, we present a distributed iterative method to let each agent i estimate its associated entries within $\mathbf{x}^\star$ (5). We use a Jacobi Over–Relaxation scheme and discuss the selection of its parameter h. From all the possible vectors $\mathbf{x}^\star$, we prove that the agents' estimates converge to an expression that depends on the weighted centroid of the initial estimates.

Matrix Υ in (5) is decomposed into a matrix D with the $p \times p$ *blocks* in the main diagonal of Υ, and a matrix N with the remaining elements,

$$\begin{aligned}&\Upsilon = D - N, \text{ with } D = \text{blkDiag}(\Upsilon_{11}, \Upsilon_{22}, \dots, \Upsilon_{nn}), \\ &N = \begin{bmatrix} N_{11} & \dots & N_{1n} \\ \vdots & \ddots & \vdots \\ N_{n1} & \dots & N_{nn} \end{bmatrix}, \text{ where, } \forall i,j \in \mathcal{V}: \\ &N_{ij} = \begin{cases} \Sigma_{\mathbf{z}_e}^{-1} & \text{if } e = (i,j) \in \mathcal{E} \text{ or } e = (j,i) \in \mathcal{E}, \\ 0 & \text{otherwise.} \end{cases}\end{aligned} \quad (12)$$

From this, the JOR system equations for multi–dimensional noisy centroid–based localization used to compute $\mathbf{x}^\star$ (5) in a distributed way are:

$$\mathbf{x}(k+1) = M_{\text{JOR}}\ \mathbf{x}(k) + hD^{-1}\eta, \text{ with}$$
$$M_{\text{JOR}} = (1-h)\mathbf{I}_{np} + hD^{-1}N. \tag{13}$$

Matrix M_{JOR} remains constant during the iterations. The method is distributed and each agent i runs (13) to estimate its entries $x_i(k) \in \mathbb{R}^p$ within $\mathbf{x}(k)$ using local information and from its neighbors:

$$x_i(k+1) = (1-h)x_i(k) + h\Upsilon_{ii}^{-1}\Bigg(\sum_{e\in\overline{\mathcal{E}}_i} \Sigma_{\mathbf{z}_e}^{-1}\mathbf{z}_e - \sum_{e\in\underline{\mathcal{E}}_i} \Sigma_{\mathbf{z}_e}^{-1}\mathbf{z}_e + \sum_{e\in(\underline{\mathcal{E}}_i\cup\overline{\mathcal{E}}_i)} \Sigma_{\mathbf{z}_e}^{-1} x_j(k)\Bigg). \tag{14}$$

The multi–dimensional noisy centroid–based distributed localization method (13) has connections with scalar weighted consensus problems [37], [28]. However, M_{JOR} in (13) does not satisfy the properties (row-stochastic, non-negative, primitive, a single eigenvalue equal to one) used in classical scalar consensus [37], [28] to establish the convergence for connected graphs. We extend several properties from scalar weighted consensus and we show that algorithm (13) makes the agents states converge to an expression that depends on weighted-centroid (Def. 4.1).

Now we present the main result in this section. The proof relies on several intermediary results provided in the Appendix.

***Theorem** 5.1 (JOR Weighted Centroid iterations):* If agents execute the multi–dimensional noisy centroid–based localization method (12)–(13) under a weakly connected graph $\mathcal{G}$, their estimates asymptotically converge to the optimal estimates relative to the centroid $\hat{\mathbf{x}}^c_{\mathcal{V}}$ (Definition 4.1), plus the weighted centroid of the initial states,

$$\lim_{k\to\infty} \mathbf{x}(k) = \hat{\mathbf{x}}^c_{\mathcal{V}} + M^c_\star \mathbf{x}(0). \tag{15}$$

Proof: We let $\mathbf{e}(k)$ be he error containing the difference between the agents estimates $\mathbf{x}(k)$ and the solution $\mathbf{x}^\star$ as in (52).

$$\mathbf{e}(k) = \mathbf{x}(k) - \mathbf{x}^\star, \qquad \mathbf{x}(k) = \mathbf{e}(k) + \mathbf{x}^\star, \tag{16}$$

Then, (12)–(13) becomes

$$\mathbf{e}(k+1) = M_{\text{JOR}}\mathbf{e}(k),\ \mathbf{e}(k) = (M_{\text{JOR}})^k\mathbf{e}(0). \tag{17}$$

Using Proposition A.2,

$$\lim_{k\to\infty} \mathbf{e}(k) = \lim_{k\to\infty} (M_{\text{JOR}})^k\mathbf{e}(0) = M^c_\star\mathbf{e}(0). \tag{18}$$

Reversing (16) we get

$$\lim_{k\to\infty} \mathbf{x}(k) = M^c_\star\mathbf{x}(0) + \Pi\mathbf{x}^\star, \tag{19}$$

with Π as in Definition 4.1 and $\mathbf{x}^\star$ as in (52). Using Lemma A.2, $\mathbf{x}^\star = \hat{\mathbf{x}}^a_{\mathcal{V}} + (\mathbf{1}\otimes\mathbf{I}_p)\mathbf{x}^\star_a$,

$$\lim_{k\to\infty} \mathbf{x}(k) = M^c_\star\mathbf{x}(0) + \Pi\hat{\mathbf{x}}^a_{\mathcal{V}} + \Pi(\mathbf{1}\otimes\mathbf{I}_p)\mathbf{x}^\star_a. \tag{20}$$

Since $\Pi(\mathbf{1}\otimes\mathbf{I}_p) = \mathbf{0}_{p\times p}$, and $\hat{\mathbf{x}}^c_{\mathcal{V}} = \Pi\hat{\mathbf{x}}^a_{\mathcal{V}}$ (Definition 4.1), we get (15). ■

VI. Conditions on h for Avoiding Ringing

The convergence of the JOR localization method is asymptotic (Theorem 5.1). Not only convergence is a necessary requirement in practice, but it is also convenient that localization estimates evolve smoothly without oscillating behaviors. This would allow using stable and predictable data in higher level methods. It is well known that, when the discrete poles of the system (i.e., the eigenvalues of the system matrix) are real and negative, the estimates have an oscillatory behavior (*ringing*) [36]. An example of a convergent system with this behavior can be seen in Fig. 1 (left). As far as we know, these issues associated to the eigenvalues are not usually discussed in the context of network localization, beyond asymptotic convergence. This can be alleviated by forcing the system eigenvalues to be real, strictly positive, and smaller than 1, as we propose next.

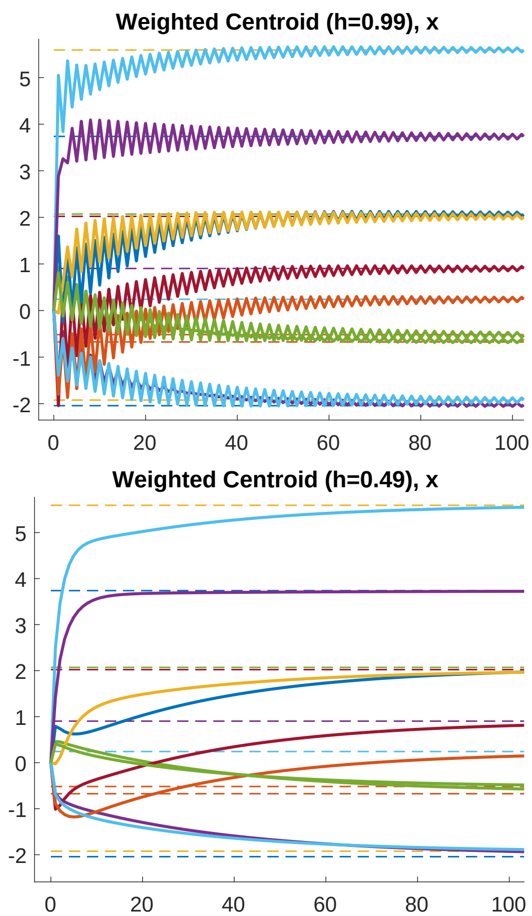

Fig. 1. Example with 10 agents in a chain graph. Evolution along iterations of the estimated $x-$coordinate relative to the weighted centroid of the team. **Top**: The ringing oscillatory behavior can be observed for $h = 0.99$. At each step, the estimates change their values sharply. **Bottom**: The ringing oscillatory behavior is removed with $h = 0.49$. The estimates converge now smoothly.

***Lemma** 6.1:* The system matrix M_{JOR} in (13) with $0 < h < 1/2$ and $\mathcal{G}$ weakly connected, has p eigenvalues equal to 1, and all its remaining eigenvalues are real, strictly positive, and smaller than 1.

Proof: The result follows from the proof of Proposition A.1, using $h < 1/2$ in (30) to get $0 < \lambda_{i,r}(M_{\text{JOR}}) \leq 1$,

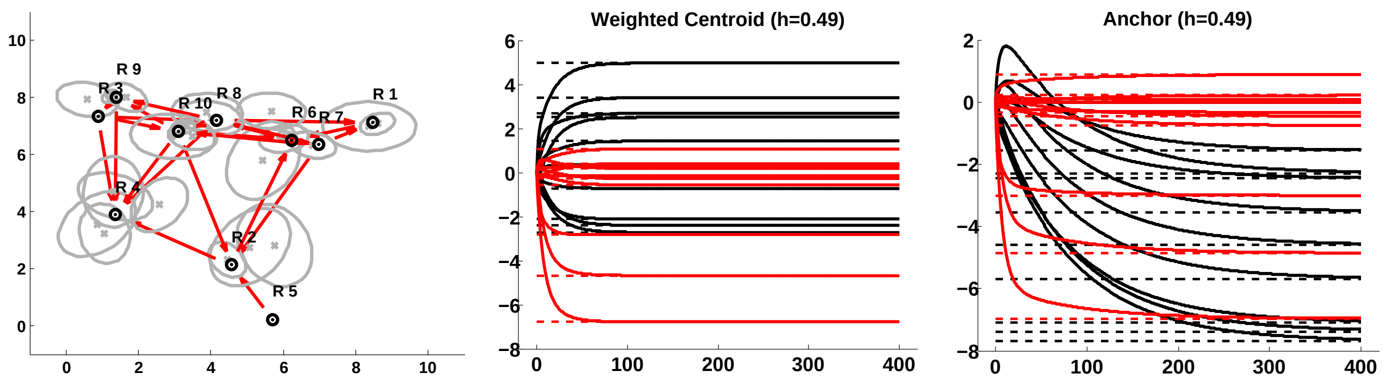

Fig. 2. **Left**: Initial scenario. **Center and Right**: Evolution of the estimated $x-$ (in black) and $y-$ coordinates (in red) of the agents positions along 400 iterations ($x-$axis) when agents run the weighted centroid (Center) and anchor–based (Right) localization methods.

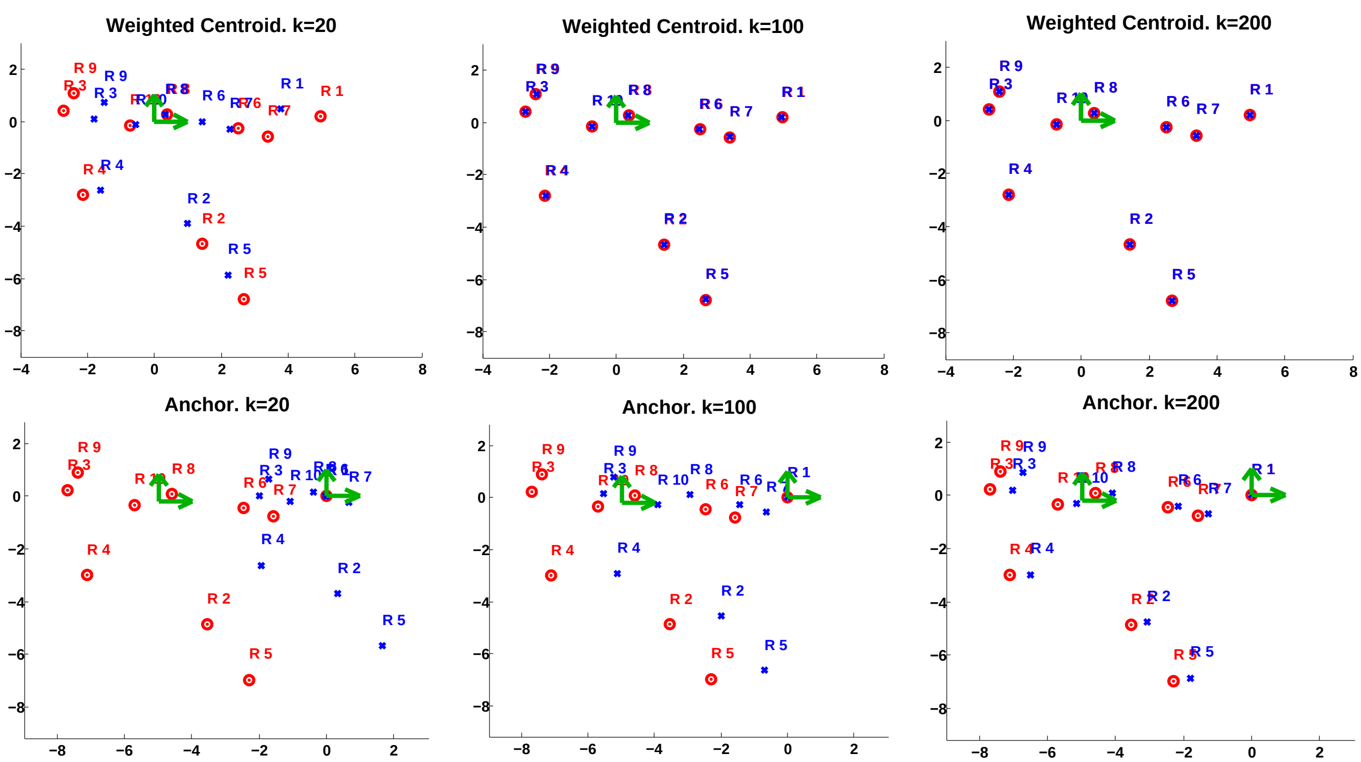

Fig. 3. Estimated agents positions (blue crosses) after $k = 20$ (Left), $k = 100$ (Center) and $k = 200$ iterations (Right) of the weighted centroid (Top) and anchor–based (Bottom) localization methods, for the initial scenario in Fig. 2. The red circles represent the goal centralized estimates in (2)–(3), with the positions expressed relative to their weighted centroid as in (7) (Top), and relative to the anchor agent $R1$ (Bottom). The green axis represent the frame used as the origin. In the Right figures, we also display with a green axis the weighted centroid, which is located at position $(-4.97, -0.19)$.

instead of (31). ∎

Observe in Fig. 1 (right) this ringing behavior is removed with h close to $1/2$. Note that the selection of $0 < h < 1/2$ does not require knowing global information of the graph. Additional information could be used to speed up the method. However, we prefer for generality to use Lemma 6.1 and avoid requirements on global data.

VII. Anchor and Centroid–based Strategies

Both anchor-based and weighted centroid localization methods are convergent for connected graphs. However, we demonstrate next that our proposed weighted centroid approach converges faster than fixing an anchor node.

Lemma *7.1:* Let $0 < h < 1/2$ and $\mathcal{G}$ be weakly connected. Let $\rho(M)$ be the spectral radius of a matrix M. The convergence rate of the anchor-based $\rho(M^a_{\mathrm{JOR}})$ and of the weighted centroid algorithms $\rho_{ess}(M_{\mathrm{JOR}}) = \rho\left(M_{\mathrm{JOR}} - M^c_{\star}\right)$, with M^a_{JOR}, M_{JOR}, $M^c_{\star}$ as in (32), (13), (6), satisfy

$$\rho_{ess}(M_{\mathrm{JOR}}) \leq \rho(M^a_{\mathrm{JOR}}). \tag{21}$$

Proof: From Lemma 6.1, and Prop. A.1 and A.2, the eigenvalues of M_{JOR} are positive, and

$$\rho_{ess}(M_{\mathrm{JOR}}) = \rho\left(M_{\mathrm{JOR}} - M^c_{\star}\right) = \lambda_{n-1,p}, \tag{22}$$

with $\lambda_{n-1,p}$ as in (40)–(42). From (34) with $0 < h < 1/2$, the eigenvalues of M^a_{JOR} are positive,

$$\text{and} \qquad \rho(M^a_{\mathrm{JOR}}) = \lambda^a_{n-1,p}, \tag{23}$$

with $\lambda^a_{n-1,p}$ as in (40). And thus, (21) can be concluded from (40), (22) and (23). ■

VIII. Simulations

Fig. 2 illustrates the difference of performance discussed in Lemma 7.1. There are 10 agents placed randomly in a 2D region of 10 × 10 meters. Fig. 2 left: Each agent i gets noisy measurements (crosses and ellipses) of the relative position of its out-neighbors j (arrows). The noise covariance matrix $\Sigma_{\mathbf{z}_e}$ depends on the relative measurements (ρ_{ij}, α_{ij} in polar coordinates) between agents i, j, with $e = (i, j)$ as follows:

$$\Sigma_{\mathbf{z}_e} = R^T_{ij} \mathrm{diag}(\sigma_1^2, \sigma_2^2) R_{ij},$$

where $\sigma_1 = 0.15\rho_{ij}$, $\sigma_2 = 0.1\rho_{ij}$, are the standard deviations in the parallel and perpendicular directions of the arrow, and

$$R_{ij} = \begin{bmatrix} \cos(\alpha_{ij}) & \sin(\alpha_{ij}) \\ -\sin(\alpha_{ij}) & \cos(\alpha_{ij}) \end{bmatrix}.$$

We plot the uncertainty ellipses with the 95% of noisy measurements ($\pm 2\sigma_1, \pm 2\sigma_2$), centered around the noisy measurement. Circles represent instead the true initial robot positions. Note that the covariance matrices are not fully diagonal. For instance, for the measurement between agents $i = 5$ and $j = 2$, we have

$$\Sigma_{\mathbf{z}_e} = \begin{bmatrix} 0.0659 & -0.0273 \\ -0.0273 & 0.0954 \end{bmatrix}.$$

Fig. 2 center: Agents run the distributed weighted centroid localization method with $h = 0.49$ in order to avoid ringing (Lemma 6.1). The estimates converge fast and without exhibiting the ringing oscillatory behavior. Fig. 2 right: The estimated agent positions when fixing node 1 as an anchor at the $(0, 0)$ position, with clearly slower convergence than using weighted centroid approach. Fig. 3 shows the evolution of the estimated agents positions for the methods in Fig. 2. After few steps, the estimated positions (blue crosses) obtained with the distributed weighted centroid localization (Fig. 3, top) are very close to the goal centralized estimates (red circles). Fig. 3 bottom: after $k = 200$ iterations, the estimated agent positions (blue circles) when fixing node 1 as an anchor at the $(0, 0)$ position, have not converged yet to the goal centralized estimates (red circles).

IX. Conclusions

We presented a distributed method that allows a set of agents to estimate their positions, expressed relative to the weighted centroid, using noisy relative measurements of the positions of their neighbors. Our method is based on the Jacobi Over–Relaxation (JOR), and it includes a tuning parameter h. A novel feature of our JOR based distributed method is that relative state measurements can be multidimensional with covariance matrices not fully diagonal. Thus, our approach is more flexible and practical, since very often the measurements are fused from several sensors. We also defined the conditions that guarantee smoother performance of the estimates. This is a desirable feature if, for example, the estimates are used in a higher level task sensitive to oscillating signals. Additionally, we proved our weighted-centroid method converges faster than its counterpart based on a fixed anchor.

***Proposition** A.1 (Eigenvalues of JOR):* For weakly connected graphs $\mathcal{G}$, with $0 < h < 1$, the JOR system matrix M_{JOR} in (13) has p eigenvalues equal to 1, and the remaining eigenvalues are real and have modulus strictly smaller than one.

Proof: From (12)–(13), M_{JOR} equals

$$M_{\mathrm{JOR}} = \mathbf{I}_{np} - hD^{-1}\Upsilon, \tag{24}$$

with Υ as in (5). From (5) we have

$$\begin{aligned} \Upsilon \cdot (\mathbf{1}_n \otimes \mathbf{I}_p) &= \mathbf{0}_{np\times p}, \text{ and thus} \\ M_{\mathrm{JOR}} \cdot (\mathbf{1}_n \otimes \mathbf{I}_p) &= \mathbf{1}_n \otimes \mathbf{I}_p, \end{aligned} \tag{25}$$

then M_{JOR} has at least p eigenvalues equal to one.

Next, we study if the modulus of the remaining eigenvalues is strictly less than 1. We pay attention to the centroid–based M_{JOR} system matrix in (13). We first consider its associated Jacobi matrix $D^{-1}N$. According to [38, Notation 2.3], which studies the convergence of block iterative methods, matrix Υ in (5) is of type Z^p_n and also of type $\hat{Z}^p_n$ for a connected graph. Matrix Υ also satisfies [38, Condition (3.2), Proposition 3.1]:

$$\Upsilon_{ii} + \frac{1}{2} \sum_{j\neq i, j=1}^{n} (\Upsilon_{ij} + \Upsilon_{ji})^T \geq 0, \forall i \in \mathcal{V}. \tag{26}$$

This comes from the fact that, from (5), we have $\Upsilon_{ji} = \Upsilon^T_{ij}$ and $\Upsilon_{ij} + \Upsilon^T_{ji} = 2\Upsilon_{ij}$, so that (26) gives

$$\Upsilon_{ii} + \sum_{j\neq i, j=1}^{n} \Upsilon_{ij} = \mathbf{0}_{p\times p} \geq 0, \forall i \in \mathcal{V}. \tag{27}$$

Therefore, according to [38, Proposition 4.6],

$$\rho(D^{-1}N) \leq 1, \tag{28}$$

with Υ, D, and N as in (12).

The eigenvalues of the centroid-based JOR system matrix (13) and of the centroid–based Jacobi matrix $D^{-1}N$ are related $\forall i \in \mathcal{V}, r = 1, \ldots, p$ by

$$\lambda_{i,r}(M_{\mathrm{JOR}}) = 1 - h + h\lambda_{i,r}(D^{-1}N), \tag{29}$$

Since $h > 0$, and using (28), we get:

$$\begin{aligned} -h \leq h\lambda_{i,r}(D^{-1}N) &\leq h, \\ 1 - 2h \leq \lambda_{i,r}(M_{\mathrm{JOR}}) &\leq 1 - h + h = 1. \end{aligned} \tag{30}$$

Now, using $h < 1$, we get that, $\forall i \in \mathcal{V}, r = 1, \ldots, p$,

$$-1 < \lambda_{i,r}(M_{\mathrm{JOR}}) \leq 1. \tag{31}$$

From (31), we discard the existence of eigenvalues equal to -1. From (25), M_{JOR} has *at least* p eigenvalues equal to 1. Next, we prove that *exactly* p eigenvalues are equal to 1, and all the other eigenvalues are strictly smaller than 1.

The proof uses results from [38], [12], that refer to anchor–based methods, together with relations between eigenvalues of anchor–based and centroid–based systems. We let $M^a_{\rm JOR}$, Υ^a, D^a, N^a be as $M_{\rm JOR}$, Υ, D, N in (12)–(13) but removing the rows and columns associated to the anchor agent. The anchor–based JOR system matrix $M^a_{\rm JOR}$ is related to the anchor–based Jacobi matrix $(D^a)^{-1}N^a$ as follows:

$$\begin{aligned} M^a_{\rm JOR} &= (1-h)\mathbf{I}_{(n-1)p} + h(D^a)^{-1}N^a, \\ \lambda^a_{i,r}(M^a_{\rm JOR}) &= 1-h+h\lambda^a_{i,r}((D^a)^{-1}N^a), \end{aligned} \tag{32}$$

$\forall i = 1,\ldots,(n-1), r = 1,\ldots,p$. The convergence of the anchor-based Jacobi method was proved in [12, Theorem 2], using results from [38], and concluding that, for connected graphs, $\rho((D^a)^{-1}N^a) < 1$. Besides, since $h > 0$,

$$-h < h\lambda^a_{i,r}((D^a)^{-1}N^a) < h. \tag{33}$$

Following a similar reasoning as in (28)–(31), we conclude that, for connected graphs,

$$-1 < 1-2h < \lambda^a_{i,r}(M^a_{\rm JOR}) < 1, \tag{34}$$

$\forall i = 1,\ldots,(n-1), r = 1,\ldots,p.$

Now, we use a simplified notation for the eigenvalues of matrices $M^a_{\rm JOR}$ and $M_{\rm JOR}$. We define $\forall r = 1,\ldots,p$,

$$\begin{aligned} \lambda_{i,r} &= \lambda_{i,r}(M_{\rm JOR}), \text{ with } i = 1,\ldots,n, \text{ and} \\ \lambda^a_{i,r} &= \lambda^a_{i,r}(M^a_{\rm JOR}), \text{ with } i = 1,\ldots,n-1. \end{aligned} \tag{35}$$

We consider these eigenvalues are sorted as follows:

$$\begin{aligned} &\lambda_{1,1} \le \ldots \lambda_{1,p} \le \cdots \le \lambda_{n,1} \le \cdots \le \lambda_{n,p}, \\ &\lambda^a_{1,1} \le \ldots \lambda^a_{1,p} \le \cdots \le \lambda^a_{n-1,1} \le \cdots \le \lambda^a_{n-1,p}. \end{aligned} \tag{36}$$

From (31), (34), for connected graphs:

$$\begin{aligned} &-1 < \lambda_{1,1} \le \cdots \le \cdots \le \lambda_{n-1,p} \le 1, \text{ and} \\ &\lambda_{n,1} = \cdots = \lambda_{n,p} = 1, \\ &-1 < \lambda^a_{1,1} \le \ldots \lambda^a_{n-1,p} < 1. \end{aligned} \tag{37}$$

Now we relate the eigenvalues $\lambda_{n-1,p}$ and $\lambda^a_{n-1,p}$ to prove that $\lambda_{n-1,p} < 1$ for connected graphs. We apply results of symmetric matrices. Although matrix $M_{\rm JOR}$ in (13) is not symmetric, according to [39, Definition 1.3.1], $M_{\rm JOR}$ is *similar* to the following symmetric matrix:

$$\begin{aligned} M_{\rm JOR} &\backsim D^{1/2}M_{\rm JOR}D^{-1/2} \\ &= (1-h)\mathbf{I}_{np} + hD^{-1/2}ND^{-1/2}. \end{aligned} \tag{38}$$

$M^a_{\rm JOR}$ in (32) is also similar to a symmetric matrix,

$$M^a_{\rm JOR} \backsim (1-h)\mathbf{I}_{(n-1)p} + h(D^a)^{-1/2}N^a(D^a)^{-1/2}.$$

Thus, according to [39, Corollary 1.3.4], $M_{\rm JOR}$ and $M^a_{\rm JOR}$ have the same eigenvalues as their symmetric counterparts. Thus, the eigenvalues are real. We can now use [39, Theorem 4.3.15], which applies to Hermitian matrices, using our non-symmetric matrices $M_{\rm JOR}$ and $M^a_{\rm JOR}$, which states

$$\lambda_{i,r} \le \lambda^a_{i,r} \le \lambda_{i+1,r}, \tag{39}$$

$\forall i = 1,\ldots,(n-1)$, and $r = 1,\ldots,p$. In particular, for $i = n-1$ and $r = p$,

$$\lambda_{n-1,p} \le \lambda^a_{n-1,p} \le \lambda_{n,p}. \tag{40}$$

Since, for connected graphs $\lambda^a_{n-1,p} < 1$ (37), then

$$\lambda_{n-1,p} < 1, \tag{41}$$

which, together with (31) gives

$$-1 < \lambda_{1,1} \le \cdots \le \lambda_{n-1,p} < 1, \tag{42}$$

i.e., the remaining eigenvalues of $M_{\rm JOR}$ have modulus strictly smaller than 1, concluding the proof. ■

***Lemma** A.1:* The weighting matrix in Definition 4.1 are left eigenvectors of the $M_{\rm JOR}$ system matrix (13), associated to the eigenvalue 1,

$$\mathbf{w}^T M_{\rm JOR} = \mathbf{w}^T. \tag{43}$$

Proof: With $M_{\rm JOR}$ (13) as in (24), we have

$$\begin{aligned} \mathbf{w}^T M_{\rm JOR} &= \mathbf{w}^T - h\mathbf{w}^T D^{-1}\Upsilon \\ &= \mathbf{w}^T - h(\mathbf{1}_n \otimes \mathbf{I}_p)^T\Upsilon. \end{aligned} \tag{44}$$

Since Υ is symmetric, the term $(\mathbf{1}_n \otimes \mathbf{I}_p)^T\,\Upsilon$ vanishes as in (25), and we finally get (43). ■

Now, we focus on the *system matrix*, and show the convergence of its powers. The following proposition is used in the proof of Theorem 5.1.

***Proposition** A.2 (Convergence of the powers of the JOR matrix):* Let $M_{\rm JOR}$ be the system matrix associated to the JOR iterations as in (13), associated to a weakly connected graph $\mathcal{G}$. Let $M^c_\star$ be as in Definition 4.1, and let $0 < h < 1$. Then,

$$\lim_{k\to\infty}(M_{\rm JOR})^k = M^c_\star. \tag{45}$$

Proof: As discussed after (37), the system matrix $M_{\rm JOR}$ is *similar* ([39, Definition 1.3.1]) to a matrix $D^{1/2}M_{\rm JOR}D^{-1/2}$ which is symmetric, and thus ([39, Theorem 4.1.5]) diagonalizable. Thus, ([39, Observation 1.3.2], [39, Theorem 1.3.7]), the system matrix $M_{\rm JOR}$ in (13) is diagonalizable.

Let $\mathbf{w}$ be the as in Def. 4.1. We let V_L and V_R be respectively left and right eigenvectors of $M_{\rm JOR}$,

$$\begin{aligned} V_L^T &= \begin{bmatrix} \mathbf{w}^T \\ V^T_{L_{n-1,p}} \\ \vdots \\ V^T_{L_{1,1}} \end{bmatrix} = \begin{bmatrix} \mathbf{w}^T \\ \bar{V}^T_L \end{bmatrix}, \\ V_R &= \begin{bmatrix} (\mathbf{1}_n \otimes \mathbf{I}_p) & V_{R_{n-1,p}} & \ldots & V_{R_{1,1}} \end{bmatrix} \\ &= \begin{bmatrix} (\mathbf{1}_n \otimes \mathbf{I}_p) & \bar{V}_R \end{bmatrix}. \end{aligned} \tag{46}$$

From Lemma A.1 and eq. (25), $\mathbf{w}$ and $(\mathbf{1}_n \otimes \mathbf{I}_p)$ are left and right eigenvectors of $M_{\rm JOR}$ associated to the eigenvalue 1. $V_{L_{1,1}}, V_{L_{1,p}}\ldots, V_{L_{n-1,p}}$ have been chosen to be orthogonal to $\mathbf{w}$, and $V_{R_{1,1}}, V_{R_{1,p}}\ldots, V_{R_{n-1,p}}$ are orthogonal to $(\mathbf{1}_n \otimes \mathbf{I}_p)$. Note that $[V_{n,1},\ldots,V_{n,p}] = \mathbf{w}$ are not necessarily orthogonal among them.

Let λ_M be the diagonal matrix with the eigenvalues of $M_{\rm JOR}$. Then,

$$\begin{aligned} &V_L^T M_{\rm JOR} = \lambda_M V_L^T, \quad \text{and} \quad M_{\rm JOR} V_R = V_R \lambda_M, \\ &(M_{\rm JOR})^k = V_R \lambda_M^k V_R^{-1} = (V_L^T)^{-1} \lambda_M^k V_L^T, \\ &(M_{\rm JOR})^{2k} = V_R \lambda_M^k (V_L^T V_R)^{-1} \lambda_M^k V_L^T. \end{aligned} \tag{47}$$

We will focus first on the term $(V_L^T V_R)^{-1}$. From [39, Theorem 1.4.7],

$$V_{L_{i,r}}^T V_{R_{j,s}} = 0, \text{ for all } \lambda_{i,r}(M_{\rm JOR}) \neq \lambda_{j,s}(M_{\rm JOR}),$$

and thus

$$V_L^T V_R = \begin{bmatrix} \mathbf{w}^T(\mathbf{1}_n \otimes \mathbf{I}_p) & \mathbf{0} \\ \mathbf{0} & \bar{V}_L^T \bar{V}_R \end{bmatrix}, \text{ and} \tag{48}$$

$$(V_L^T V_R)^{-1} = \begin{bmatrix} (\mathbf{w}^T(\mathbf{1}_n \otimes \mathbf{I}_p))^{-1} & \mathbf{0} \\ \mathbf{0} & (\bar{V}_L^T \bar{V}_R)^{-1} \end{bmatrix}.$$

Now we pay attention to λ_M. From Proposition A.1, for a connected graph,

$$\begin{aligned} &\lambda_{n,1}(M_{\rm JOR}) = \ldots, \lambda_{n,p}(M_{\rm JOR}) = 1, \\ &-1 < \lambda_{1,1}(M_{\rm JOR}) \leq \cdots \leq \lambda_{n-1,p}(M_{\rm JOR}) < 1, \end{aligned} \tag{49}$$

Thus, $\lambda_M = \mathrm{diag}(\lambda_{n,p}(M_{\rm JOR}), \ldots, \lambda_{1,1}(M_{\rm JOR}))$, satisfies:

$$\lim_{k\to\infty} \lambda_M^k = \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{50}$$

Therefore, using (48),

$$\begin{aligned} \lim_{k\to\infty} (M_{\rm JOR})^k &= \lim_{k\to\infty} (M_{\rm JOR})^{2k} \\ &= \lim_{k\to\infty} V_R \lambda_M^k (V_L^T V_R)^{-1} \lambda_M^k V_L^T \\ &= V_R \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} (V_L^T V_R)^{-1} \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} V_L^T \\ &= [(\mathbf{1}_n \otimes \mathbf{I}_p), \mathbf{0}] (V_L^T V_R)^{-1} \begin{bmatrix} \mathbf{w}^T \\ \mathbf{0} \end{bmatrix} \\ &= (\mathbf{1}_n \otimes \mathbf{I}_p)(\mathbf{w}^T(\mathbf{1}_n \otimes \mathbf{I}_p))^{-1} \mathbf{w}^T, \end{aligned} \tag{51}$$

as $M_\star^c$ in Definition 4.1, giving (45). ■

Next, we show that all the solutions of (12)–(13) are the goal centralized estimates $\hat{\mathbf{x}}_{\mathcal{V}}^a$ (2)–(3), up to an additive term, which is equivalent to expressing $\hat{\mathbf{x}}_{\mathcal{V}}^a$ relative to a different coordinate frame. This result is used in the proof of Theorem 5.1.

***Lemma** A.2 (Solutions of the JOR equations):* The vectors $\mathbf{x}^\star$ satisfying (12)–(13), i.e.,

$$\mathbf{x}^\star = M_{\rm JOR}\mathbf{x}^\star + hD^1\eta, \text{ are given by} \tag{52}$$

$$\mathbf{x}^\star = \hat{\mathbf{x}}_{\mathcal{V}}^a + (\mathbf{1}_n \otimes \mathbf{I}_p)\mathbf{x}_a^\star \tag{53}$$

for all possible $\mathbf{x}_a^\star \in \mathbb{R}^p$, with $\hat{\mathbf{x}}_{\mathcal{V}}^a$ as in (3).

Proof: Since all $\mathbf{x}^\star$ satisfying (52) also satisfy (5), we focus on (5). The relation between the incidence matrices, with and without the anchor row is

$$\mathcal{A} = \begin{bmatrix} -\mathbf{1}_{n-1}^T \\ \mathbf{I}_{n-1} \end{bmatrix} \mathcal{A}^a, \qquad \mathcal{A}^a = \begin{bmatrix} \mathbf{0}_{n-1} & \mathbf{I}_{n-1} \end{bmatrix} \mathcal{A}.$$

$\Upsilon \mathbf{x}^\star = \eta$ (5) can be expressed, distinguishing between the elements of $\mathbf{x}^\star = [\mathbf{x}_a^\star, (\mathbf{x}_{\mathcal{V}^a}^\star)^T]^T$,

$$(\mathcal{A} \otimes \mathbf{I}_p)\Sigma_{\mathbf{z}}^{-1}(\mathcal{A}^a \otimes \mathbf{I}_p)^T(\mathbf{x}_{\mathcal{V}^a}^\star - (\mathbf{1}_{n-1} \otimes \mathbf{I}_p)\mathbf{x}_a^\star) = \eta,$$

giving two rows of equations. The first row is redundant (it equals the second row, multiplied by $-\mathbf{1}_{n-1}^T \otimes \mathbf{I}_p$). Using Υ^a, η^a (2), the second row is:

$$\Upsilon^a(\mathbf{x}_{\mathcal{V}^a}^\star - (\mathbf{1}_{n-1} \otimes \mathbf{I}_p)\mathbf{x}_a^\star) = \eta^a. \tag{54}$$

Since $\Upsilon^a \hat{\mathbf{x}}_{\mathcal{V}^a}^a = \eta^a$ (4), then all $\mathbf{x}^\star$ satisfy

$$\hat{\mathbf{x}}_{\mathcal{V}^a}^a = \mathbf{x}_{\mathcal{V}^a}^\star - (\mathbf{1}_{n-1} \otimes \mathbf{I}_p)\mathbf{x}_a^\star, \tag{55}$$

and $\hat{\mathbf{x}}_{\mathcal{V}}^a = \mathbf{x}^\star - (\mathbf{1}_n \otimes \mathbf{I}_p)\mathbf{x}_a^\star$ as in (53). ■

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