Effective fractal dimension in the hyperspace and the space of probability distributions (informal talk)

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Why algorithmic randomness?

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- detect randomness

Why algorithmic randomness?

- detect randomness
- produce randomness

Why algorithmic randomness?

- detect randomness
- produce randomness
- mimic randomness

Information content in a separable space

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Definition

Let $x \in X, n \in \mathbb{N}$. The Kolmogorov complexity of x at precision n is

$$\mathrm{K}_n^f(x) = \inf \left\{ \mathrm{K}(q) \,\middle|\, q \in D, \rho(x,q) \leq 2^{-n} \right\}.$$

Effective dimension in a separable space

(X, ρ) is a separable metric space, D is a dense set, and
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$$\mathrm{cdim}^f(x) = \liminf_n \frac{\mathrm{K}_n^f(x)}{r}.$$

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Both definitions relativize to any oracle B by using $\mathrm{K}^B(w)$

Point to set principle for separable X

((X,ρ) is a separable metric space, D is a dense set and $f : \{0,1\}^* \twoheadrightarrow D$)

Theorem (ptsp separable spaces)

Let $E \subseteq X$. Then

$$\dim_{\mathrm{H}}(E) = \min_{B \subseteq \{0,1\}^*} \mathrm{cdim}^{f,B}(E).$$

PTSP

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- Let us answer questions on $\dim_{\mathrm{H}}$ using effective dimension

PTSP

- Let us answer questions on $\dim_{\mathrm{H}}$ using effective dimension
- Questions on $\dim_{\mathrm{H}}(E)$ are questions on the randomness of $x \in E$

The hyperspace

- Let (X, ρ) be a separable metric space
- Let $\mathcal{K}(X)$ be the set of nonempty compact subsets of X together with the Hausdorff metric $\mathrm{dist_H}$ defined as follows

$$\mathrm{dist_H}(U, V) = \max\left\{\sup_{x \in U} \rho(x, V), \sup_{y \in V} \rho(y, U)\right\}.$$

$(\rho(a, B) = \inf\{\rho(a, b) | b \in B\})$

Known result

Theorem (McClure 1996)

Let $E \subseteq X$ be **self-similar**. *Let $\psi_s(t) = 2^{-1/t^s}$. Then*

$$\dim_{\mathrm{H}}^{\psi}(\mathcal{K}(E)) \leq \dim_{\mathrm{H}}(E).$$

Constructive exact dimension

Definition

Let $x \in X$. The *f-constructive dimension$^\varphi$ of x* is

$$\mathrm{cdim}^{f,\varphi}(x) = \inf\{s \mid \exists^\infty n\, \mathrm{K}^f_n(x) \leq \log(1/\varphi_s(2^{-n}))\}.$$

Definition

Let $x \in X$. The *f-constructive strong dimension$^\varphi$ of x* is

$$\mathrm{cDim}^{f,\varphi}(x) = \inf\{s \mid \forall^\infty n\, \mathrm{K}^f_n(x) \leq \log(1/\varphi_s(2^{-n}))\}.$$

Theorem (Hyperspace dimension theorem)

Let $E \subseteq X$ be an analytic set.
Let φ be a gauge family, let $\tilde{\varphi}_s(t) = 2^{-1/\varphi_s(t)}$. Then

$$\dim_{\mathrm{P}}^{\tilde{\varphi}}(\mathcal{K}(E)) \geq \dim_{\mathrm{P}}^{\varphi}(E).$$

Prokhorov metric space

Let $P(X)$ be the set of Borel probability measures on X

$$d_P(\mu,\nu) = \inf\{\alpha > 0 \,|\, \mu(A) \le \nu(A_\alpha) + \alpha \wedge \nu(A) \le \mu(A_\alpha) + \alpha\}$$

$A_\alpha = \{x \,|\, \rho(A, x) < \alpha\}$, $\emptyset_\alpha = \emptyset$
Notice that $d_P(\delta_x, \delta_y) = \min(\rho(x, y), 1)$.

Information content in Prokhorov metric space

((X, ρ) is a separable metric space, D is a dense set and $f : \{0,1\}^* \twoheadrightarrow D$)

$$\mathcal{M} = \left\{ \alpha_1 \delta_{a_1} + \ldots + \alpha_k \delta_{a_k} \,\middle|\, k \in \mathbb{N}, a_i \in D, \alpha_i \in \mathbb{Q} \cap [0,1], \sum_{j=1}^{k} \alpha_k = 1 \right\}$$

Why Prokhorov metric space I

Definition

$\Gamma \subseteq P(X)$ is tight if for any $\epsilon > 0$ there is K compact with

$$\mu(K) \geq 1 - \epsilon$$

for any $\mu \in \Gamma$.

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Theorem (Prokhorov theorem)

For $\Gamma \subseteq P(X)$, $\overline{\Gamma}$ is compact if and only if Γ is tight

Why Prokhorov metric space II

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Theorem (Riesz representation theorem)

Let (X, ρ) *be compact and Hausdorff.*
If $\varphi : C(X) \to \mathbb{R}$ *is positive and* $\|\varphi\| = 1$ *then there is a unique* $\mu \in P(X)$ *with*

$$\varphi(f) = \int f d\mu$$

for all $f \in C(X)$.

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