Curves That Must Be Retraced

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Abstract

We exhibit a polynomial time computable plane curve $\mathbf{\Gamma}$ that has finite length, does not intersect itself, and is smooth except at one endpoint, but has the following property. For every computable parametrization f of $\mathbf{\Gamma}$ and every positive integer m, there is some positive-length subcurve of $\mathbf{\Gamma}$ that f retraces at least m times. In contrast, every computable curve of finite length that does not intersect itself has a constant-speed (hence non-retracing) parametrization that is computable relative to the halting problem.

1 Introduction

A curve is a mathematical model of the path of a particle undergoing continuous motion. Specifically, in a Euclidean space $\mathbb{R}^n$, a curve is the range Γ of a continuous function $f : [a, b] \to \mathbb{R}^n$ for some $a < b$. The function f, called a *parametrization* of Γ, clearly contains more information than the pointset Γ, namely, the precise manner in which the particle "traces" the points $f(t) \in \Gamma$ as t, which is often considered a time parameter, varies from a to b. When the particle's motion is algorithmically governed, the parametrization must be computable (as a function on the reals, see below).

This paper shows that the geometry of a curve Γ may force every *computable* parametrization f of Γ to retrace various parts of its path (i.e., "go back and forth along Γ") many times, even when Γ is an efficiently computable, smooth, finite-length curve that does not intersect itself. In fact, our main theorem exhibits a plane curve $\mathbf{\Gamma} \subseteq \mathbb{R}^2$ with the following properties.

1. $\mathbf{\Gamma}$ is *simple*, i.e., it does not intersect itself.

2. $\mathbf{\Gamma}$ is *rectifiable*, i.e., it has finite length.

3. $\mathbf{\Gamma}$ is *smooth except at one endpoint*, i.e., $\mathbf{\Gamma}$ has a tangent at every interior point and a 1-sided tangent at one endpoint, and these tangents vary continuously along $\mathbf{\Gamma}$.

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4. $\mathbf{\Gamma}$ is *polynomial time computable* in the strong sense that there is a polynomial time computable position function $\vec{s} : [0,1] \to \mathbb{R}^2$ such that the velocity function $\vec{v} = \vec{s}'$ and the acceleration function $\vec{a} = \vec{v}'$ are polynomial time computable; the total distance traversed by $\vec{s}$ is finite; and $\vec{s}$ parametrizes $\mathbf{\Gamma}$, i.e., range$(\vec{s}) = \mathbf{\Gamma}$.

5. $\mathbf{\Gamma}$ *must be retraced* in the sense that every parametrization $f : [a,b] \to \mathbb{R}^2$ of $\mathbf{\Gamma}$ that is computable in *any* amount of time has the following property. For every positive integer m, there exist disjoint, closed subintervals $I_0, \ldots, I_m$ of $[a,b]$ such that the curve $\Gamma_0 = f(I_0)$ has positive length and $f(I_i) = \Gamma_0$ for all $1 \leq i \leq m$. (Hence f retraces Γ_0 at least m times.)

The terms "computable" and "polynomial time computable" in properties 4 and 5 above refer to the "bit-computability" model of computation on reals formulated in the 1950s by Grzegorczyk [9] and Lacombe [17], extended to feasible computability in the 1980s by Ko and Friedman [13] and Kreitz and Weihrauch [16], and exposited in the recent paper by Braverman and Cook [4] and the monographs [20, 14, 23, 5]. As will be shown here, condition 4 also implies that the pointset $\mathbf{\Gamma}$ is polynomial time computable in the sense of Brattka and Weihrauch [2]. (See also [23, 3, 4].)

A fundamental and useful theorem of classical analysis states that every simple, rectifiable curve Γ has a *normalized constant-speed parametrization*, which is a one-to-one parametrization $f : [0,1] \to \mathbb{R}^n$ of Γ with the property that $f([0,t])$ has arclength tL for all $0 \leq t \leq 1$, where L is the length of Γ. (A simple, rectifiable curve Γ has exactly two such parametrizations, one in each direction, and standard terminology calls either of these *the* normalized constant-speed parametrization $f : [0,1] \to \mathbb{R}^n$ of Γ. The constant-speed parametrization is also called the *parametrization by arclength* when it is reformulated as a function $f : [0,L] \to \mathbb{R}^n$ that moves with constant speed 1 along Γ.) Since the constant-speed parametrization does not retrace any part of the curve, our main theorem implies that this classical theorem is not entirely constructive. Even when a simple, rectifiable curve has an efficiently computable parametrization, the constant-speed parametrization need not be computable.

In addition to our main theorem, we prove that every simple, rectifiable curve Γ in $\mathbb{R}^n$ with a computable parametrization has the following two properties.

I. The length of Γ is lower semicomputable.

II. The constant-speed parametrization of Γ is computable relative to the length of Γ.

These two things are not hard to prove if the computable parametrization is one-to-one, (in fact, they follow from results of Müller and Zhao [19] in this case) but our results hold even when the computable parametrization retraces portions of the curve many times.

Taken together, I and II have the following two consequences.

1. The curve $\mathbf{\Gamma}$ of our main theorem has a finite length that is lower semi-computable but not computable. (The existence of polynomial-time computable curves with this property was first proven by Ko [15].)

2. Every simple, rectifiable curve Γ in $\mathbb{R}^n$ with a computable parametrization has a constant-speed parametrization that is Δ^0_2-computable, i.e., computable relative to the halting problem. Hence, the existence of a constant-speed parametrization, while not entirely constructive, is constructive relative to the halting problem.

2 Length, Computability, and Complexity of Curves

In this section we summarize basic terminology and facts about curves. As we use the terms here, a *curve* is the range Γ of a continuous function $f : [a,b] \to \mathbb{R}^n$ for some $a < b$. The function f is called a *parametrization* of Γ. Each curve clearly has infinitely many parametrizations.

A curve is *simple* if it has a parametrization that is one-to-one, i.e., the curve "does not intersect itself". The length of a simple curve Γ is defined as follows. Let $f : [a,b] \overset{1-1}{\to} \mathbb{R}^n$ be a one-to-one parametrization of Γ. For each *dissection* $\vec{t}$ of $[a,b]$, i.e., each tuple $\vec{t} = (t_0, \ldots, t_m)$ with $a = t_0 < t_1 < \ldots < t_m = b$, define the *$f$-$\vec{t}$-approximate length* of Γ to be

$$\mathcal{L}_{\vec{t}}^{f}(\Gamma) = \sum_{i=0}^{m-1} |f(t_{i+1}) - f(t_i)|.$$

Then the *length* of Γ is

$$\mathcal{L}(\Gamma) = \sup_{\vec{t}} \mathcal{L}_{\vec{t}}^{f}(\Gamma),$$

where the supremum is taken over all dissections $\vec{t}$ of $[a,b]$. It is easy to show that $\mathcal{L}(\Gamma)$ does not depend on the choice of the one-to-one parametrization f, i.e. that the length is an intrinsic property of the pointset Γ.

In sections 4 and 5 of this paper we use a more general notion of length, namely, the 1-dimensional Hausdorff measure $\mathcal{H}^1(\Gamma)$, which is defined for *every* set $\Gamma \subseteq \mathbb{R}^n$. We refer the reader to [7] for the definition of $\mathcal{H}^1(\Gamma)$. It is well known that $\mathcal{H}^1(\Gamma) = \mathcal{L}(\Gamma)$ holds for every simple curve Γ.

A curve Γ is *rectifiable*, or *has finite length*, if $\mathcal{L}(\Gamma) < \infty$. In sections 4 and 5 we use the notation $\mathcal{RC}$ for the set of all rectifiable simple curves.

Definition. Let $f : [a,b] \to \mathbb{R}^n$ be continuous.

1. For $m \in \mathbb{Z}^+$, f has *m-fold retracing* if there exist disjoint, closed subintervals $I_0, \ldots, I_m$ of $[a,b]$ such that the curve $\Gamma_0 = f(I_0)$ has positive length and $f(I_i) = \Gamma_0$ for all $1 \leq i \leq m$.

2. f is *non-retracing* if f does not have 1-fold retracing.

3. f has *bounded retracing* if there exists $m \in \mathbb{Z}^+$ such that f does not have m-fold retracing.

4. f has *unbounded retracing* if f does not have bounded retracing, i.e., if f has m-fold retracing for all $m \in \mathbb{Z}^+$.

We now review the notions of computability and complexity of a real-valued function. An *oracle* for a real number t is any function $O_t : \mathbb{N} \to \mathbb{Q}$ with the property that $|O_t(s) - t| \leq 2^{-s}$ holds for all $s \in \mathbb{N}$ given in unary. A function $f : [a,b] \to \mathbb{R}^n$ is *computable* if there is an oracle Turing machine M with the following property. For every $t \in [a,b]$ and every precision parameter $r \in \mathbb{N}$ given in unary, if M is given r as input and *any* oracle O_t for t as its oracle, then M outputs a rational point $M^{O_t}(r) \in \mathbb{Q}^n$ such that $|M^{O_t}(r) - f(t)| \leq 2^{-r}$. A function $f : [a,b] \to \mathbb{R}^n$ is *computable in polynomial time* if there is an oracle machine M that does this in time polynomial in $r + l$, where l is the maximum length of the query responses provided by the oracle.

An *oracle* for a function $f : [a,b] \to \mathbb{R}^n$ is any function $\mathcal{O}_f : ([a,b] \cap \mathbb{Q}) \times \mathbb{N} \to \mathbb{Q}^n$ with the property that $|\mathcal{O}_f(q,r) - f(q)| \leq 2^{-r}$ holds for all $q \in [a,b] \cap \mathbb{Q}$ and $r \in \mathbb{N}$. A decision problem

A is *Turing reducible* to a function $f : [a, b] \to \mathbb{R}^n$, and we write $A \leq_{\mathrm{T}} f$, if there is an oracle Turing machine M such that, for every oracle $\mathcal{O}_f$ for f, $M^{\mathcal{O}_f}$ decides A. It is easy to see that, if f is computable, then $A \leq_{\mathrm{T}} f$ if and only if A is decidable.

A curve is *computable* if it has a parametrization $f : [a, b] \to \mathbb{R}^n$, where $a, b \in \mathbb{Q}$ and f is computable. A curve is *computable in polynomial time* if it has a parametrization that is computable in polynomial time.

3 An Efficiently Computable Curve That Must Be Retraced

This section presents our main theorem, which is the existence of a smooth, rectifiable, simple plane curve $\mathbf{\Gamma}$ that is parametrizable in polynomial time but not computably parametrizable in any amount of time without unbounded retracing. We begin with a precise construction of the curve $\mathbf{\Gamma}$, followed by a brief intuitive discussion of this construction. The rest of the section is devoted to proving that $\mathbf{\Gamma}$ has the desired properties.

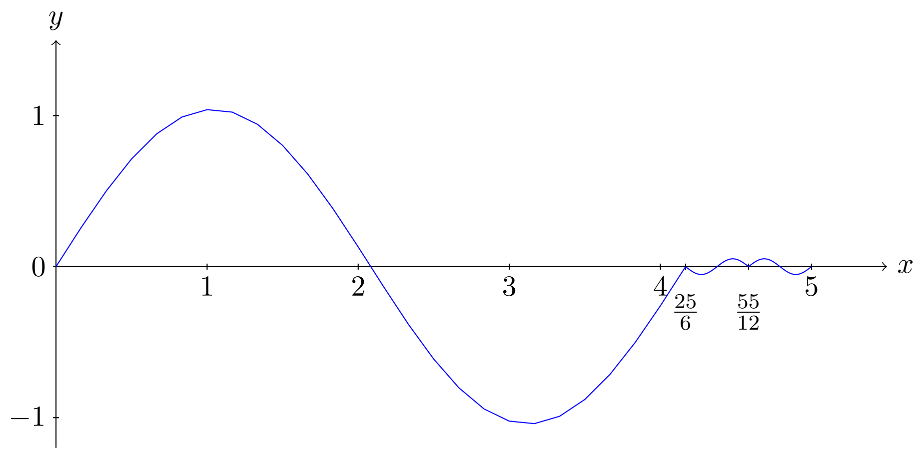

Figure 3.1: $\psi_{0,5,1}$

Construction 3.1. (1) For each $a, b \in \mathbb{R}$ with $a < b$, define the functions $\varphi_{a,b}, \xi_{a,b} : [a, b] \to \mathbb{R}$ by

$$\varphi_{a,b}(t) = \frac{b-a}{4} \sin \frac{2\pi(t-a)}{b-a}$$

and

$$\xi_{a,b}(t) = \begin{cases} -\varphi_{a,\frac{a+b}{2}}(t) & \text{if } a \leq t \leq \frac{a+b}{2} \\ \varphi_{\frac{a+b}{2},b}(t) & \text{if } \frac{a+b}{2} \leq t \leq b. \end{cases}$$

(2) For each $a, b \in \mathbb{R}$ with $a < b$ and each positive integer n, define the function $\psi_{a,b,n} : [a, b] \to \mathbb{R}$ by

$$\psi_{a,b,n}(t) = \begin{cases} \varphi_{a,d_0}(t) & \text{if } a \leq t \leq d_0 \\ \xi_{d_{i-1},d_i}(t) & \text{if } d_{i-1} \leq t \leq d_i, \end{cases}$$

where

$$d_i = \frac{a+5b}{6} + i\frac{b-a}{6n}$$

for $0 \leq i \leq n$. (See Figure 3.1.)

(3) Fix a standard enumeration $M_1, M_2, \ldots$ of (deterministic) Turing machines that take positive integer inputs. For each positive integer n, let $\tau(n)$ denote the number of steps executed by M_n on input n. It is well known that the *diagonal halting problem*

$$K = \left\{ n \in \mathbb{Z}^+ \mid \tau(n) < \infty \right\}$$

is undecidable.

(4) Define the horizontal and vertical acceleration functions $a_x, a_y : [0,1] \to \mathbb{R}$ as follows. For each $n \in \mathbb{N}$, let

$$t_n = \int_0^n e^{-x} dx = 1 - e^{-n},$$

noting that $t_0 = 0$ and that t_n converges monotonically to 1 as $n \to \infty$. Also, for each $n \in \mathbb{Z}^+$, let

$$t_n^- = \frac{t_{n-1} + 4t_n}{5}, \quad t_n^+ = \frac{6t_n - t_{n-1}}{5},$$

noting that these are symmetric about t_n and that $t_n^+ \leq t_{n+1}^-$.

(i) For $0 \leq t \leq 1$, let

$$a_x(t) = \begin{cases} -2^{-(n+\tau(n))}\xi_{t_n^-, t_n^+}(t) & \text{if } t_n^- \leq t < t_n^+ \\ 0 & \text{if no such } n \text{ exists,} \end{cases}$$

where $2^{-\infty} = 0$.

(ii) For $0 \leq t < 1$, let

$$a_y(t) = \psi_{t_{n-1}, t_n, n}(t),$$

where n is the unique positive integer such that $t_{n-1} \leq t < t_n$.

(iii) Let $a_y(1) = 0$.

(Note that $a_x(t)$ is positive if and only if such an n exists and is an element of K. We have thus encoded K into the geometry of $\mathbf{\Gamma}$.)

(5) Define the horizontal and vertical velocity and position functions $v_x, v_y, s_x, s_y : [0,1] \to \mathbb{R}$ by

$$v_x(t) = \int_0^t a_x(\theta) d\theta, \quad v_y(t) = \int_0^t a_y(\theta) d\theta,$$

$$s_x(t) = \int_0^t v_x(\theta) d\theta, \quad s_y(t) = \int_0^t v_y(\theta) d\theta.$$

(6) Define the vector acceleration, velocity, and position functions $\vec{a}, \vec{v}, \vec{s} : [0,1] \to \mathbb{R}^2$ by

$$\vec{a}(t) = (a_x(t), a_y(t)),$$
$$\vec{v}(t) = (v_x(t), v_y(t)),$$
$$\vec{s}(t) = (s_x(t), s_y(t)).$$

(7) Let $\mathbf{\Gamma} = \text{range}(\vec{s})$.

Intuitively, a particle at rest at time $t = a$ and moving with acceleration given by the function $\varphi_{a,b}$ moves forward, with velocity increasing to a maximum at time $t = \frac{a+b}{2}$ and then decreasing back to 0 at time $t = b$. The vertical acceleration function a_y, together with the initial conditions $v_y(0) = s_y(0) = 0$ implied by (5), thus causes a particle to move generally upward (i.e., $s_y(t_0) < s_y(t_1) < \cdots$), coming to momentary rests at times $t_1, t_2, t_3, \ldots$. Between two consecutive such stopping times t_{n-1} and t_n, the particle's vertical acceleration is controlled by the function $\psi_{t_{n-1},t_n,n}$. This function causes the particle's vertical motion to do the following between times t_{n-1} and t_n.

(i) From time t_{n-1} to time $\frac{t_{n-1}+5t_n}{6}$, move upward from elevation $s_y(t_{n-1})$ to elevation $s_y(t_n)$.

(ii) From time $\frac{t_{n-1}+5t_n}{6}$ to time t_n, make n round trips to a lower elevation $s \in (s_y(t_{n-1}), s_y(t_n))$.

In the meantime, the horizontal acceleration function a_x, together with the initial conditions $v_x(0) = s_x(0) = 0$ implied by (5), ensure that the particle remains on or near the y-axis. The deviations from the y-axis are simply described: The particle moves to the right from time $\frac{t_{n-1}+4t_n}{5}$ through the completion of the n round trips described in (ii) above and then moves to the y-axis between times t_n and $\frac{6t_n - t_{n-1}}{5}$. The amount of lateral motion here is regulated by the coefficient $2^{-(n+\tau(n))}$. If $\tau(n) = \infty$ (i.e., $n \notin \mathrm{K}$), then there is no lateral motion, and the n round trips in (ii) are retracings of the particle's path. If $\tau(n) < \infty$ (i.e., $n \in \mathrm{K}$), then these n round trips are "forward" motion along a curvy part of $\mathbf{\Gamma}$. We have thus encoded the halting problem K into the retracing/non-retracing behavior of our parametrization of $\mathbf{\Gamma}$. In fact, $\mathbf{\Gamma}$ contains points of arbitrarily high curvature, but the particle's motion is kinematically realistic in the sense that the acceleration vector $\vec{a}(t)$ is polynomial time computable, hence continuous and bounded on the interval $[0, 1]$. Figure 3.2 illustrates the path of the particle from time t_{n-1} to t_{n+1} with $n = 1$ and hypothetical (model dependent!) values $\tau(1) = 1$ and $\tau(2) = 2$.

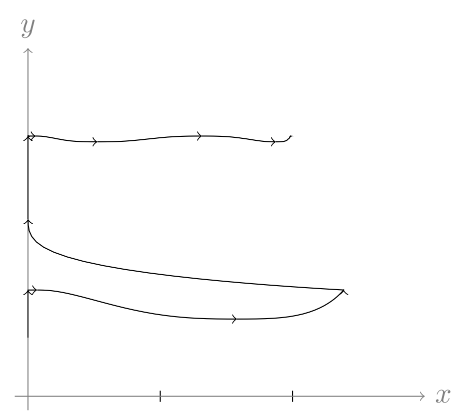

Figure 3.2: Example of $\vec{s}(t)$ from t_0 to t_2

The rest of this section is devoted to proving the following theorem concerning the curve $\mathbf{\Gamma}$.

Theorem 3.2. *(main theorem). Let $\vec{a}, \vec{v}, \vec{s}$, and $\mathbf{\Gamma}$ be as in Construction 3.1.*

1. *The functions $\vec{a}, \vec{v}$, and $\vec{s}$ are Lipschitz and computable in polynomial time, hence continuous and bounded.*

2. *The total length, including retracings, of the parametrization $\vec{s}$ of $\mathbf{\Gamma}$ is finite and computable in polynomial time.*

3. *The curve $\mathbf{\Gamma}$ is simple, rectifiable, and smooth except at one endpoint.*

4. *Every computable parametrization $f : [a, b] \to \mathbb{R}^2$ of $\mathbf{\Gamma}$ has* unbounded retracing.

For the remainder of this section, we use the notation of Construction 3.1.

The following two observations facilitate our analysis of the curve $\mathbf{\Gamma}$. The proofs are routine calculations.

Observation 3.3. *For all $n \in \mathbb{Z}^+$, if we write*

$$d_i^{(n)} = \frac{t_{n-1} + 5t_n}{6} + i\frac{t_n - t_{n-1}}{6n}$$

and

$$e_i^{(n)} = d_i^{(n)} + \frac{t_n - t_{n-1}}{12n}$$

for all $0 \le i < n$, then

$$t_{n-1} < t_n^- < d_0^{(n)} < e_0^{(n)} < d_1^{(n)} < e_1^{(n)} < \cdots < d_{n-1}^{(n)} < e_{n-1}^{(n)} < t_n < t_n^+ < t_{n+1}^-.$$

Observation 3.4. *For all $a, b \in \mathbb{R}$ with $a < b$,*

$$\int_a^b \int_a^b \varphi_{a,b}(\theta) d\theta dt = \frac{(b-a)^3}{8\pi}.$$

We now proceed with a quantitative analysis of the geometry of $\mathbf{\Gamma}$. We begin with the horizontal component of $\vec{s}$.

Lemma 3.5. *1. For all $t \in [0,1] - \bigcup_{n \in K}(t_n^-, t_n^+)$, $v_x(t) = s_x(t) = 0$.*

2. For all $n \in K$ and $t \in (t_n^-, t_n)$, $v_x(t) > 0$.

3. For all $n \in K$ and $t \in (t_n, t_n^+)$, $v_x(t) < 0$.

4. For all $n \in \mathbb{Z}^+$, $s_x(t_n) = \frac{(e-1)^3}{1000\pi e^{3n}} 2^{-(n+\tau(n))}$.

5. $s_x(1) = 0$.

Proof. Parts 1-3 are routine by inspection and induction. For $n \in \mathbb{Z}^+$, Observation 3.4 tells us that

$$\begin{aligned} s_x(t_n) &= \frac{(t_n - t_n^-)^3}{8\pi} 2^{-(n+\tau(n))} \\ &= \frac{(\frac{1}{5}(t_n - t_{n-1}))^3}{8\pi} 2^{-(n+\tau(n))} \\ &= \frac{(\frac{1}{5}((e-1)e^{-n}))^3}{8\pi} 2^{-(n+\tau(n))} \\ &= \frac{(e-1)^3}{1000\pi e^{3n}} 2^{-(n+\tau(n))} \end{aligned}$$

so 4 holds. This implies that $s_x(t_n) \to 0$ as $n \to \infty$, whence 5 follows from 1,2, and 3. ∎

The following lemma analyzes the vertical component of $\vec{s}$. We use the notation of Observation 3.3, with the additional proviso that $d_n^{(n)} = t_n$.

Lemma 3.6. *1. For all $n \in \mathbb{Z}^+$ and $t \in (t_{n-1}, d_0^{(n)})$, $v_y(t) > 0$.*

2. For all $n \in \mathbb{Z}^+$, $0 \le i < n$, and $t \in (d_i^{(n)}, e_i^{(n)})$, $v_y(t) < 0$.

3. For all $n \in \mathbb{Z}^+$, $0 \le i < n$, and $t \in (e_i^{(n)}, d_{i+1}^{(n)})$, $v_y(t) > 0$.

4. For all $n \in \mathbb{Z}^+$, $0 \le i < n$, and $t \in \{e_i^{(n)}, d_i^{(n)}, t_n\}$, $v_y(t) = 0$.

5. For all $n \in \mathbb{Z}^+$ and $0 \le i \le n$, $s_y(d_i^{(n)}) = s_y(d_0^{(n)})$.

6. For all $n \in \mathbb{Z}^+$ and $0 \le i < n$, $s_y(e_i^{(n)}) = s_y(e_0^{(n)})$.

7. For all $n \in \mathbb{N}$, $s_y(t_n) = \frac{5^3(e-1)^3}{6^3 \cdot 8\pi} \sum_{i=1}^n \frac{1}{e^{3i}}$.

8. For all $n \in \mathbb{Z}^+$, $s_y(e_0^{(n)}) = s_y(t_n) - \frac{(e-1)^3}{12^3 n^3 8\pi e^{3n}}$.

9. $s_y(1) = \frac{5^3(e-1)^3}{6^3 \cdot 8\pi(e^3-1)}$.

Proof. Parts 1-6 are clear by inspection and induction. By 4. and Observation 3.4,

$$\begin{aligned} s_y(t_n) - s_y(t_{n-1}) &= s_y(d_0^{(n)}) - s_y(t_{n-1}) \\ &= \frac{[\frac{5}{6}(t_n - t_{n-1})]^3}{8\pi} = \frac{[\frac{5}{6}((e-1)e^{-n})]^3}{8\pi} \\ &= \frac{5^3(e-1)^3}{6^3 \cdot 8\pi e^{3n}} \end{aligned}$$

for all $n \in \mathbb{Z}^+$, so 6 holds by induction. Also by 4 and Observation 3.4,

$$\begin{aligned} s_y(t_n) - s_y(e_0^{(n)}) &= s_y(d_0^{(n)}) - s_y(e_0^{(n)}) \\ &= \frac{[\frac{1}{12n}(t_n - t_{n-1})]^3}{8\pi} = \frac{[\frac{1}{12n}((e-1)e^{-n})]^3}{8\pi} \\ &= \frac{(e-1)^3}{12^3 n^3 8\pi e^{3n}}, \end{aligned}$$

so 7 holds. Finally, by 6,

$$s_y(1) = \frac{5^3(e-1)^3}{6^3 8\pi(e^3-1)},$$

i.e., 8 holds. □

By Lemmas 3.5 and 3.6, we see that $\vec{s}$ parametrizes a curve from $\vec{s}(0) = (0,0)$ to $\vec{s}(1) = (0, \frac{5^3(e-1)^3}{6^3 8\pi(e^3-1)})$.

It is clear from Observation 3.3 and Lemmas 3.5 and 3.6 that the curve Γ does not intersect itself. We thus have the following.

Corollary 3.7. Γ *is a simple curve from $\vec{s}(0) = (0,0)$ to $\vec{s}(1) = (0, \frac{5^3(e-1)^3}{6^3 8\pi(e^3-1)})$.*

Proof. Let $\vec{s}' : [0,1] \to \mathbb{R}^2$ be such that

$$\vec{s}'(t) = \begin{cases} \vec{s}(t_n^+)\frac{t-t_n^-}{t_n^+-t_n^-} + \vec{s}(t_n^-)\frac{t_n^+-t}{t_n^+-t_n^-} & t \in (t_n^-, t_n^+), n \notin K, \\ \vec{s}(t) & \text{otherwise.} \end{cases}$$

Note that by construction of $\vec{s}$, retracing happens along y-axis between $(0, s_y(t_n^-))$ and $(0, s_y(t_n^+))$ only when $t \in (t_n^-, t_n^+)$ for $n \notin K$. In $\vec{s}'$, for all $n \notin K$, $\vec{s}'$ maps (t_n^-, t_n^+) to the vertical line segment between $(0, s_y(t_n^-))$ and $(0, s_y(t_n^+))$ linearly. Otherwise, $\vec{s}'(t) = \vec{s}(t)$. Hence, $\vec{s}'(0) = (0,0)$, $\vec{s}'(1) = (0, \frac{5^3(e-1)^3}{6^3 8\pi(e^3-1)})$, and $\vec{s}'$ is a one-to-one parametrization of $\Gamma = \text{range}(\vec{s})$, although $\vec{s}'$ is not computable. Therefore Γ is a simple curve. $\square$

Lemma 3.8. *The functions $\vec{a}, \vec{v}$, and $\vec{s}$ are Lipschitz, hence continuous, on $[0,1]$.*

Proof. It is clear by differentiation that $Lip(\varphi_{a,b}) = \frac{\pi}{2}$ for all $a, b \in \mathbb{R}$ with $a < b$. It follows by inspection that $Lip(a_x) \le \frac{\pi}{4}$ and $Lip(a_y) = \frac{\pi}{2}$, whence

$$Lip(\vec{a}) \le \sqrt{Lip(a_x)^2 + Lip(a_y)^2} \le \frac{\pi\sqrt{5}}{4}.$$

Thus $\vec{a}$ is Lipschitz, hence continuous (and bounded), on $[0,1]$. It follows immediately that $\vec{v}$ and $\vec{s}$ are Lipschitz, hence continuous, on $[0,1]$. $\square$

Since every Lipschitz parametrization has finite total length [1], and since the length of a curve cannot exceed the total length of any of its parametrizations, we immediately have the following.

Corollary 3.9. *The total length, including retracings, of the parametrization $\vec{s}$ is finite. Hence the curve $\mathbf{\Gamma}$ is rectifiable.*

Lemma 3.10. *The curve $\mathbf{\Gamma}$ is smooth except at the endpoint $\vec{s}(1)$.*

Proof. We have seen that $\mathbf{\Gamma}([0, t_1^-])$ is simply a segment of the y-axis, and that the vector velocity function $\vec{v}$ is continuous on $[0,1]$. Since the set

$$Z = \{t \in (0,1) \mid \vec{v}(t) = 0\}$$

has no accumulation points in $(0,1)$, it therefore suffices to verify that, for each $t^* \in Z$,

$$\lim_{t \to t^{*-}} \frac{\vec{v}(t)}{|\vec{v}(t)|} = \lim_{t \to t^{*+}} \frac{\vec{v}(t)}{|\vec{v}(t)|}, \tag{3.1}$$

i.e., that the left and right tangents of $\mathbf{\Gamma}$ coincide at $\vec{s}(t^*)$. But this is clear, because Lemmas 3.5 and 3.6 tell us that

$$Z = \left\{ t_n \;\middle|\; n \in \mathbb{Z}^+ \text{ and } \tau(n) = \infty \right\},$$

and both sides of (3.1) are $(0,1)$ at all t^* in this set. $\square$

Lemma 3.11. *The functions $\vec{a}, \vec{v}$, and $\vec{s}$ are computable in polynomial time. The total length including retracings, of $\vec{s}$ is computable in polynomial time.*

Proof. This follows from Observation 3.4, Lemmas 3.5 and 3.6, and the polynomial time computability of $f(n) = \sum_{i=1}^n e^{-3i}$. $\square$

Note that, barring the unlikely possibility that all #P functions are computable in polynomial time, the integrals of polynomial-time computable functions are not always polynomial-time computable [14]. However, in our case, the functions are defined piecewise. For each piece, we showed that the integrals are easily computable by giving polynomial-time computable closed form formula for them. Since the piece of the function decays exponentially, to compute these integrals, we only need to sum the first polynomially many pieces to a polynomial precision.

Definition. A *modulus of uniform continuity* for a function $f : [a, b] \to \mathbb{R}^n$ is a function $h : \mathbb{N} \times \mathbb{N}$ such that, for all $s, t \in [a, b]$ and $r \in \mathbb{N}$,

$$|s - t| \leq 2^{-h(r)} \implies |f(s) - f(t)| \leq 2^{-r}.$$

It is well known (e.g., see [14]) that every computable function $f : [a, b] \to \mathbb{R}^n$ has a modulus of uniform continuity that is computable.

The next lemma is the crucial one. Part (4) of Construction 3.1 encodes the halting problem K into the retracing/non-retracing behavior of our parametrization of $\mathbf{\Gamma}$. We now show that *any* parametrization of $\mathbf{\Gamma}$ that has bounded retracing *must* entail a solution to K.

Lemma 3.12. *Let $f : [a, b] \to \mathbb{R}^2$ be a parametrization of $\mathbf{\Gamma}$. If f has bounded retracing and a computable modulus of uniform continuity, then* $\mathrm{K} \leq_{\mathrm{T}} f_y$*, where f_y is the vertical component of f.*

Proof. Assume the hypothesis. Then there exist $m \in \mathbb{Z}^+$ and $h : \mathbb{N} \to \mathbb{N}$ such that f does not have m-fold retracing and h is a computable modulus of uniform continuity for f. Note that h is also a modulus of uniform continuity for f_y.

input $n \in \mathbb{Z}^+$;
if $n \leq m$ **then**
use a finite lookup table to accept if $n \in \mathrm{K}$ and reject if $n \notin \mathrm{K}$
else
begin
  r:= the least positive integer such that $2^{3-r} < s_y(t_n) - s_y(e_0^{(n)})$;
  Δ:=$2^{-h(r)}$;
  for $0 \leq j \leq (b - a)/\Delta$ **do**
  begin
    τ_j:=$a + \Delta \cdot j$;
    call j **high** if $|\mathcal{O}_g(\tau_j, r) - s_y(t_n)| < 2^{1-r}$
    call j **low** if $|\mathcal{O}_g(\tau_j, r) - s_y(e_0^{(n)})| < 2^{1-r}$
  end;
  if there exist $0 < j_0 < \cdots < j_m$ such that j_i is high for all even i and low for all odd i
  then accept
  else reject
end.

Figure 3.3: Algorithm for $M^{\mathcal{O}_g}(n)$ in the proof of Lemma 3.12.

Let M be an oracle Turing machine that, given an oracle $\mathcal{O}_g$ for a function $g : [a, b] \to \mathbb{R}$, implements the algorithm in Figure 3.3. The key properties of this algorithm's choice of r and Δ are that the following hold when $g = f_y$.

(i) For each time t with $f_y(t) = s_y(t_n)$, there is a nearby time τ_j with j high. Similarly for $f_y(t) = s_y(e_0^{(n)})$ and j low.

(ii) For each high j, $|f_y(\tau_j) - s_y(t_n)| \leq 3 \cdot 2^{-r}$. Similarly for each low j and $s_y(e_0^{(n)})$.

(iii) No j can be both high and low.

Now let $n \in \mathbb{Z}^+$. We show that $M^{\mathcal{O}_{f_y}}(n)$ accepts if $n \in \mathrm{K}$ and rejects if $n \notin \mathrm{K}$. This is clear if $n \leq m$, so assume that $n > m$.

If $n \in \mathrm{K}$, then Observation 3.3, Lemma 3.5, and Lemma 3.6 tell us that $M^{\mathcal{O}_{f_y}}(n)$ accepts. If $n \notin \mathrm{K}$, then the fact that f does not have m-fold retracing tells us that $M^{\mathcal{O}_{f_y}(n)}$ rejects. □

Proof of Theorem 3.2. Part 1 follows from Lemmas 3.8 and 3.11. Part 2 follows from Lemma 3.11. Part 3 follows from Corollaries 3.7 and 3.9 and Lemma 3.10. Part 4 follows from Lemma 3.12, the fact that every computable function $g : [a, b] \to \mathbb{R}^2$ has a computable modulus of uniform continuity, and the fact that A is decidable wherever $A \leq_{\mathrm{T}} g$ and g is computable. □

4 Lower Semicomputability of Length

In this section we prove that every computable curve Γ has a lower semicomputable length. Our proof is somewhat involved, because our result holds even if every computable parametrization of Γ is retracing.

Construction 4.1. Let $f : [0, 1] \to \mathbb{R}^n$ be a computable function. Given an oracle Turing machine M that computes f and a computable modulus $m : \mathbb{N} \to \mathbb{N}$ of the uniform continuity of f, the (M, m)-cautious polygonal approximator of range(f) is the function $\pi_{M,m} : \mathbb{N} \to \{\textit{polygonal paths}\}$ computed by the following algorithm.

> **input** $r \in \mathbb{N}$;
> $S := \{\}$; // S may be a multi-set
> **for** i:=0 **to** $2^{m(r)}$ **do**
> $a_i := i2^{-m(r)}$;
>   use M to compute the first rational x_i with
> $|x_i - f(a_i)| \leq 2^{-(r+m(r)+1)}$;
>   add x_i to S;
> output a longest path inside a minimum spanning tree of S.

Definition. Let (X, d) be a metric space. Let $\Gamma \subseteq X$ and $\epsilon > 0$. Let

$$\Gamma(\epsilon) = \left\{ p \in X \;\middle|\; \inf_{p' \in \Gamma} d(p, p') \leq \epsilon \right\}$$

be the *Minkowski sausage* of Γ with radius ϵ.

Let $d_{\mathrm{H}} : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}$ be such that for all $\Gamma_1, \Gamma_2 \in \mathcal{P}(X)$

$$d_{\mathrm{H}}(\Gamma_1, \Gamma_2) = \inf \{\epsilon \mid \Gamma_1 \subseteq \Gamma_2(\epsilon) \text{ and } \Gamma_2 \subseteq \Gamma_1(\epsilon)\}.$$

Note that d_{H} is the *Hausdorff distance* function.

Let $\mathcal{K}(X)$ be the set of nonempty compact subsets of X. Then $(\mathcal{K}(X), d_{\mathrm{H}})$ is a metric space [6].

Theorem 4.2. *(Frink [8], Michael [18]). Let (X, d) be a compact metric space. Then $(\mathcal{K}(X), d_{\mathrm{H}})$ is a compact metric space.*

Definition. Let $\mathcal{RC}$ be the set of all simple rectifiable curves in $\mathbb{R}^n$.

Theorem 4.3. *([22] page 55). Let $\Gamma \in \mathcal{RC}$. Let $\{\Gamma_n\}_{n\in\mathbb{N}} \subseteq \mathcal{RC}$ be a sequence of rectifiable curves such that $\lim_{n\to\infty} d_{\mathrm{H}}(\Gamma_n, \Gamma) = 0$. Then $\mathcal{H}^1(\Gamma) \le \liminf_{n\to\infty} \mathcal{H}^1(\Gamma_n)$.*

This theorem has the following consequence.

Theorem 4.4. *Let $\Gamma \in \mathcal{RC}$. For all $\epsilon > 0$, there exists $\delta > 0$ such that for all $\Gamma' \in \mathcal{RC}$, if $d_{\mathrm{H}}(\Gamma, \Gamma') < \delta$, then $\mathcal{H}^1(\Gamma') > \mathcal{H}^1(\Gamma) - \epsilon$.*

In the following, we prove a few technical lemmas that lead to Lemma 4.9, which plays an important role in proving Theorem 4.10.

Lemma 4.5. *Let $\Gamma \in \mathcal{RC}$. Let $p_0, p_1 \in \Gamma$ be its two endpoints. Let $\Gamma' \subsetneq \Gamma$ such that $p_0, p_1 \in \Gamma'$. Then $\Gamma' \notin \mathcal{RC}$.*

Proof. If Γ' is not closed, then we are done. Assume that Γ' is closed. Let γ be a one-to-one parametrization of Γ such that $\gamma(0) = p_0$ and $\gamma(1) = p_1$.

Since $\Gamma' \neq \Gamma$ and $p_0, p_1 \in \Gamma'$, $\gamma^{-1}(\Gamma') \subseteq I_0 \cup I_1$ for some $I_0 \subseteq [0,1]$ and $I_1 \subseteq [0,1]$ that are closed and disjoint.

And thus, for any continuous function $\gamma' : [0,1] \to \mathbb{R}^n$, $\gamma'^{-1}(\gamma(I_0))$ and $\gamma'^{-1}(\gamma(I_1))$ are closed and disjoint. Therefore, for any continuous function $\gamma' : [0,1] \to \mathbb{R}^n$, $\gamma'^{-1}(\Gamma') \neq [0,1]$, i.e., $\Gamma' \notin \mathcal{RC}$. □

Lemma 4.6. *Let $\Gamma \in \mathcal{RC}$. Let $\Gamma' \subseteq \Gamma$ be a connected compact set. Then $\Gamma' \in \mathcal{RC}$.*

Proof. Let γ be a one-to-one parametrization of Γ.

Let $a = \inf\{\gamma^{-1}(\Gamma')\}$ and let $b = \sup\{\gamma^{-1}(\Gamma')\}$.

Let $\gamma' : [0,1] \to \mathbb{R}^n$ be such that for all $t \in [0,1]$

$$\gamma'(t) = \gamma(a + t(b - a)).$$

Then γ' defines a curve and we show that $\gamma'([0,1]) = \Gamma'$.

It is clear that $\Gamma' \subseteq \gamma'([0,1])$. Since Γ' is compact, we know that $\gamma'(0), \gamma'(1) \in \Gamma'$.

Suppose for some $t' \in (0,1)$, $\gamma'(t') \notin \Gamma'$. Since Γ' is compact, there exists $\epsilon > 0$ such that $\gamma'([t'-\epsilon, t'+\epsilon]) \cap \Gamma' = \varnothing$. Then $\Gamma' \subseteq \gamma'([0, t'-\epsilon)) \cup \gamma'((t'+\epsilon, 1])$. Since γ' is one-one,

$$d_{\mathrm{H}}(\gamma'([0, t'-\epsilon)), \gamma'((t'+\epsilon, 1])) > 0.$$

Hence,

$$d_{\mathrm{H}}(\Gamma' \cap \gamma'([0, t'-\epsilon)), \Gamma' \cap \gamma'((t'+\epsilon, 1])) > 0.$$

Thus, Γ' cannot be connected.

Therefore, if Γ' is connected, then $\Gamma' = \gamma'([0,1])$ and hence $\Gamma' \in \mathcal{RC}$. □

Lemma 4.7. *Let $\Gamma_0, \Gamma_1, \ldots$ be a convergent sequence of compact sets in compact metric space (X, d) that is eventually connected. Let $\Gamma = \lim_{n\to\infty} \Gamma_n$. Then Γ is connected.*

Proof. We prove the contrapositive.

Assume that Γ is not connected. Then there exists open sets $A, B \subseteq X$ such that $A \cap B = \varnothing$, $\Gamma \cap A \neq \varnothing$, $\Gamma \cap B \neq \varnothing$, and $\Gamma \subseteq A \cup B$.

Then $(\Gamma \cap A) \cap (\Gamma \cap B) = \varnothing$, thus $d_{\mathrm{H}}(\Gamma \cap A, \Gamma \cap B) > 0$. Let

$$\delta = d_{\mathrm{H}}(\Gamma \cap A, \Gamma \cap B).$$

Since $\lim_{n\to\infty} \Gamma_n = \Gamma$, let n_0 be such that for all $n \geq n_0$,

$$d_{\mathrm{H}}(\Gamma_n, \Gamma) \leq \tfrac{\delta}{3}.$$

It is clear that

$$(\Gamma \cap A)(\tfrac{\delta}{3}) \cap \Gamma_n \neq \varnothing,$$
$$(\Gamma \cap B)(\tfrac{\delta}{3}) \cap \Gamma_n \neq \varnothing,$$

and

$$\Gamma_n \subseteq (\Gamma \cap A)(\tfrac{\delta}{3}) \cup (\Gamma \cap B)(\tfrac{\delta}{3}).$$

By the definition of δ,

$$d_{\mathrm{H}}((\Gamma \cap A)(\tfrac{\delta}{3}), (\Gamma \cap B)(\tfrac{\delta}{3})) \geq \tfrac{\delta}{3}.$$

Thus Γ_n is not connected for all $n \geq n_0$. □

Lemma 4.8. *Let $\Gamma \in \mathcal{RC}$ and let $\gamma : [0, 1] \to \Gamma$ be a parametrization of Γ. Let*

$$L(\Gamma, \epsilon) = \inf \left\{ \mathcal{H}^1(\Gamma') \;\middle|\; \Gamma' \in \mathcal{RC} \text{ and } \Gamma' \subseteq \Gamma(\epsilon) \text{ and } \gamma(0), \gamma(1) \in \Gamma' \right\}.$$

Then

$$\lim_{\epsilon\to 0^+} L(\Gamma, \epsilon) = \mathcal{H}^1(\Gamma).$$

Proof. It is clear that $\lim_{\epsilon\to 0^+} L(\Gamma, \epsilon) \leq \mathcal{H}^1(\Gamma)$. It suffices to show that $\lim_{\epsilon\to 0^+} L(\Gamma, \epsilon) \geq \mathcal{H}^1(\Gamma)$.

Let $\delta > 0$. For each $i \in \mathbb{N}$, let

$$S_i = \left\{ \Gamma' \in \mathcal{RC} \;\middle|\; \Gamma' \subseteq \Gamma(\tfrac{1}{i}) \text{ and } \gamma(0), \gamma(1) \in \Gamma' \right\},$$

where γ is a parametrization of Γ. Note that if $i_2 < i_1$, then $S_{i_1} \subseteq S_{i_2}$. Let $\Gamma_0, \Gamma_1, \ldots$ be an arbitrary sequence such that for all $i \in \mathbb{N}$, $\Gamma_i \in S_{k_i}$, and $k_0, k_1, \cdots \in \mathbb{N}$ is a strictly increasing sequence.

Since for all $i \in \mathbb{N}$, Γ_i is compact and connected, by Theorem 4.2 and Lemma 4.7, there is at least one cluster point and every cluster point is a connected compact set. Let Γ' be a cluster point. It is clear that $\Gamma' \subseteq \Gamma$. Then by Lemma 4.6, $\Gamma' \in \mathcal{RC}$. It is also clear that $\gamma(0), \gamma(1) \in \Gamma'$ by definition of S_i. Thus by Lemma 4.5, $\Gamma' = \Gamma$.

By Theorem 4.3, $\liminf_{n\to\infty} \mathcal{H}^1(\Gamma_n) \geq \mathcal{H}^1(\Gamma') = \mathcal{H}^1(\Gamma)$. Then by Theorem 4.4, this implies that for all sufficiently large $i \in \mathbb{N}$,

$$(\forall \Gamma'' \in S_i)\mathcal{H}^1(\Gamma'') \geq \mathcal{H}^1(\Gamma) - \delta.$$

Therefore, for all sufficiently large $i \in \mathbb{N}$, $L(\Gamma, \frac{1}{i}) \geq \mathcal{H}^1(\Gamma) - \delta$. Since $\delta > 0$ is arbitrary,

$$\lim_{\epsilon\to 0^+} L(\Gamma, \epsilon) \geq \mathcal{H}^1(\Gamma).$$

□

Lemma 4.9. *Let $\Gamma \in \mathcal{RC}$ and let $f : [0,1] \to \Gamma$ be a parametrization of Γ. Let*

$$L(\Gamma, \epsilon, p_1, p_2) = \inf \left\{ \mathcal{H}^1(\Gamma') \;\middle|\; \Gamma' \in \mathcal{RC} \text{ and } \Gamma' \subseteq \Gamma(\epsilon) \text{ and } p_1, p_2 \in \Gamma' \right\}.$$

Then

$$\lim_{\epsilon \to 0^+} \sup_{p_1, p_2 \in \Gamma(\epsilon)} L(\Gamma, \epsilon, p_1, p_2) = \mathcal{H}^1(\Gamma).$$

Proof. For every $p \in \Gamma(\epsilon)$, there exists a point $p' \in \Gamma$ such that $\|p, p'\| \leq \epsilon$ and line segment $[p, p'] \subseteq \Gamma(\epsilon)$. Thus it is clear that for all $p_1, p_2 \in \Gamma(\epsilon)$, $L(\Gamma, \epsilon, p_1, p_2) \leq 2\epsilon + \mathcal{H}^1(\Gamma)$. Therefore,

$$\lim_{\epsilon \to 0^+} \sup_{p_1, p_2 \in \Gamma(\epsilon)} L(\Gamma, \epsilon, p_1, p_2) \leq \mathcal{H}^1(\Gamma).$$

For the other direction, observe that

$$\lim_{\epsilon \to 0^+} \sup_{p_1, p_2 \in \Gamma(\epsilon)} L(\Gamma, \epsilon, p_1, p_2) \geq \lim_{\epsilon \to 0^+} L(\Gamma, \epsilon).$$

Applying Lemma 4.8 completes the proof. □

Theorem 4.10. *Let $\Gamma \in \mathcal{RC}$ such that $\Gamma = \gamma([0,1])$, where γ is a continuous function. (Note that γ may not be one-one.) Let $S(a) = \{\gamma(x_i) \mid x_i \in a\}$ for all dissection a of $[0,1]$. Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of dissections of Γ such that*

$$\lim_{n \to \infty} \mathrm{mesh}(a_n) = 0.$$

Then

$$\lim_{n \to \infty} \mathcal{H}^1(LMST(a_n)) = \mathcal{H}^1(\Gamma),$$

where $LMST(a)$ is the longest path inside the minimum Euclidean spanning tree of $S(a)$ and $\mathrm{mesh}(a)$ *is the maximum of $|x_i - x_{i+1}|$ for all i such that $x_i, x_{i+1} \in a$.*

Proof. For all $n \in \mathbb{N}$, let

$$\epsilon_n = 2d_{\mathrm{H}}(\Gamma, S(a_n)).$$

Note that since γ is uniformly continuous and $\lim_{n \to \infty} \mathrm{mesh}(a_n) = 0$, $\lim_{n \to \infty} \epsilon_n = 0$.

Let $w = 2\epsilon_n$.

Claim. *Let T be a Euclidean spanning tree of $S(a)$. If T has an edge that is not inside $\Gamma(w)$, then T is not a minimum spanning tree.*

Proof of Claim. Let E be an edge of T such that $E \nsubseteq \Gamma(w)$. Then $\mathcal{H}^1(E) > 2w$. Removing E from T will break T into two subtrees T_1, T_2. By the definition of ϵ_n and the continuity of γ, there exists $s_1, s_2 \in S(a)$ with $\|s_1 - s_2\| \leq \epsilon_n$ such that $s_1 \in T_1$ and $s_2 \in T_2$.

It is clear that $T_1 \cup T_2 \cup \{(s_1, s_2)\}$ is also a Euclidean spanning tree of $S(a)$ and $\mathcal{H}^1(T_1 \cup T_2 \cup \{(s_1, s_2)\}) < \mathcal{H}^1(T)$, i.e., T is not minimum. □

Let T be a minimum Euclidean spanning tree of $S(a)$. Let L be the longest path inside T. Then $L \subseteq T \subseteq \Gamma(w)$. Note that $\mathcal{H}^1(L) \leq \mathcal{H}^1(\Gamma)$, because the length of the longest path inside T is no more than that of the the linearly-structured spanning tree formed by connecting the the points in

$S(a)$ in the order of the dissection a. Let p_0, p_1 be the two endpoints of Γ. Since L is the longest path inside T and p_0, p_1 are each within ϵ_n distance to some point in $S(a_n)$,

$$L(\Gamma, w, p_0, p_1) \leq 2\epsilon_n + \mathcal{H}^1(L).$$

By Lemma 4.9,

$$\lim_{w \to 0^+} L(\Gamma, w, p_0, p_1) = \mathcal{H}^1(\Gamma).$$

Then

$$\lim_{n\to\infty} \mathcal{H}^1(LMST(a_n)) = \mathcal{H}^1(\Gamma).$$

□

This result implies that when the sampling density is high, the number of leaves in the minimum spanning tree is asymptotically smaller than the total number of nodes.

For a rectifiable curve Γ, if we somehow have access to a one-to-one parametrization of it, we produce polygonal approximation of Γ by taking dissections of the parametrization. For each such dissection, the polygonal approximation is forms a minimum spanning tree over the dissecting points and it is actually the trivial longest path in the MST. By the previous theorem, even when we don't have access to a one-to-one parametrization, we can still use the length of the longest paths in MSTs over dissections of Γ to approximate the length of Γ. These observations give us the proof of the following main result of this section.

Theorem 4.11. *Let $\gamma : [0,1] \to \mathbb{R}^n$ be computable such that $\Gamma = \gamma([0,1]) \in \mathcal{RC}$. Then $\mathcal{H}^1(\Gamma)$ is lower semicomputable.*

Proof. Let the function f, M, and m in Construction 4.1 be γ, a computation of γ, and its computable modulus respectively.

For each input $r \in \mathbb{N}$, let $\pi_{M,m}(r)$ be the longest path L_r in $MST(S_r)$ (the minimum spanning tree of the S_r), where S_r is the set of points sampled by $\pi_{M,m}(r)$. Let $l_r = \mathcal{H}^1(L_r) - 2^{-r}$. Note that l_r is computable from $r \in \mathbb{N}$. We show that for all $r \in \mathbb{N}$, $l_r \leq \mathcal{H}^1(\Gamma)$ and $\lim_{r\to\infty} l_r = \mathcal{H}^1(\Gamma)$.

Let $\tilde{f}$ be a one-one parametrization of Γ. Let $\pi : \{0, \ldots, 2^{m(r)}\} \to \{0, \ldots, 2^{m(r)}\}$ be a permutation of $\{0, \ldots, 2^{m(r)}\}$ such that for all $i, j \in \{0, \ldots, 2^{m(r)}\}$,

$$i < j \implies \tilde{f}^{-1}(f(a_{\pi(i)})) < \tilde{f}^{-1}(f(a_{\pi(j)})).$$

Let $\hat{\Gamma}_r$ be the polygonal curve connecting the points $f(a_{\pi(0)}), f(a_{\pi(1)}), \ldots, f(a_{\pi(2^{m(r)})})$ in order. Then $\hat{\Gamma}_r$ is a polygonal approximation of Γ and $\mathcal{H}^1(\hat{\Gamma}_r) \leq \mathcal{H}^1(\Gamma)$. Let $\bar{\Gamma}_r$ be the polygonal curve connecting the points in S_r in the order of $x_{\pi(0)}, x_{\pi(1)}, \ldots, x_{\pi(2^{m(r)})}$.

Due to the approximation induced by the computation in Construction 4.1,

$$\mathcal{H}^1(\bar{\Gamma}_r) \leq \mathcal{H}^1(\hat{\Gamma}_r) + 2^{-r}.$$

Then it is clear that

$$\mathcal{H}^1(L_r) = \mathcal{H}^1(LMST(S_r)) \leq \mathcal{H}^1(\bar{\Gamma}_r) \leq \mathcal{H}^1(\hat{\Gamma}_r) + 2^{-r}.$$

Thus

$$l_r \leq \mathcal{H}^1(\hat{\Gamma}_r).$$

Let $\hat{S}_r = \{f(a_0), f(a_1), \ldots, f(a_{2^{m(r)}})\}$. Note that $\hat{S}_r$ may be a multi-set. By Theorem 4.10,

$$\lim_{r\to\infty} LMST(\hat{S}_r) = \mathcal{H}^1(\Gamma).$$

Let

$$\epsilon_r = 2d_{\mathrm{H}}(\Gamma, S_r).$$

By Contruction 4.1,

$$\lim_{r\to\infty} \epsilon_r = 0.$$

Let $w_r = 2\epsilon_r$, and let T_r be a minimum Euclidean spanning tree of S_r. Let L_r be the longest path inside T_r. By the Claim in Theorem 4.10, $L \subseteq T \subseteq \Gamma(w_r)$.

By an essentially identical argument as the one in the proof of Theorem 4.10,

$$\lim_{r\to\infty} l_r = \lim_{r\to\infty} \mathcal{H}^1(LMST(S_r)) = \mathcal{H}^1(\Gamma),$$

which completes the proof. □

5 Δ_2^0-Computability of the Constant-Speed Parametrization

In this section we prove that every computable curve Γ has a constant speed parametrization that is Δ_2^0-computable.

Theorem 5.1. *Any computable rectifiable curve has a constant speed parametrization that is computable in its length.*

More precisely, Theorem 5.1 can be rephrased as follows: Let $\Gamma = \gamma^*([0,1]) \in \mathcal{RC}$, where γ^* is computable. (γ^* may not be one-one.) Let $l = \mathcal{H}^1(\Gamma)$ and O_l be an oracle such that for all $n \in \mathbb{N}$, $|O_l(n) - l| \leq 2^{-n}$. Let f be a computation of γ^* with modulus m. Let γ be the constant speed parametrization of Γ. Then γ is computable with oracle O_l.

A constant speed parametrization of Γ requires the computation to allocate time for the parametrization to spend on traversing each piece of Γ in proportion to the length of the piece. The main obstacle in computing a constant speed parametrization of Γ lies in that it is impossible to estimate accurately the length of any part of Γ. Having access to the length of the curve allows us to approximate the curve to sufficient precision (by our choice) so that we have a good enough view of how long different parts of Γ are and hence remove the obstacle in computing a constant speed parametrization of Γ.

Proof. On input k as the precision parameter for computation of the curve and a rational number $x \in [0,1] \cap \mathbb{Q}$, we output a point $f_k(x) \in \mathbb{R}^n$ such that $|f_k(x) - \gamma(x)| \leq 2^{-k}$.

Without loss of generality, assume that $\mathcal{H}^1(\Gamma) > 1000 \cdot 2^{-k}$, and let $\delta = 2^{-(4+k)}$. Run f as in Construction 4.1 with increasingly larger precision parameter $r > -\log\delta$ until

$$\mathcal{H}^1(LMST(a)) > \mathcal{H}^1(\Gamma) - \tfrac{\delta}{2}$$

and the shortest distance between the two endpoints of $LMST(a)$ inside the polygonal sausage around $LMST(a)$ with width $2d = 2 \cdot 2^{-r}$ is at least $\mathcal{H}^1(\Gamma) - \frac{\delta}{2}$. This can be achieved by using Euclidean shortest path algorithms [12, 11].

Let $d_k \leq 2^{-(4+k)}$ be the largest d such that the above conditions are satisfied, which is assured by Theorem 4.11 and Lemma 4.9. Let $\mathcal{S}$ be the polygonal sausage around $LMST(a)$ with width $2d_k$. For $p_1, p_2 \in \mathcal{S}$, let $d_{\mathcal{S}}(p_1, p_2) =$ the shortest distance between p_1 and p_2 inside $\mathcal{S}$. Note that $\mathcal{S}$ is connected. Let f_k be the constant speed parametrization of $LMST(a)$ and γ be the constant speed parametrization of Γ. Without loss of generality, assume that $\|\gamma(0) - f_k(0)\| < \|\gamma(1) - f_k(0)\|$ and $\|\gamma(1) - f_k(1)\| < \|\gamma(0) - f_k(1)\|$, since we can hardcode approximate locations of $\gamma(0)$ and $\gamma(1)$ such that when d_k is sufficiently small, we can decide whether a sampled point is closer to $\gamma(0)$ or $\gamma(1)$. As we now prove

$$\lim_{k\to\infty} \{f_k(0), f_k(1)\} = \{\gamma(0), \gamma(1)\}.$$

Note that for each $s \in S$ such that $s \notin LMST(a)$, there exists $p \in LMST(a) \cap S$ such that the shortest path from s to p in $MST(a)$ has length less than $\frac{\delta}{2}$, i.e., $d_{MST(a)}(s,p) < \frac{\delta}{2}$, since $\mathcal{H}^1(LMST(a)) > \mathcal{H}^1(\Gamma) - \frac{\delta}{2}$ and $\mathcal{H}^1(MST(a)) \leq \mathcal{H}^1(\Gamma)$.

Let $\delta_0 = d_{\mathcal{S}}(\gamma(0), f_k(0))$. Let s_0 be the closest point to $\gamma(0)$ in $S \cap LMST(a)$. Then $d_{\mathcal{S}}(\gamma(0), s_0) \leq \frac{\delta}{2} + d_k$. Then $d_{LMST(a)}(s_0, f_k(0)) \geq \delta_0 - \frac{\delta}{2} - d_k$. Since $s_0 \in S \cap LMST(a)$ and we assume $\mathcal{H}^1(\Gamma) > 1000 \cdot 2^{-k}$,

$$d_{\mathcal{S}}(s_0, \gamma(1)) \leq \mathcal{H}^1(LMST(a)) - \delta_0 + \tfrac{\delta}{2} + d_k + \tfrac{\delta}{2} + d_k = \mathcal{H}^1(LMST(a)) - \delta_0 + \delta + 2d_k.$$

Then

$$\begin{aligned} d_{\mathcal{S}}(\gamma(0), \gamma(1)) &\leq \mathcal{H}^1(LMST(a)) - \delta_0 + \delta + 2d_k + \tfrac{\delta}{2} + d_k \\ &< \mathcal{H}^1(LMST(a)) - \delta_0 + \tfrac{3\delta}{2} + 3d_k. \end{aligned}$$

And hence

$$d_{\mathcal{S}}(\gamma(0), \gamma(1)) \leq \mathcal{H}^1(\Gamma) - \delta_0 + 2\delta + 3d_k. \tag{5.1}$$

By the choice of d_k, we have that $d_{\mathcal{S}}(f_k(0), f_k(1)) \geq \mathcal{H}^1(\Gamma) - \frac{\delta}{2}$. Now, note that for any two points $p_1, p_2 \in \Gamma$,

$$d_{\mathcal{S}}(p_1, p_2) \leq \frac{\mathcal{H}^1(\Gamma) + d_{\mathcal{S}}(\gamma(0), \gamma(1))}{2},$$

since we can put them in half of a loop. Therefore

$$d_{\mathcal{S}}(f_k(0), f_k(1)) \leq \frac{\mathcal{H}^1(\Gamma) + d_{\mathcal{S}}(\gamma(0), \gamma(1))}{2}.$$

Thus

$$d_{\mathcal{S}}(\gamma(0), \gamma(1)) \geq \mathcal{H}^1(\Gamma) - \delta. \tag{5.2}$$

By (5.1) and (5.2), we have

$$\delta_0 \leq 3\delta + 3d_k \leq 6\delta < 2^{-k}, \tag{5.3}$$

i.e.,

$$\|f_k(0) - \gamma(0)\| \leq d_{\mathcal{S}}(f_k(0), \gamma(0)) \leq 6\delta < 2^{-k}. \tag{5.4}$$

Similarly,

$$\|f_k(1) - \gamma(1)\| \leq d_{\mathcal{S}}(f_k(1), \gamma(1)) \leq 6\delta < 2^{-k}. \tag{5.5}$$

Now we proceed to show that for all $t \in (0,1)$, $\|f_k(t) - \gamma(t)\| < 10\delta$ with $f(0)$ being at most 6δ from $\gamma(0)$ inside $\mathcal{S}$ and $f(1)$ being at most 6δ from $\gamma(1)$ inside $\mathcal{S}$. Let $\Delta_k = \|f_k(t) - \gamma(t)\|$, and let $s_f \in S \cap LMST(a)$ be such that $|f_k^{-1}(s_f) - t|$ is minimized. Then $d_{LMST(a)}(f_k(t), s_f) \le d_k$, since every edge in $MST(a)$ is at most d_k long. Let $s'_\gamma \in S \cap \Gamma$ be such that $|\gamma^{-1}(s'_\gamma) - t|$ is minimized. Then $d_\Gamma(\gamma(t), s'_\gamma) \le d_k$, since we sample S using d_k as the density parameter. Let $s_\gamma \in S \cap LMST(a)$ such that $d_{MST(a)}(s_\gamma, s'_\gamma)$ is minimized. Then $d_{MST(a)}(s_\gamma, s'_\gamma) \le \frac{\delta}{2}$, since $\mathcal{H}^1(MST(a)) \ge \mathcal{H}^1(\Gamma) - \frac{\delta}{2}$. Then $\|f_k(t) - s_\gamma\| \ge \Delta_k - (\frac{\delta}{2} + d_k) = \Delta_k - \frac{\delta}{2} - d_k$. Note that $d_{LMST(a)}(s_f, s_\gamma) \ge \|s_f - s_\gamma\| \ge \Delta_k - \frac{\delta}{2} - 2d_k$.

Without loss of generality, assume that distance from s_γ to $f_k(0)$ along $LMST(a)$ is $\Delta_k - \frac{\delta}{2} - d_k$ more than the distance from $f_k(t)$ to $f_k(0)$. Otherwise, we simply look from the $\gamma(1)$ and $f_k(1)$ side instead. Now note the following.

(i) The path traced by γ from $\gamma(0)$ to $\gamma(t)$ has length $t \cdot \mathcal{H}^1(\Gamma)$.

(ii) The shortest distance between $\gamma(t)$ to s_γ inside $\Gamma \cup MST(a)$ is at most $d_k + \frac{\delta}{2}$.

(iii) The path traced by f_k from s_γ to $f_k(1)$ has length

$$d_{LMST(a)}(s_\gamma, f_k(1)) \le \mathcal{H}^1(LMST(a)) - [t(\mathcal{H}^1(\Gamma) - \tfrac{\delta}{2}) - d_k + \Delta_k - \tfrac{\delta}{2} - d_k].$$

(iv) The shortest distance from $\gamma(1)$ to $f_k(1)$ inside $\mathcal{S}$ is at most 6δ.

It follows that the distance from $\gamma(0)$ to $\gamma(1)$ inside $\mathcal{S}$ is at most

$$\begin{aligned}
&t \cdot \mathcal{H}^1(\Gamma) + d_k + \tfrac{\delta}{2} + \mathcal{H}^1(LMST(a)) - [t(\mathcal{H}^1(\Gamma) - \tfrac{\delta}{2}) - d_k + \Delta_k - \tfrac{\delta}{2} - d_k] + 6\delta \\
&\le \mathcal{H}^1(LMST(a)) + 3d_k + 8\delta - \Delta_k \\
&\le \mathcal{H}^1(\Gamma) + 11\delta - \Delta_k.
\end{aligned}$$

By (5.2), we have

$$\Delta_k \le 12\delta < 2^{-k}.$$

□

Corollary 5.2. *Let Γ be a computable curve with the property described in property 4 of Theorem 3.2. Then the length of Γ, i.e., $\mathcal{H}^1(\Gamma)$, is not computable.*

Proof. We prove the contrapositive. Let Γ be a curve with a computable parametrization with a computable length $\mathcal{H}^1(\Gamma)$. Then by Theorem 5.1, we can use the Turing machine that computes $\mathcal{H}^1(\Gamma)$ as the oracle in the statement of Theorem 5.1 and obtain a Turing machine that computes the constant speed parametrization of Γ. Therefore, Γ does not have the property described in item 4 of Theorem 3.2. □

6 Conclusion

As we have noted, Ko [15] has proven the existence of computable curves with finite, but uncomputable lengths, and the curve $\mathbf{\Gamma}$ of our main theorem is one such curve. In the recent paper [10], we have given a precise characterization of those points in $\mathbb{R}^n$ that lie on computable curves of finite length. With these things in mind, an earlier draft of this paper posed the following.

Question. Is there a point $x \in \mathbb{R}^n$ such that x lies on a computable curve of finite length but not on any computable curve of computable length?

Rettinger and Zheng [21] have recently answered this question affirmatively.

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