Effective dimension: from computation to fractal geometry and number theory

Elvira Mayordomo

Universidad de Zaragoza, Iowa State University

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  - Semicomputable dimension, where the point to set principle gives a way back to classical fractal geometry
  - Finite-state dimension and its connection with Borel normality and number theory
  - More speculative connections

Resource-bounded fractal dimension

- (Lutz 2003) Used in Computational Complexity, every class has a dimension (E is linear exponential time)

$$\begin{array}{rl} \dim_{\mathrm{p}}(X) & \dim(X|E) \\ \mathrm{Dim}_{\mathrm{p}}(X) & \mathrm{Dim}(X|E) \\ \dim_{\mathrm{pspace}}(X) & \dim(X|ESPACE) \\ \mathrm{Dim}_{\mathrm{pspace}}(X) & \mathrm{Dim}(X|ESPACE) \end{array}$$

- dim and Dim are dual concepts that correspond to effectivization of Hausdorff and packing dimension, respectively
- Gambling definition (for space bounds it can be information theory based)

Resource-bounded fractal dimension results

Theorem (Harkins, Hitchcock 2011)

$\dim(\mathrm{P_{btt}}(\mathrm{P_{ctt}}(\mathrm{DENSE}^c)) \mid \mathrm{E}) = 0$

Theorem (Harkins, Hitchcock 2011)

$\mathrm{E} \not\subseteq \mathrm{P_{btt}}(\mathrm{P_{ctt}}(\mathrm{DENSE}^c))$

Theorem (Fortnow et al 2011)

$\mathrm{Dim}(\mathrm{E} \mid \mathrm{ESPACE}) = 0$ *or 1*

Constructive dimension

- Semicomputable gambling or Kolmogorov complexity [Lutz 2003b, Mayordomo 2002]
- $\mathrm{K}(w) = \min\{|y| \mid U(y) = w\}$ (U is a fixed universal Turing Machine)

Definition

Let $x \in \{0,1\}^\infty$

$$\dim(x) = \liminf_n \frac{\mathrm{K}(x \upharpoonright n)}{n},$$

$$\mathrm{Dim}(x) = \limsup_n \frac{\mathrm{K}(x \upharpoonright n)}{n}.$$

Definition

Let $E \subseteq \{0,1\}^\infty$,

$$\dim(E) = \sup_{x \in E} \dim(x).$$

Constructive dimension

- Partial randomness when compared to Martin-Löf randomness: $x \in \{0,1\}^\infty$ is M-L random iff there is a c such that for every n, $\mathrm{K}(x \upharpoonright n) > n - c$

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- (Doty 2008) For each $x \in \{0,1\}^\infty$, $\epsilon > 0$ with $\dim(x) > 0$ there is a y with $x \equiv_{\mathrm{T}} y$ and $\mathrm{Dim}(y) > 1 - \epsilon$

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- (Downey Hirschfeldt 2006) for other connections to randomness

Point to set principle

Definition

Let $x \in \mathbb{R}^n, r \in \mathbb{N}$. The *Kolmogorov complexity of x at precision r* is

$$\mathrm{K}_r(x) = \inf\left\{\mathrm{K}(q) \,\middle|\, q \in \mathbb{Q}^n, |x - q| \leq 2^{-r}\right\}.$$

$$\mathrm{dim}(x) = \liminf_r \frac{\mathrm{K}_r(x)}{r},$$

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Definition

Let $E \subseteq \mathbb{R}^n$,

$$\mathrm{dim}(E) = \sup_{x \in E} \mathrm{cdim}(x).$$

Point to set principle

Theorem (Lutz Lutz 2018)

Let $E \subseteq \mathbb{R}^n$. Then

$$\dim_{\mathrm{H}}(E) = \min_{B \subseteq \{0,1\}^*} \dim^B(E).$$

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Let $E \subseteq \mathbb{R}^n$. Then

$$\dim_{\mathrm{P}}(E) = \min_{B \subseteq \{0,1\}^*} \mathrm{Dim}^B(E).$$

Application of point to set principle to fractal geometry

Theorem (Marstrand 1954)

Let $E \subseteq \mathbb{R}^2$ *be an analytic set with* $\dim_{\mathrm{H}}(E) = s$*. Then for almost every* $\theta \in (0, 2\Pi)$, $\dim_{\mathrm{H}}(p_\theta E) = \min\{s, 1\}$

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PTSP take home message

- Kolmogorov complexity arguments are far from trivial
- Useful results. There is already a paper [Orponen 2020] with an alternative (not easier) geometrical proof of [Lutz Stull 2018]
- Many open problems in fractal geometry to attack, also in spaces different from Euclidean

Finite-state dimension

- (Dai et al 2004) Finite-state effectivization through gambling and compression
- (Doty Moser 2006) Kolmogorov complexity style definition. Let $x \in \Sigma^\infty$

$$\dim_{\mathrm{FS}}(x) = \inf_{M \mathrm{FS}} \liminf_{n} \frac{\mathrm{K}_M(x \upharpoonright n)}{n}$$

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- $b \in \mathbb{N}$, $D_b = \left\{ nb^{-m} \,\middle|\, n, m \in \mathbb{N} \right\}$
- Let $x \in \mathbb{R}$, $r \in \mathbb{N}$

$$\mathrm{K}_r^{b,M}(x) = \inf \left\{ \mathrm{K}_M(q) \,\middle|\, q \in \mathcal{D}_b, |x - q| \leq 2^{-r} \right\}.$$

$$\dim_{\mathrm{FS}}^b(x) = \inf_{M \mathrm{FS}} \liminf_r \frac{\mathrm{K}_r^{b,M}(x)}{r}$$

Borel normality

- Let $x \in \mathbb{R}$, $b \in \mathbb{N}$, x is b-**normal** if $(b^n x)$ is uniformly distributed mod 1.
  That is, for $(u, v) \subseteq [0, 1)$,

$$\lim_N \frac{\#\{n \leq N \mid b^n x \in (u, v)\}}{N} = (v - u)$$

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- x is b-normal iff $\dim_{\mathrm{FS}}^b(x) = 1$
  (based on [Schnorr Stimm 1972])
- Borel normality is base-dependent, so finite-state dimension is too
- x is **absolutely normal** if x is b-normal for every b

α-Borel normality

- For α probability distribution on $\{0, \ldots, b-1\}$, x is α-b-**normal** if $(b^n x)$ is α-distributed mod 1

α-Borel normality

- For α probability distribution on $\{0, \ldots, b-1\}$, x is α-b-**normal** if $(b^n x)$ is α-distributed mod 1
- (Huang et al 2020) extension of [Schnorr Stimm 1972] to α-normality
  Robust gambling characterization of normality, gambling success on x in terms of divergence between α and empirical distribution of $(b^n x)$

How far are Finite-state dimension and constructive dimension?

Fourier dimension

- Given a Borel measure μ on $\mathbb{R}$,

$$\hat{\mu}(u) = \int e^{-2\pi i u x} d\mu(x)$$

- μ is s-Fourier if $\hat{\mu}(u) \leq c|u|^{-s/2}$

$$\dim_{\mathrm{F}} E = \sup \{ s \leq 1 \mid \text{there exists } s\text{-Fourier } \mu \text{ with } \mu(E) = 1 \}$$

- $\dim_{\mathrm{F}} E \leq \dim_{\mathrm{H}}(E)$

Fourier dimension connections

- How do we effectivize Fourier dimension?

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- $\dim_{\mathrm{F}}(E) > s$ implies μ-a.e. $x \in E$ is absolutely normal (for μ s-Fourier)

Fourier dimension connections

- How do we effectivize Fourier dimension?
- $\dim_F(E) > s$ implies μ-a.e. $x \in E$ is absolutely normal (for μ s-Fourier)
- (Lyons 1983) (not quite)
  - $\dim_F(E) = 0$ implies there is a b s.t. for each $x \in E$ there is a nonuniform γ s.t. x is b-γ-normal
  - $\dim_F(E) > 0$ implies that for every b there is $x \in E$ that is b-normal or has no b-asymptotic distribution

Conclusions

- Constructive/effective dimension is a useful tool in fractal geometry through the point to set principle
- In particular in spaces different from Euclidean
- Finite state dimension gives a very robust characterization of Borel normality
- We need to clarify the connections of Fourier dimension with normality/finite-state dimension

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