Construction of an absolutely normal real number in polynomial time

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Abstract

We construct an absolutely normal number using time very close to quadratic in the number of output digits.

1 Warning

This proof is part of a joint paper with Jack H. Lutz on Alan Turing and normality. Therefore we don’t include a proper introduction.

Strauss [10] proved that almost every real number that is computable in polynomial time is absolutely normal. The measure conservation theorem of resource-bounded measure [7] automatically derives from this proof explicit examples of absolutely normal numbers that are computable in polynomial time.

Very recently three efficient constructions of absolutely normal numbers have been obtained simultaneously, [1], [5], and this result.

2 Preliminaries

For any natural number $k \geq 2$, we let $\Sigma_k = \{0, \ldots, k-1\}$ be a $k$-symbol alphabet. $\Sigma_k^*$ denotes the set of finite strings over alphabet $\Sigma_k$, $\Sigma_k^\infty$ denotes infinite sequences over alphabet $\Sigma_k$.

For $0 \leq i \leq j$, we write $x[i \ldots j]$ for the string consisting of the $i$-th through the $j$-th symbols of $x$. We use $\lambda$ for the empty string.

Definition. Let $s \in [0, \infty)$.

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1. An \( s \)-gale on \( \Sigma^k \) is a function \( d : \Sigma^*_k \to [0, \infty) \) satisfying
\[
d(w) = k^{-s} \sum_{a \in \Sigma_k} d(wa)
\]
for all \( w \in \Sigma^*_k \).

2. A martingale is a 1-gale, that is, a function \( d : \Sigma^*_k \to [0, \infty) \) satisfying
\[
d(w) = \sum_{a \in \Sigma_k} d(wa)
\]
for all \( w \in \Sigma^*_k \).

**Definition.** Let \( s \in [0, \infty) \) and \( d \) be an \( s \)-gale. We say that \( d \) succeeds on a sequence \( S \in \Sigma^\infty_k \) if
\[
\limsup_{n \to \infty} d(S[0 \ldots n]) = \infty.
\]
The success set of \( d \) is
\[
S^\infty[d] = \{ S \in \Sigma^\infty_k \mid d \text{ succeeds on } S \}.
\]

**Definition.** We say that a function \( d : \Sigma^*_k \to [0, \infty) \cap \mathbb{Q} \) is exactly \( t(n) \)-computable if \( d(w) \) is computable in time \( t(|w|) \).

**Definition.** We say that a function \( d : \Sigma^*_k \to [0, \infty) \cap \mathbb{Q} \) is \( \text{on-line } t(n) \)-computable if \( d(w) \) is computable in time \( t(|w|) \) and for every \( w \in \Sigma^*_k \), \( a \in \Sigma_k \), the computation of \( d(wa) \) starts with a computation of \( d(w) \).

**Definition.** We say that an \( s \)-gale \( d : \Sigma^*_k \to [0, \infty) \cap \mathbb{Q} \) is FS-computable if \( d(w) \) is computable by a finite-state gambler as defined in [4].

**Definition.** Let \( X \subseteq \Sigma^\infty_k \), The on-line \( t(n) \)-dimension of \( X \) is
\[
\dim_{\text{ol-}t(n)}(X) = \inf \left\{ s \in [0, \infty) \mid \text{there is an exactly on-line } O(t(n))-\text{computable } s \text{-gale } d \text{ s.t. } X \subseteq S^\infty[d] \right\}
\]

**Definition.** Let \( X \subseteq \Sigma^\infty_k \), The FS-dimension of \( X \) is
\[
\dim_{FS}(X) = \inf \left\{ s \in [0, \infty) \mid \text{there is a FS-computable } s \text{-gale } d \text{ s.t. } X \subseteq S^\infty[d] \right\}
\]

We will use \( \dim_{\Delta}^{(k)}(X) \) to refer to the \( \Delta \)-dimension of \( X \subseteq \Sigma^\infty_k \) when we want to stress that the underlying sequence space is \( \Sigma^\infty_k \).

**Definition.** Let \( x \in \Sigma^\infty_k \), let \( c \in \mathbb{N} \). We say that \( x \) is on-line \( n^c \)-random if no exactly on-line \( O(n^c) \)-computable martingale succeeds on \( x \).

For a complete introduction and motivation of effective dimension see [8].
2.1 Representations of Reals

We will use infinite sequences over \( \Sigma_k \) to represent real numbers in \([0,1)\). For this, we associate each string \( w \in \Sigma_k^* \) with the half-open interval \([w]_k\) defined by \([w]_k = [x, x + k^{-|w|}]\), for \( x = \sum_{i=1}^{|w|} w[i-1]k^{-i} \). Each real number \( \alpha \in [0,1) \) is then represented by the unique sequence \( S_k(\alpha) \in \Sigma_k^\infty \) satisfying

\[
w \subseteq S_k(\alpha) \iff \alpha \in [w]_k
\]

for all \( w \in \Sigma_k^* \). We have

\[
\alpha = \sum_{i=1}^\infty S_k(\alpha)[i-1]k^{-i}
\]

and the mapping \( \alpha \mapsto S_k(\alpha) \) is a bijection from \([0,1)\) to \( \Sigma_k^\infty \) (notice that \([w]_k\) being half-open prevents double representations). If \( x \in \Sigma_k^\infty \) then \( \text{real}_k(x) = \alpha \) such that \( x = S_k(\alpha) \). Therefore we always have that \( \text{real}_k(S_k(\alpha)) = \alpha \) and \( S_k(\text{real}_k(x)) = x \).

**Definition.** A number \( \alpha \in [0,1) \) is \( k \)-normal \([2]\), if for every \( w \in \Sigma_k^* \),

\[
\lim_{n \to \infty} \frac{1}{n} \left\{ i < n \mid S_k(\alpha)[i..i + |w| - 1] = w \right\} = k^{-|w|}.
\]

That is, \( \alpha \) is \( k \)-normal if every string \( w \) has asymptotic frequency \( k^{-|w|} \) in \( S_k(\alpha) \).

**Definition.** A number \( \alpha \in [0,1) \) is absolutely normal \([2]\), if for every \( k \in \mathbb{N} \) \( \alpha \) is \( k \)-normal. That is, \( \alpha \) is absolutely normal if for every \( k \), every string \( w \) has asymptotic frequency \( k^{-|w|} \) in \( S_k(\alpha) \).

The following theorem is proven in \([3]\) and also follows from \([9]\).

**Theorem 2.1** A number \( \alpha \in [0,1) \) is \( k \)-normal if and only if \( \dim_{FS}(S_k(\alpha)) = 1 \).

3 Main result

In this section we prove or main result. We need the following lemma with the conversion of FS-gales on \( \Sigma_l \) to p-gales on \( \Sigma_k \) which is a careful generalization of lemma 3.1 in \([6]\).

**Lemma 3.1** Let \( k, l \geq 2 \), let \( c \in \mathbb{N} \), let \( s \in \mathbb{Q} \). For any FS-computable \( s \)-gale \( d \) on \( \Sigma_l \) and rational \( s' > s \), there is an exactly on-line \( O(n^2) \)-computable \( s' \)-gale \( d' \) on \( \Sigma_k \) such that \( \text{real}_l(S^\infty([d])) \subseteq \text{real}_k(S^\infty([d'])) \).
As a consequence we can compare FS and polynomial-time dimension for different representations of the same number.

**Theorem 3.2** Let $\alpha \in [0,1)$, let $k, l \geq 2$. Then
\[ \dim_{\text{ol-}n^2}^{(k)}(S_k(\alpha)) \leq \dim_{\text{FS}}^{(l)}(S_l(\alpha)). \]

**Theorem 3.3** Let $\alpha \in [0,1]$. If $S_2(\alpha)$ is on-line $n^2$-random then $\alpha$ is absolutely normal.

**Proof.** If $S_2(\alpha)$ is on-line $n^2$-random then $\dim_{\text{ol-}n^2}^{(2)}(S_2(\alpha)) = 1$. By theorem 3.2, for every $l \in \mathbb{N}$, $\dim_{\text{FS}}^{(l)}(S_l(\alpha)) = 1$, and (by theorem 2.1) $\alpha$ is absolutely normal.

**Theorem 3.4** There is an algorithm computing the first $n$ bits of $S_2(\alpha)$ in time $n^2 \log^* n$, for $\alpha$ absolutely normal.

**Proof.** Let $d$ be an exactly on-line $n^2 \log^* n$-computable martingale that is universal for all exactly on-line $O(n^2)$-computable martingales. Then any $x \not\in S^\infty[d]$ is on-line $n^2$-random.

We construct $x \not\in S^\infty[d]$ by martingale diagonalization in time $n^2 \log^* n$ for the first $n$ bits of $x$.

**References**


