Construction of an absolutely normal real number in polynomial time

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Abstract

We construct an absolutely normal number using time very close to quadratic in the number of output digits.

1 Warning

This proof is part of a joint paper with Jack H. Lutz on Alan Turing and normality. Therefore we don't include a proper introduction.

Strauss [10] proved that almost every real number that is computable in polynomial time is absolutely normal. The measure conservation theorem of resource-bounded measure [7] automatically derives from this proof explicit examples of absolutely normal numbers that are computable in polynomial time.

Very recently three efficient constructions of absolutely normal numbers have been obtained simultaneously, [1], [5], and this result.

2 Preliminaries

For any natural number $k \geq 2$, we let $\Sigma_k = \{0, \ldots, k-1\}$ be a k-symbol alphabet. Σ_k^* denotes the set of finite strings over alphabet Σ_k, Σ_k^∞ denotes infinite sequences over alphabet Σ_k.

For $0 \leq i \leq j$, we write $x[i \ldots j]$ for the string consisting of the i-th through the j-th symbols of x. We use λ for the empty string.

Definition. Let $s \in [0, \infty)$.

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1. An *s-gale* on Σ_k is a function $d : \Sigma_k^* \to [0, \infty)$ satisfying

$$d(w) = k^{-s} \sum_{a \in \Sigma_k} d(wa)$$

for all $w \in \Sigma_k^*$.

2. A *martingale* is a 1-gale, that is, a function $d : \Sigma_k^* \to [0, \infty)$ satisfying

$$d(w) = \frac{\sum_{a \in \Sigma_k} d(wa)}{k}$$

for all $w \in \Sigma_k^*$.

Definition. Let $s \in [0, \infty)$ and d be an s-gale. We say that d *succeeds* on a sequence $S \in \Sigma_k^\infty$ if

$$\limsup_{n \to \infty} d(S[0 \ldots n]) = \infty.$$

The success set of d is

$$S^\infty[d] = \{S \in \Sigma_k^\infty \mid d \text{ succeeds on } S\}.$$

Definition. We say that a function $d : \Sigma_k^* \to [0, \infty) \cap \mathbb{Q}$ is *exactly $t(n)$-computable* if $d(w)$ is computable in time $t(|w|)$.

Definition. We say that a function $d : \Sigma_k^* \to [0, \infty) \cap \mathbb{Q}$ is *exactly on-line $t(n)$-computable* if $d(w)$ is computable in time $t(|w|)$ and for every $w \in \Sigma_k^*$, $a \in \Sigma_k$, the computation of $d(wa)$ starts with a computation of $d(w)$.

Definition. We say that an s-gale $d : \Sigma_k^* \to [0, \infty) \cap \mathbb{Q}$ is *FS-computable* if $d(w)$ is computable by a finite-state gambler as defined in [4].

Definition. Let $X \subseteq \Sigma_k^\infty$, The *on-line $t(n)$-dimension* of X is

$$\dim_{\text{ol-}t(n)}(X) = \inf \left\{ s \in [0, \infty) \;\middle|\; \begin{array}{l} \text{there is an exactly on-line} \\ O(t(n))\text{-computable } s\text{-gale } d \text{ s.t.} \quad X \subseteq S^\infty[d] \end{array} \right\}$$

Definition. Let $X \subseteq \Sigma_k^\infty$, The *FS-dimension* of X is

$$\dim_{\text{FS}}(X) = \inf \left\{ s \in [0, \infty) \;\middle|\; \begin{array}{l} \text{there is a FS-computable } s\text{-gale } d \text{ s.t.} \\ X \subseteq S^\infty[d] \end{array} \right\}$$

We will use $\dim_\Delta^{(k)}(X)$ to refer to the Δ-dimension of $X \subseteq \Sigma_k^\infty$ when we want to stress that the underlying sequence space is Σ_k^∞.

Definition. Let $x \in \Sigma_k^\infty$, let $c \in \mathbb{N}$. We say that x is *on-line n^c-random* if no exactly on-line $O(n^c)$-computable martingale succeeds on x.

For a complete introduction and motivation of effective dimension see [8].

2.1 Representations of Reals

We will use infinite sequences over Σ_k to represent real numbers in [0,1). For this, we associate each string $w \in \Sigma_k^*$ with the half-open interval $[w]_k$ defined by $[w]_k = [x, x + k^{-|w|})$, for $x = \sum_{i=1}^{|w|} w[i-1]k^{-i}$. Each real number $\alpha \in [0,1)$ is then represented by the unique sequence $S_k(\alpha) \in \Sigma_k^\infty$ satisfying

$$w \sqsubseteq S_k(\alpha) \iff \alpha \in [w]_k$$

for all $w \in \Sigma_k^*$. We have

$$\alpha = \sum_{i=1}^{\infty} S_k(\alpha)[i-1]k^{-i}$$

and the mapping $\alpha \mapsto S_k(\alpha)$ is a bijection from $[0,1)$ to Σ_k^∞ (notice that $[w]_k$ being half-open prevents double representations). If $x \in \Sigma_k^\infty$ then $real_k(x) = \alpha$ such that $x = S_k(\alpha)$. Therefore we always have that $real_k(S_k(\alpha)) = \alpha$ and $S_k(real_k(x)) = x$.

Definition. A number $\alpha \in [0,1)$ is *k-normal* [2], if for every $w \in \Sigma_k^*$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ i < n \;\middle|\; S_k(\alpha)[i..i+|w|-1] = w \right\} \right| = k^{-|w|}.$$

That is, α is k-normal if every string w has asymptotic frequency $k^{-|w|}$ in $S_k(\alpha)$.

Definition. A number $\alpha \in [0,1)$ is *absolutely normal* [2], if for every $k \in \mathbb{N}$ α is k-normal. That is, α is absolutely normal if for every k, every string w has asymptotic frequency $k^{-|w|}$ in $S_k(\alpha)$.

The following theorem is proven in [3] and also follows from [9].

Theorem 2.1 *A number $\alpha \in [0,1)$ is k-normal if and only if $\dim_{\mathrm{FS}}(S_k(\alpha)) = 1$.*

3 Main result

In this section we prove or main result. We need the following lemma with the conversion of FS-gales on Σ_l to p-gales on Σ_k which is a careful generalization of lemma 3.1 in [6].

Lemma 3.1 *Let $k, l \geq 2$, let $c \in \mathbb{N}$, let $s \in \mathbb{Q}$. For any FS-computable s-gale d on Σ_l and rational $s' > s$, there is an exactly on-line $O(n^2)$-computable s'-gale d' on Σ_k such that $\mathrm{real}_l(S^\infty[d])) \subseteq \mathrm{real}_k(S^\infty[d'])$.*

As a consequence we can compare FS and polynomial-time dimension for different representations of the same number.

Theorem 3.2 *Let $\alpha \in [0,1)$, let $k, l \geq 2$. Then*

$$\dim^{(k)}_{ol\text{-}n^2}(S_k(\alpha)) \leq \dim^{(l)}_{\mathrm{FS}}(S_l(\alpha)).$$

Theorem 3.3 *Let $\alpha \in [0,1]$. If $S_2(\alpha)$ is on-line n^2-random then α is absolutely normal.*

Proof.

If $S_2(\alpha)$ is on-line n^2-random then $\dim^{(2)}_{\text{ol-}n^2}(S_2(\alpha)) = 1$. By theorem 3.2, for every $l \in \mathbb{N}$, $\dim^{(l)}_{\mathrm{FS}}(S_l(\alpha)) = 1$, and (by theorem 2.1) α is absolutely normal.

□

Theorem 3.4 *There is an algorithm computing the first n bits of $S_2(\alpha)$ in time $n^2 \log^* n$, for α absolutely normal.*

Proof. Let d be an exactly on-line $n^2 \log^* n$-computable martingale that is universal for all exactly on-line $O(n^2)$-computable martingales. Then any $x \notin S^\infty[d]$ is on-line n^2-random.

We construct $x \notin S^\infty[d]$ by martingale diagonalization in time $n^2 \log^* n$ for the first n bits of x.

□

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