On controllability of timed continuous Petri nets

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Abstract. Continuous Petri Nets is a subclass of hybrid models representing relaxed views of discrete events systems, in which timing may adopt different semantics. Even if no semantics is strictly superior, we proved in [1] that for an important subclass of models infinite servers semantics provides always a better approximation of the of the underlying discrete model than finite servers. This paper then concentrates on controllability under this semantics. First we propose a notion of controllability over subsets of the reachable polytope, and provide a necessary and sufficient condition for markings with no null elements (interior points); later the transformation of an arbitrary initial marking into an interior one is done. The technically more involved part of the paper is the extension of those results to the case in which some transitions are non controllable. An interesting point is that all characterisations are structural (i.e., depend only on the structure and firing speeds of the timed continuous net).

1 Introduction

Petri Nets constitute a well-known paradigm useful to model discrete event systems. In many practical cases, an enumeration approach has to be used to verify some properties of net models. Unfortunately, for highly marked systems, even for bounded, the reachability graph can be so large that many properties cannot be analyzed. This problem is known as the state explosion problem. Systems that frequently appear in practice, for instance in manufacturing, telecommunications, traffic or logistic, lead to Petri net models with many states. So, to analyze such systems fluidification has been proposed.

Fluidification constitutes a relaxation technique to study discrete systems through a "similar" but continuous model. Using fluid models, more analytical techniques can be used for the analysis of some interesting properties. In Petri Nets, fluidification has been introduced from different perspectives ([2, 3]). Here we consider the approach adopted in [4]. In this work, timed continuous Petri net (TCPN) models under infinite server semantics are considered. The continuous model thus obtained is piecewise linear with bounded and nonnegative inputs.

In recent years, a lot of research has been done on controllability of switched linear systems. For instance, [5] and [6] give sufficient and necessary conditions for controllability of 3-dimensional systems and single switching sequence systems, respectively, but always under the assumption of unconstrained inputs. Classic works ([7]) and [8] deal with controllability on linear systems (non piecewise) with bounded and nonnegative inputs, respectively. Camlibel [9] has extended the results of Brammer to a particular class of piecewise linear systems known as linear complementary systems. In [10] optimal control of switched piecewise affine autonomous systems is studied, assuming that the decision variables are the switching instants and the sequence of operating modes.

However, in timed continuous Petri net (TCPN) systems switching is not controllable. Moreover, in [11] it is shown that these systems are not controllable in the classical sense. In [12] it is proven that for Join Free Timed Continuous Petri Nets there exists an invariant set, named Controllability Space (CS), in which the system exhibits the controllability property, i.e. any state of CS is reachable from any other state of CS. In that work, the set CS is characterized.

Here the study of controllability and reachability properties for general timed continuous Petri nets under infinite server semantics is addressed. A controllability notion is presented for TCPN systems. It deals with the possibility that the state evolves from any state, of a given set, to another; so it is an appropriate adaptation of the classical controllability concept of linear continuous systems (see [13]). Based on this controllability definition, the particular structure of TCPN systems allows to obtain a structural characterization of TCPN exhibiting this property. It is worth to remark that this property does not depend on the initial marking, but on the structure and timing of the net as it is proved in the sequel.

This work is organized as follows: in Section 2 an overview of continuous and timed continuous Petri nets is presented, while in Section 3, a concept of controllability is formally introduced. In Section 4, necessary and sufficient conditions for controllability are given, under the hypothesis that all transitions are controllable, while the controllability of systems with uncontrollable transitions is studied in Section 5. Finally, some conclusions are presented in Section 6.

2   Basic concepts

The structure $\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post} \rangle$ of *continuous Petri nets* is the same as the structure of discrete PN. That is, P is a finite set of places, T is a finite set of transitions with $P \cap T = \emptyset$, $\mathbf{Pre}$ and $\mathbf{Post}$ are $|P| \times |T|$ sized, natural valued, *pre- and post- incidence matrices*. The main difference is in the evolution rule, since in continuous PN firing is not restricted to be done in integer amounts, and so the marking is not forced to be integer. More precisely, a transition t is *enabled* at $\mathbf{m}$ iff for every $p \in {}^{\bullet}t$, $\mathbf{m}[p] > 0$, and its *enabling degree* is $enab(t, \mathbf{m}) = \min_{p \in {}^{\bullet}t} \{ \mathbf{m}[p] / \mathbf{Pre}[p, t] \}$. The firing of t in a certain amount $\alpha \leq enab(t, \mathbf{m})$ leads to a new marking $\mathbf{m}' = \mathbf{m} + \alpha \cdot \mathbf{C}[P, t]$, where $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$ is the token-flow matrix. Right and left rational annullers of $\mathbf{C}$ are called *T*-

and *P-flows*, respectively. If there exists $\mathbf{y} > \mathbf{0}$ ($\mathbf{x} > \mathbf{0}$) such that $\mathbf{y} \cdot \mathbf{C} = \mathbf{0}$ ($\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$), the net is said to be *conservative* (*consistent*). A set of places Σ is a *siphon* iff $^{\bullet}\Sigma \subseteq \Sigma^{\bullet}$ (the set of input transitions is contained in the set of output transitions). For reachability, as in [14], the limit concept is used, and a marking reached in the limit of an infinitely long sequence is considered reachable.

For the timing interpretation we will use a first order (or deterministic) approximation of the discrete case ([4]). Hence, a *Timed Continuous Petri Net* (TCPN) is a continuous PN together with a vector $\boldsymbol{\lambda} \in \mathbb{R}^{|T|}_{>0}$. Here infinite server semantics is considered, thus the flow through a timed transition t is the product of the speed, $\boldsymbol{\lambda}[t]$, and $enab(t, \mathbf{m})$, the instantaneous enabling, i.e., $\mathbf{f}(\mathbf{m})[t] = \boldsymbol{\lambda}[t] \cdot enab(t, \mathbf{m}) = \boldsymbol{\lambda}[t] \cdot \min_{p \in \bullet t}\{\mathbf{m}[p]/\mathbf{Pre}[p, t]\}$. For the flow to be well defined, every transition must have at least one input place, hence in the following we will assume $\forall t \in T, |^{\bullet}t| \geq 1$. The "*min*" in the definition leads to the concept of *configurations*: a configuration assigns to a transition one place that for some markings will control its firing rate (i.e. it is constraining that transition). A good upper bound for the number of configurations is $\prod_{t \in T} |^{\bullet}t|$.

The flow through the transitions can be written in a vectorial form as $\mathbf{f}(\mathbf{m}) = \boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbf{m}$ (see [11]), where $\boldsymbol{\Lambda}$ is a diagonal matrix whose elements are those of $\boldsymbol{\lambda}$, and $\boldsymbol{\Pi}(\mathbf{m})$ is the configuration operator matrix, defined by elements as

$$\boldsymbol{\Pi}(\mathbf{m})[i, j] = \begin{cases} \frac{1}{\mathbf{Pre}[p_j, t_i]} & \text{if } p_j \text{ is constrainting } t_i \\ 0 & \text{otherwise} \end{cases}$$

If more than one place is constraining the flow of a transition at a given marking, any of them can be used, but only one is taken.

Control action may only be a reduction of the flow through the transitions. Transitions in which a control action can be applied are called controllable. The effective flow through a controllable transition can be represented as: $\mathbf{f}_i(\tau) = \boldsymbol{\lambda}(t_i) \cdot enab(\tau)[t_i] - u(\tau)[t_i]$, where $0 \leq u(\tau)[t_i] \leq \lambda(t_i) \cdot enab(\tau)[t_i]$.

The control vector $\mathbf{u} \in \mathbb{R}^{|T|}$ is defined such that $\mathbf{u}_i$ represents the control action on t_i. If t_i is not controllable then $\mathbf{u}_i = 0$. The set of all controllable transitions is denoted by T_c, and the set of uncontrollable transitions is denoted by T_{nc}. Then, $T_c \cap T_{nc} = \varnothing$ and $T_c \cup T_{nc} = T$.

The behavior of a TCPN forced system is described by the state equation:

$$\begin{aligned} \overset{\bullet}{\mathbf{m}} &= \mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbf{m} - \mathbf{C}\mathbf{u} \\ \mathbf{0} &\leq \mathbf{u} \leq \boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbf{m} \end{aligned} \tag{1}$$

Given a marking trajectory, an input $\mathbf{u}(\mathbf{m})$ (as a function of $\mathbf{m}$) such that $\mathbf{0} \leq \mathbf{u}(\mathbf{m}) \leq \boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbf{m}$, and $\forall t_i \in T_{nc} \; \mathbf{u}_i = 0$ along the marking trajectory, is called *suitably bounded*. Notice that if an input is not suitably bounded for a marking trajectory, then it cannot be applied in this. A marking $\mathbf{m}$ for which $\exists \mathbf{u}$ suitably bounded such that $\mathbf{C}[\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbf{m} - \mathbf{u}] = \mathbf{0}$ is called *equilibrium marking*.

Marking $\mathbf{m}_2$ is said to be reachable from $\mathbf{m}_1$ if there exists an input $\mathbf{u}$ that transfers the marking from $\mathbf{m}_1$ to $\mathbf{m}_2$ in either finite or infinite time (lim-reachable) and it is suitably bounded. A marking reachable from the initial

one is simply called reachable. The set of reachable markings can be defined for autonomous continuous PN and TCPN systems [14]. In the sequel, the term reachability *always* refer to timed systems.

3   Controllability definition

Whenever a TCPN system has *P-flows*, linear dependencies between marking variables appear, introducing state invariants. So, systems with P-flows are not controllable in the classical sense [11]. However, we are interested in the study of controllability "over" this invariant. In the sequel, we refer to this state invariant as $Class(\mathbf{m_0})$, since it is the equivalence class of $\mathbf{m_0}$ under the relation β defined as: $\mathbf{m_1}\beta\mathbf{m_2}$ iff $\mathbf{B_y}^T\mathbf{m_1} = \mathbf{B_y}^T\mathbf{m_2}$, where $\mathbf{B_y}$ is a basis of P-flows.

Notice that, for a general TCPN system, every reachable marking belongs to $Class(\mathbf{m_0})$. The set $Class(\mathbf{m_0})$ can be divided into subsets of markings associated to the same configuration, which are named *regions* and are denoted by $\Re_i = \{\mathbf{m} \in Class(\mathbf{m_0})|\mathbf{\Pi(m)} = \mathbf{\Pi}_i\}$. Notice that such regions are convex sets, and inside each one, the state equation (1) is linear ($\mathbf{\Pi(m)}$ is constant). In the sequel, let us denote by $int(Class(\mathbf{m_0}))$ $(int(\Re_i))$ the set of interior markings of $Class(\mathbf{m_0})$ $(\Re_i)$ considering the space generated by the columns of $\mathbf{C}$.

Next, linear systems controllability definition is recalled [13].

Definition 1. *A state equation is fully controllable if there exists an input such that for any two states $\mathbf{x}_1$ and $\mathbf{x}_2$ of the state space, it is possible to transfer the state from $\mathbf{x}_1$ to $\mathbf{x}_2$ in finite time.*

Notice that this definition cannot be applied to TCPN systems because the set of reachable markings never compose a vector space, as it is inside $Class(\mathbf{m_0})$. Moreover, in TCPN systems the input must be suitably bounded (i.e. $\mathbf{0} \leq \mathbf{u} \leq \mathbf{\Lambda\Pi(m)m}$, $\mathbf{u}_i = 0, \forall t_i \in T_{nc}$). Therefore, the following adaptation of the classical controllability definition is proposed.

Definition 2. *The TCPN system $\langle\mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0}\rangle$ is controllable with bounded input (BIC) over $S \subseteq Class(\mathbf{m_0})$ if for any two markings $\mathbf{m}_1, \mathbf{m}_2 \in S$ there exists an input $\mathbf{u}_{12}$ that transfers the system from $\mathbf{m}_1$ to $\mathbf{m}_2$ in finite or infinite time, and it is suitably bounded (i.e. $\mathbf{0} \leq \mathbf{u}_{12} \leq \mathbf{\Lambda\Pi(m)m}$, and $\forall t_i \in T_{nc}$ $u_i = 0$ along the marking trajectory).*

Besides, if $S = Class(\mathbf{m_0})$ then the system is said to be fully controllable with bounded input (BIFC).

It is important to remark that controllability is a *structural* property. Even when, under this definition, it is said that a system is controllable over some subset of $Class(\mathbf{m_0})$, is the dynamical behavior of the system, which is determined by the structure and timing, that makes the system be controllable or not.

4 The case where all transitions are controllable

In [14] reachability is studied for untimed continuous Petri net systems. An important result introduced in that paper is that a marking $\mathbf{m}$ is reachable iff $\exists \boldsymbol{\sigma} \geq \mathbf{0}$ such that $\mathbf{m} = \mathbf{m_0} + \mathbf{C}\boldsymbol{\sigma} \geq \mathbf{0}$ and the transitions in the support of $\boldsymbol{\sigma}$ are fireable. This result can be extended for TCPN in the following way:

Proposition 1. *Let* $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ *be a* TCPN *system in which* $T_c = T$. *A marking* $\mathbf{m_1} \in Class(\mathbf{m_0})$ *is reachable from* $\mathbf{m_0} \in int(Class(\mathbf{m_0}))$ *iff* $\exists \boldsymbol{\sigma} \geq \mathbf{0}$ *such that* $\mathbf{C}\boldsymbol{\sigma} = (\mathbf{m_1} - \mathbf{m_0})$.

The following proposition gives a necessary and sufficient condition for controllability over the interior of $Class(\mathbf{m_0})$.

Proposition 2. *Let* $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ *be a* TCPN *system, and let* S *be defined as* $S = \{\mathbf{m} \in Class(\mathbf{m_0}) | \mathbf{m} > \mathbf{0}\}$. *The system* $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ *is* BIC *over* S *iff the net is consistent.*

*Proof.* In [11] a reachability theorem is introduced, according to which, if the net is consistent then the system is BIC over S. Now, for the other implication, consider any vector $\mathbf{d} \in span(\mathbf{C})$ and a marking $\mathbf{m_1} \in S$. Then, there exists a scalar $\beta > 0$ such that $\mathbf{m_1} + \beta \mathbf{d} \geq \mathbf{0}$. Let $\mathbf{m_2} = \mathbf{m_1} + \beta \mathbf{d}$, then $\mathbf{m_2} \in Class(\mathbf{m_0})$. Since the system is BIC over the interior of $Class(\mathbf{m_0})$, $\mathbf{m_2}$ is a particular solution of the fundamental equation, so $(\mathbf{m_2} - \mathbf{m_1}) = \beta \mathbf{d} = \mathbf{C}\boldsymbol{\sigma}$, where $\boldsymbol{\sigma} \geq \mathbf{0}$. Therefore, $\forall \mathbf{d} \in span(\mathbf{C}), \exists \boldsymbol{\sigma}$ such that $\mathbf{C}\boldsymbol{\sigma} = \mathbf{d}$. Besides, this property implies that $\exists \mathbf{x} > \mathbf{0}$ such that $\mathbf{C}\mathbf{x} = \mathbf{0}$, i.e. the net is consistent. □

Notice that the condition for controllability (consistency) is purely structural. Actually, the TCPN is BIC over the interior of $Class(\mathbf{m_0})$ iff the TCPN is BIC over the interior of $Class(\mathbf{m_1})$, for every $\mathbf{m_1} \geq \mathbf{0}$. Next proposition gives a condition to transfer the marking from the border of $Class(\mathbf{m_0})$ to its interior.

Proposition 3. *Let* $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ *be a* TCPN *system. An input* $\mathbf{u}$, *such that every enabled transition is always fired, transfers the marking from* $\mathbf{m_0}$ *to some* $\mathbf{m}_f$, *where* $\mathbf{m}_f$ *has not null elements, iff there are not empty siphons at* $\mathbf{m_0}$.

Proof. Suppose that $\mathbf{m_0}$ has null elements. Let us define an input $\mathbf{u}$ such that for any enabled transition t_j, $\mathbf{u}_j < [\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbf{m}]_j$. So, a transition t_j is such that $[\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbf{m} - \mathbf{u}]_j = 0$ iff there is an input place to t_j, named p_i, without tokens. In the same way, p_i cannot win tokens iff there exist unmarked input places at all the input transitions of p_i. So, a place p_i has not tokens at time τ and remains without tokens for future time, iff for every input transition to that place there exists an input place without tokens for all time. Repeating this reasoning to these new input places, it can be seen that p_i remains unmarked iff it belongs to an unmarked siphon. □

The following theorem introduces necessary and sufficient conditions for controllability over $Class(\mathbf{m_0})$. The proof is immediate from previous propositions.
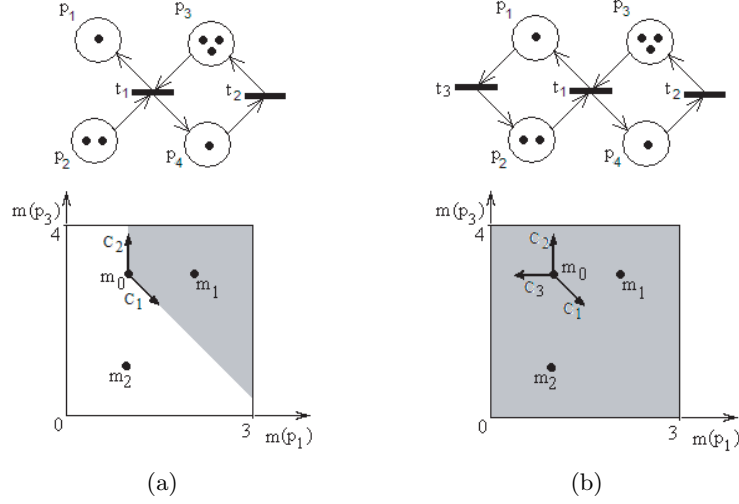
Fig. 1. (a) TCPN that is not BIFC. (b) A BIFC TCPN system.

Theorem 1. *The* TCPN *system* $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ *is* BIFC *iff* $\mathcal{N}$ *is consistent and there do not exist empty siphons at any marking in* $Class(\mathbf{m_0})$.

Example 1. Consider the TCPN systems of Figure 1, where $\mathbf{m}_0 = [1, 2, 3, 1]^T$, $\mathbf{m}_1 = [2, 1, 3, 1]^T$ and $\mathbf{m}_2 = [1, 2, 1, 3]^T$. Since both systems have 2 P-semiflows, only the marking of two places (in this example, places p_1 and p_3) are needed to represent the whole state.

Given the system in Figure 1(a), since there exists a vector $\boldsymbol{\sigma} \geq \mathbf{0}$ such that $\mathbf{C}\boldsymbol{\sigma} = (\mathbf{m}_1 - \mathbf{m_0})$, according to Proposition 1, $\mathbf{m}_1$ is reachable. On the other hand, since $\nexists \boldsymbol{\sigma} \geq \mathbf{0}$ such that $\mathbf{C}\boldsymbol{\sigma} = (\mathbf{m}_2 - \mathbf{m_0})$, then $\mathbf{m}_2$ is not reachable. Therefore it is not BIFC. The same conclusion (i.e. the system is not BIFC) can be obtained using Theorem 1. The shadowed area in Figure 1(a) corresponds to the set of reachable markings, notice that it is the convex defined by vectors $\mathbf{C}_1$ and $\mathbf{C}_2$, which represent the columns of $\mathbf{C}$ restricted to p_1 and p_3.

Now, consider the system of Figure 1(b). This system is consistent, so, according to Proposition 2, it is BIC over the interior of $Class(\mathbf{m_0})$. Therefore $\mathbf{m}_1$ and $\mathbf{m}_2$ are reachable from $\mathbf{m_0}$. Besides, since at the border markings of $Class(\mathbf{m_0})$ there are not unmarked siphons, according to Theorem 1, the system is BIFC.

5   Controllability with uncontrollable transitions

The study of controllability with uncontrollable transitions is more complicated than previous case. In this, consistency is no longer sufficient to guarantee controllability over the interior of $Class(\mathbf{m_0})$. Actually, due to the boundedness of the input, a system with uncontrollable transitions is never controllable over

$Class(\mathbf{m_0})$. So, in this case, it is first necessary to define a suitable set of markings, and then, study the controllability over it. In this work only sets of equilibrium markings are considered, because they represent the "stationary operating points" of the modeled system.

The set of all equilibrium markings is defined as:

$$E_q S = \{m \in Class(\mathbf{m_0}) | \exists \mathbf{u} \text{ suitably bounded such that } \mathbf{C}(\mathbf{\Lambda\Pi}(\mathbf{m})\mathbf{m} - \mathbf{u}) = 0\}$$

Notice that if all transitions are controllable then $E_q S = Class(\mathbf{m_0})$. The set of all equilibrium markings in the i-th region is defined as $E_i = \{\mathbf{m} | \mathbf{m} \in E_q S \cap \Re_i\}$.

In the sequel, the following notation is adopted. Let $\mathbf{m}^q \in E_q S$. An equilibrium input for $\mathbf{m}^q$ is a vector $\mathbf{u}^q$ such that $\mathbf{C}(\mathbf{\Lambda\Pi}(\mathbf{m}^q)\mathbf{m}^q - \mathbf{u}^q) = \mathbf{0}$ and it is suitably bounded. The equilibrium flow through the transitions for $\mathbf{m}^q$ and $\mathbf{u}^q$ is denoted as $\mathbf{w}^q$, i.e. $\mathbf{w}^q = \mathbf{\Lambda\Pi}(\mathbf{m}^q)\mathbf{m}^q - \mathbf{u}^q$.
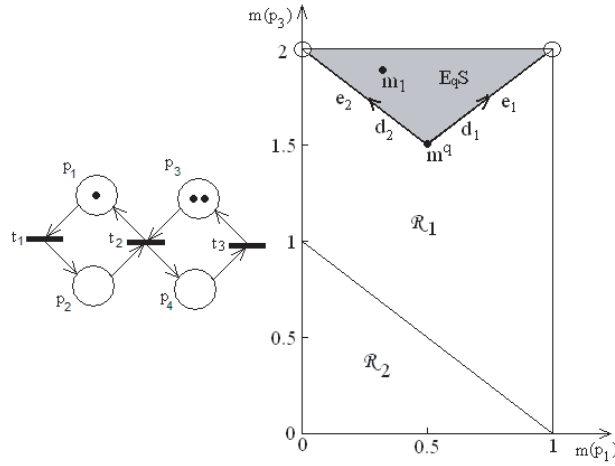
Fig. 2. The set $E_q S$, and subsets E_i, E_i^+ and E_i^*.

Example 2. Consider the system of figure 2 where $\mathbf{\Lambda} = \mathbf{I}$ and $T_c = \{t_1, t_2\}$. There exist two possible configurations: the configuration in which transition t_2 is constrained by place p_2, denoted by $\mathcal{C}_1$, and the configuration where t_2 is constrained by p_3, denoted by $\mathcal{C}_2$. $\Re_1$ and $\Re_2$ are the regions related to $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. The whole triangle with all its edges and vertices corresponds to E_1. Actually, in this example $E_1 = E_q S$. Since none of the equilibrium markings belongs to the second configuration, then $E_2 = \varnothing$.

Since the system is linear inside each region, we will first investigate the controllability over each E_i. For that, it is necessary to represent a given E_i in a matrix form. The following definition introduces this representation.

Definition 3. *Let* $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ *be a* TCPN *system. A* Generator *of* $E_i \neq \varnothing$ *is a full column rank matrix* $\mathbf{G}_i$*, of order* $|P| \times q$ *(where* q *is a suitable natural number), such that:*

1. $\forall \mathbf{m}_1, \mathbf{m}_2 \in E_i$, the vector $(\mathbf{m}_1 - \mathbf{m}_2)$ is in the range of $\mathbf{G}_i$ (i.e. it is a linear combination of the columns of $\mathbf{G}_i$).
2. $\mathbf{G}_i$ is minimal, i.e. if one of its columns is removed then (1) is no longer true.

Notice that $\mathbf{G}_i$ is a kind of basis for E_i, but, formally speaking, E_i is not a vector space, then, it does not have a basis.

Coming back to the system of figure 2, a generator of E_1 is given by

$$\mathbf{G}_1 = \begin{bmatrix} 0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \end{bmatrix}^T$$

The restriction of the columns of $\mathbf{G}_1$ to the places p_1 and p_3 are represented in figure 2 by vectors $\mathbf{d}_1$ and $\mathbf{d}_2$.

In order to deal with the variable boundedness of the input, controllability is studied through the reachability over neighborhoods of equilibrium markings, because the bounds of the input are almost constant in these. Let us detail this idea. Consider again the system of figure 2. Let $\mathbf{m}_1$ be a marking in the interior E_1. The evolution of the system, seen from $\mathbf{m}_1$, is described by:

$$(\overset{\bullet}{\mathbf{m} - \mathbf{m}_1}) = \mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m} - \mathbf{m}_1) - \mathbf{C}(\mathbf{u} - \mathbf{u}_1)$$

where $(\mathbf{u} - \mathbf{u}_1)$ is the new input. Since $\mathbf{u}_1$ is such that $[\boldsymbol{\Lambda}\boldsymbol{\Pi}\mathbf{m}_1]_j > \mathbf{u}_{1j} > 0$ for every $t_j \in T_c$ ($\mathbf{m}_1$ is in the interior of E_1), then the entries of $(\mathbf{u} - \mathbf{u}_1)$, related to the controllable transitions, can be settled as either negative or positive values, at least at the markings in a small enough neighborhood of $\mathbf{m}_1$. So, the reachability in such neighborhood can be studied through the classical Kalman's reachability condition (see [13]). However, the Kalman's condition cannot be directly applied for all equilibrium markings. Consider the marking $\mathbf{m}^q$, depicted in figure 2, instead $\mathbf{m}_1$. The equilibrium input for this marking is $\mathbf{u}^q = \mathbf{0}$, so, the entries of the input $(\mathbf{u} - \mathbf{u}^q)$ can only be settled as nonnegative values (to apply the Kalman's condition it is necessary that the input could take either positive or negative values). Therefore, it is important to know at which markings and at which entries the input can take negative values. For that, the following definitions are introduced.

Definition 4. *Let* T_c *be the set of controllable transitions. A transition* $t_j \in T_c$ *is said to be* fully controllable *at* E_i *if there exists an equilibrium marking* $\mathbf{m}^q \in E_i$ *with an equilibrium input* $\mathbf{u}^q$ *such that* $[\boldsymbol{\Lambda}\boldsymbol{\Pi}_i\mathbf{m}^q]_j > \mathbf{u}_j^q > 0$*. In other case,* t_j *is said to be* partially controllable*. The set of fully (partially) controllable transitions at* E_i *is denoted as* T_{cf}^i *(* T_{cp}^i *).*

Definition 5. *The subset of* E_i*, in which the equilibrium flow can be positive, is defined as*

$$E_i^+ = \{ \mathbf{m}^q \in E_i | \exists \mathbf{u}^q \ such \ that \ \mathbf{w}^q > \mathbf{0} \}$$

The subset of E_i, in which the equilibrium flow can be positive and the entries of the input related to transitions T^i_{cf} are positive, is defined as

$$E^*_i = \{\mathbf{m}^q \in E^+_i | \exists \mathbf{u}^q \text{ such that } \mathbf{u}^q_j > 0, \ \forall t_j \in T^i_{cf}\}$$

Notice that $\forall \mathbf{m}^q \in E^+_i \ \exists \mathbf{u}^q$ such that $\mathbf{u}^q_j < [\mathbf{\Lambda\Pi}_i \mathbf{m}^q]_j, \ \forall t_j \in T_c$. For all marking $\mathbf{m}^q \in E^*_i$ the input $\mathbf{u}$, at a neighbor marking $\mathbf{m}$, can be increased or decreased with respect to $\mathbf{u}^q$, at those entries related to the transitions in T^i_{cf} (i.e. fully controllable transitions). On the other hand, the entries of $\mathbf{u}$ related to the transitions in T^i_{cp} can only be increased with respect to $\mathbf{u}^q$ (i.e. partially controllable). For instance, the interior of the triangle in figure 2 corresponds to E^*_1, while the union of E^*_1 and the edges e_1 and e_2 (without the circled points) corresponds to E^+_1. Besides, $T^1_{cf} = T_c$ and $T^1_{cp} = \varnothing$. Therefore, for any marking in the interior of the triangle (i.e. for any $\mathbf{m}_1 \in E^*_1$) the input of the system expressed from it (i.e. $(\mathbf{u} - \mathbf{u}_1)$), can be settled as either a positive or a negative value at those entries related to the transitions $\{t_1, t_2\}$ (i.e. T^1_{cf}).

Remark 1. E^*_i, E^+_i, E_i are convex sets, and $E^*_i \subseteq E^+_i \subseteq E_i$. The markings of $\{E^+_i - E^*_i\}$ are limit points, not interiors, of E^*_i (in the space generated by $\mathbf{G}_i$).

In the sequel, let us denote as $\mathbf{C}_c$, $\mathbf{C}^i_{cf}$ and $\mathbf{C}^i_{cp}$ the matrices built with the columns of $\mathbf{C}$ related to the transitions that belong to T_c, T^i_{cf} and T^i_{cp}, respectively. In the same way, denote with $\mathbf{u}_c$, $\mathbf{u}^i_{cf}$ and $\mathbf{u}^i_{cp}$ the vectors built with the entries of $\mathbf{u}$ related to the transitions that belong to T_c, T^i_{cf} and T^i_{cp}, respectively. In this way

$$\mathbf{Cu} = \mathbf{C}_c \mathbf{u}_c = \begin{bmatrix} \mathbf{C}^i_{cf} & \mathbf{C}^i_{cp} \end{bmatrix} \begin{bmatrix} \mathbf{u}^i_{cf} \\ \mathbf{u}^i_{cp} \end{bmatrix}$$

Now, we are ready to introduce the following theorem, which gives a sufficient and necessary condition for controllability over a set E^+_i.

Theorem 2. *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a TCPN system. Consider E^+_i such that $E^+_i \cap int(\Re_i) \neq \varnothing$ and let $\mathbf{G}_i$ be a generator of it.*

The system is BIC over E^+_i, considering all marking trajectories in $\Re_i$, iff there exist an index k and a matrix $\mathbf{b} \geq \mathbf{0}$ such that

$$Cont^k \left(\mathbf{C\Lambda\Pi}_i, \begin{bmatrix} -\mathbf{C}^i_{cf} & \mathbf{C}^i_{cf} & -\mathbf{C}^i_{cp} \end{bmatrix} \right) \cdot \mathbf{b} = \begin{bmatrix} \mathbf{G}_i & -\mathbf{G}_i \end{bmatrix}$$

where the matrix function $Cont^k(\mathbf{A}, \mathbf{B}) = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^k\mathbf{B} \end{bmatrix}$.

Proof. The proof is large and to improve readability it is shown in the appendix.

This theorem also includes the result introduced in previous section, in which consistency is sufficient to guarantee controllability over $int(Class(\mathbf{m_0}))$, where $T_c = T$. This is easy to see noting that matrix $\begin{bmatrix} -\mathbf{C}^i_{cf} & \mathbf{C}^i_{cf} & -\mathbf{C}^i_{cp} \end{bmatrix}$ includes all columns of the incidende matrix, and since the net is consistent, there always exists $\mathbf{b} \geq 0$ that fulfills the condition (consider $k = 0$).

The condition of previous theorem could be difficult to check, since there is no bound for the index k. Next corollary separates this condition into a necessary and a sufficient conditions that can be checked in polynomial time.

Corollary 1. *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a TCPN system. Consider some E_i^+ such that $E_i^+ \cap int(\Re_i) \neq \varnothing$, as previously defined, and let $\mathbf{G}_i$ be a generator of it. Then:*

1. *If $\exists \mathbf{b}$ such that $Cont^{|P|-1}(\mathbf{C\Lambda\Pi}_i, -\mathbf{C}_{cf}^i) \cdot \mathbf{b} = \mathbf{G}_i$, then the system is controllable over E_i^+. Besides, if $T_{cf}^i = T_c$ then it is also a necessary condition for controllability over E_i^+, considering all the marking trajectories in $\Re_i$.*
2. *If $\nexists \mathbf{b}$ such that $Cont^{|P|-1}(\mathbf{C\Lambda\Pi}_i, -\mathbf{C}_c) \cdot \mathbf{b} = \mathbf{G}_i$, then the system is not controllable over E_i^+, considering all marking trajectories in $\Re_i$.*

Proof. Statement 1). Suppose that $\exists \mathbf{b}$ such that $Cont^{|P|-1}(\mathbf{C\Lambda\Pi}_i, -\mathbf{C}_{cf}^i) \cdot \mathbf{b} = \mathbf{G}_i$. Then, $\exists \mathbf{b}' \geq \mathbf{0}$ such that $Cont^{|P|-1}(\mathbf{C\Lambda\Pi}_i, [-\mathbf{C}_{cf}^i, \mathbf{C}_{cf}^i]) \cdot \mathbf{b}' = [\mathbf{G}_i, -\mathbf{G}_i]$. So, according to Theorem 2, the system is BIC over E_i^+. On the other hand, suppose that $T_c = T_{cf}^i$, so $\mathbf{C}_{cf} = \mathbf{C}_c$. If $\mathbf{G}_i$ is not in the range of $Cont^{|P|-1}(\mathbf{C\Lambda\Pi}_i, -\mathbf{C}_{cf}^i)$ then it is not in the range of $Cont^{|P|-1}(\mathbf{C\Lambda\Pi}_i, -\mathbf{C}_c^i)$. This condition is equal to that of statement 2).

Statement 2). Suppose that $\nexists \mathbf{b}$ such that $Cont^{|P|-1}(\mathbf{C\Lambda\Pi}_i, -\mathbf{C}_c) \cdot \mathbf{b} = \mathbf{G}_i$, then $\mathbf{G}_i$ is not in the range of $Cont^k(\mathbf{C\Lambda\Pi}_i, [-\mathbf{C}_{cf}^i, \mathbf{C}_{cf}^i, -\mathbf{C}_{cp}^i])$, for $k = |P|-1$, and according to the Calley-Hamilton's theorem, $\mathbf{G}_i$ is not in the range for any index k, so, the system is not BIC over E_i^+ (Theorem 2). $\qquad\square$

Notice that previous corollary does not consider all possible cases, for that, next proposition introduces an equivalent condition to that of Theorem 2. The proof of this proposition is presented in the appendix.

Proposition 4. *Suppose that all the coefficients of the characteristic polynomial of a matrix $\mathbf{A}$ are nonnegative.*
There exist k and $\mathbf{X}_k \leq \mathbf{0}$ such that $Cont^k(\mathbf{A}, \mathbf{B}) \cdot \mathbf{X}_k = \mathbf{Y}$ iff $\exists \mathbf{X}_{2n-1} \leq \mathbf{0}$ such that $Cont^{2n-1}(\mathbf{A}, \mathbf{B}) \cdot \mathbf{X}_{2n-1} = \mathbf{Y}$, where n is the order of $\mathbf{A}$.

Example 3. Consider the system of figure 3 with $T_c = \{t_4\}$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\lambda_4 = 2$. Four configurations are realizable in the net, those are characterized by: $\mathcal{C}_1 = \{(t_2, p_2), (t_3, p_4)\}$, $\mathcal{C}_2 = \{(t_2, p_3), (t_3, p_4)\}$, $\mathcal{C}_3 = \{(t_2, p_2), (t_3, p_5)\}$ and $\mathcal{C}_4 = \{(t_2, p_3), (t_3, p_5)\}$, (the arcs (t_1, p_1) and (t_4, p_6) are also present in all the configurations). Moreover, given the initial marking of this system, $\mathcal{C}_2$ cannot occur (p_3 and p_4 cannot constraint its respective output transitions at the same time). The lines inside the polytope correspond to set E_qS. The markings in the segment $[\mathbf{m}_1, \mathbf{m}_2]$ correspond to E_1, the markings in $[\mathbf{m}_2, \mathbf{m}_3]$ correspond to E_3 and the markings in $[\mathbf{m}_3, \mathbf{m}_4]$ correspond to E_4. For this system we have that $T_{cf}^1 = T_{cf}^3 = T_{cf}^4 = \{t_4\}$, and

$$\begin{aligned}
\mathbf{G}_1 &= [\quad 0 \quad\quad 0 \quad\quad 0 \quad\quad 0 \quad -1/2 \ \ 1/2 \ ]^T \\
\mathbf{G}_3 &= [\quad 0 \quad\quad 0 \quad -1 \quad\ 1 \quad\quad 0 \quad\quad 0 \ ]^T \\
\mathbf{G}_4 &= [-1/3 \ \ 1/3 \ -1/3 \ \ 1/3 \ -1/3 \ \ 1/3 \ ]^T
\end{aligned}$$
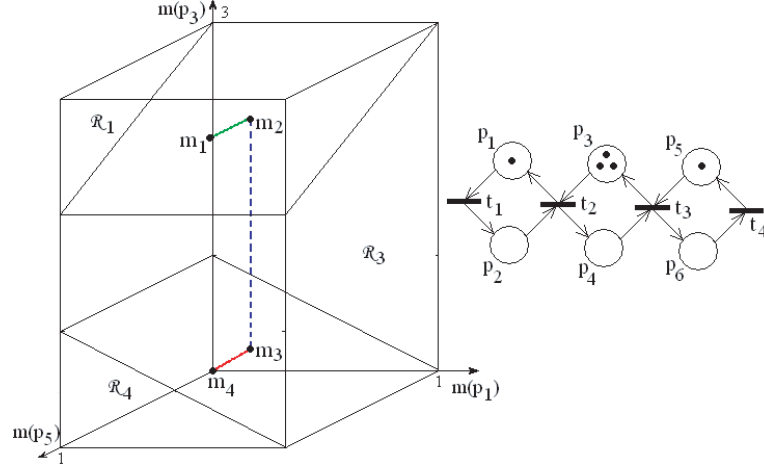
Fig. 3. TCPN system with its E_qS. Transition t_4 is the only controllable one

Since $T_{cf}^i = T_c$ for the three configurations, we can check for the condition of Corollary 1 to investigate the controllability over each E_i^+, i.e. for every E_i^+ investigate if $\mathbf{G}_i$ is in the range of $Cont^5(\mathbf{C\Lambda\Pi}_i, -\mathbf{C}_{cf}^i)$. In this case the system fulfills that condition for the three E_i^+, so, it is BIC over each one.

Now, consider the same system but with $\mathbf{\Lambda} = \mathbf{I}$. In this case, the sets E_3 and E_4 remain unchanged, and so $\mathbf{G}_3$ and $\mathbf{G}_4$, but $E_1 = \{\mathbf{m}_2\}$. Also, $T_{cf}^3 = \varnothing$ and $T_{cf}^4 = t_4$. Checking for the condition of Corollary 1 (a), it can be concluded that the system is BIC over E_4. However, since $T_{cf}^3 = \varnothing$, we cannot apply the same corollary for E_3 (the system does not fulfill the condition of neither statement a) nor b)). In such case, since the coefficients of the characteristic polynomial of $\mathbf{C\Lambda\Pi}_3$ are nonnegative, we can use Proposition 4 to investigate if the system fulfills the condition of Theorem 2 (i.e. the system is BIC over E_3^+ iff $\exists \mathbf{b} \geq \mathbf{0}$ such that $Cont^{2|P|-1}(\mathbf{C\Lambda\Pi}_3, -\mathbf{C}_{cp}^3) \cdot \mathbf{b} = [\mathbf{G}_3, -\mathbf{G}_3]$). Since there does not exist a nonnegative solution $\mathbf{b}$, we can concluded that the system is not BIC over E_3^+.

Finally, next proposition introduces sufficient conditions for controllability over the union of sets of equilibrium markings of different regions.

Proposition 5. *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a TCPN system. Consider some equilibrium sets E_1^+, E_2^+,..., E_j^+ as defined above. If the system is BIC over each one and their union (i.e. $\bigcup_{i=1}^{j} E_i^+$) is connected, then the system is BIC over the union.*

Proof. Consider two of those sets E_1^+, E_2^+ such that $E_1^+ \cap E_2^+ \neq \varnothing$. Let $\mathbf{m}^q$ be a marking such that $\mathbf{m}^q \in E_1^+ \cap E_2^+$. Since the system is controllable over E_1^+ and E_2^+, there exists a marking $\mathbf{m}_2 \in E_2^+ - E_1^+$ that is reachable, in finite time, from another marking in $\mathbf{m}_1 \in E_1^+ - E_2^+$, due to the fact that both are reachable from $\mathbf{m}^q$, in finite time, and to the continuity of the flow function. Then, any marking

of E_2^+ is reachable from any marking of E_1^+, via $\mathbf{m}_1$ and $\mathbf{m}_2$. Following a similar reasoning, it can be concluded that the system is controllable over $\bigcup_{i=1}^{j} E_i^+$.   □

Example 4. Consider the system of figure 3, where $T_c = \{t_4\}$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and $\lambda_4 = 2$. In the previous example it was shown that the system is controllable over each E_i^+. Now, since the union of E_1^+, E_3^+ and E_4^+ is connected, then, according to Proposition 5, the system is BIC over $E_1^+ \cup E_3^+ \cup E_4^+$. Notice that, the union of those sets is equal to $E_q S - \{\mathbf{m}_4\}$, i.e. the set of all the equilibrium markings inside the interior of $Class(\mathbf{m_0})$.

6   Conclusions

This work addresses the controllability of timed continuous Petri Nets ($TCPN$) from a structural point of view. The main contributions of this work are focused in defining the controllability property and its characterization. The definition of controllability for TCPN systems is introduced as an adaptation of that defined for linear continuous systems. For the case where all transitions are controllable, a polynomial characterization of controllable TCPN systems is presented. For systems with uncontrollable transitions, sufficient and necessary conditions for controllability, over subsets of equilibrium markings, are introduced, and sufficient conditions for controllability, over the union of those subsets, are given.

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Additional proofs

Proof (of Theorem 2). This proof is presented as follows: first, the state equation is rewritten, next, considering the solution of this new equation, it is proved that all equilibrium markings in a neighborhood of any marking of $E_i^* \cap int(\Re_i)$ are reachable iff the condition is fulfilled, and so, since this set can be covered by these reachable neighborhoods then the system is BIC over $E_i^* \cap int(\Re_i)$ iff the condition is fulfilled. Finally, it is proven that the system is controllable over $E_i^* \cap int(\Re_i)$ iff it is controllable over E_i^+.

If $E_i^* = \varnothing$ then consider the set E_i^+ instead of E_i^*.

Consider an equilibrium marking $(\mathbf{m}^q, \mathbf{u}^q)$, where $\mathbf{m}^q \in E_i^* \cap int(\Re_i)$. The state equation for any marking in $\Re_i$ can be expressed as:

$$\overset{\bullet}{\mathbf{m}} = (\mathbf{m} \overset{\bullet}{-} \mathbf{m}^q) = \mathbf{C\Lambda\Pi}_i(\mathbf{m} - \mathbf{m}^q) - \mathbf{C}(\mathbf{u} - \mathbf{u}^q)$$

The solution of this state equation is given by:

$$\mathbf{m}(\tau) - \mathbf{m}^q = e^{\mathbf{C\Lambda\Pi}_i\tau}(\mathbf{m_0} - \mathbf{m}^q) - \int_0^\tau e^{\mathbf{C\Lambda\Pi}_i\zeta}\mathbf{C}\left(\mathbf{u}(\tau - \zeta) - \mathbf{u}^q\right)d\zeta \quad (2)$$

Now, let us analyze the boundedness of the input. By definition of T_{cf}^i, T_{cp}^i and $\mathbf{m}^q$, the input $\mathbf{u}^q$ is such that $[\mathbf{\Lambda\Pi}_i\mathbf{m}^q]_j > \mathbf{u}_j^q > \mathbf{0} \ \forall t_j \in T_{cf}^i$, $\mathbf{u}_j^q = 0 \ \forall t_j \in T_{cp}^i$, and $\mathbf{u}_j = 0 \ \forall t_j \in T_{nc}$. Then, at any marking of a small enough neighborhood of $\mathbf{m}^q$ the value of $(\mathbf{u}(\tau - \zeta) - \mathbf{u}^q)_j$ can be settled as either positive or negative $\forall t_j \in T_{cf}^i$, as nonnegative $\forall t_j \in T_{cp}^i$, and zero $\forall t_j \in T_{nc}$. Besides, since $\mathbf{m}^q \in E_i^* \cap int(\Re_i)$, it can be demonstrated that $\mathbf{m}^q$ is an interior point of this neighborhood, considering the space generated by the columns of $Cont^k(\mathbf{C\Lambda\Pi}_i, \mathbf{C}_c)$. Therefore, considering the notation previously introduced, we can define column vectors $\mathbf{u}^+$ and $\mathbf{u}^-$ such that $\mathbf{C}_{cf}^i\left(\mathbf{u}_{cf}^i - \mathbf{u}_{cf}^{qi}\right) = \left[\mathbf{C}_{cf}^i - \mathbf{C}_{cf}^i\right] \cdot [\mathbf{u}^+ \ \mathbf{u}^-]^T$ where $\mathbf{u}^+, \mathbf{u}^- \geq \mathbf{0}$.

Then $\mathbf{Cu} = \left[\mathbf{C}_{cf}^i - \mathbf{C}_{cf}^i \ \mathbf{C}_{cp}^i\right]\mathbf{u}^*$ where $\mathbf{u}^* = [\mathbf{u}^+ \ \mathbf{u}^- \ (\mathbf{u}_{cp}^i - \mathbf{u}_{cp}^{qi})]^T \geq 0$ at any marking in a neighborhood of $\mathbf{m}^q$.

Considering $\mathbf{m_0} = \mathbf{m}^q$ and substituting previous equation into (2), we obtain:

$$\mathbf{m}(\tau) - \mathbf{m}^q = -\int_0^\tau e^{\mathbf{C\Lambda\Pi}_i\zeta}\left[\mathbf{C}_{cf}^i \ -\mathbf{C}_{cf}^i \ \mathbf{C}_{cp}^i\right]\mathbf{u}^*(\tau - \zeta)d\zeta$$

Expanding the exponential matrix and taking the constant elements out of the integral, we obtain

$$\mathbf{m}(\tau) - \mathbf{m}^q = \begin{bmatrix} \mathbf{I} \ (\mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}_i) \ (\mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}_i)^2 \ ... \end{bmatrix} \cdot$$
$$\begin{bmatrix} -\mathbf{C}_{cf}^i \ \mathbf{C}_{cf}^i \ -\mathbf{C}_{cp}^i \end{bmatrix} \begin{bmatrix} \int_0^\tau \mathbf{u}^*(\tau - \zeta)d\zeta \\ \int_0^\tau \zeta \mathbf{u}^*(\tau - \zeta)d\zeta \\ \int_0^\tau \frac{\zeta^2}{2} \mathbf{u}^*(\tau - \zeta)d\zeta \\ \vdots \end{bmatrix} \tag{3}$$

Notice that the entries of the right side vector are linearly independent and non-negative functions, for a small enough neighborhood of $\mathbf{m}^q$. Moreover, previous equation constitutes a necessary and sufficient condition for reachability.

If the condition of this theorem is fulfilled then, by definition of $\mathbf{G}_i$, for any equilibrium marking $\mathbf{m}^r$ in $\Re_i$ there exists $\mathbf{b}^r \geq \mathbf{0}$ such that

$$\mathbf{m}^r - \mathbf{m}^q = \begin{bmatrix} \mathbf{I} \ (\mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}_i) \ (\mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}_i)^2 \ ... \end{bmatrix} \cdot \begin{bmatrix} -\mathbf{C}_{cf}^i \ \mathbf{C}_{cf}^i \ -\mathbf{C}_{cp}^i \end{bmatrix} \mathbf{b}^r$$

The entries of $\mathbf{b}^r \geq \mathbf{0}$ could be as small as desired (just considering $\mathbf{m}^r$ close enough to $\mathbf{m}^q$), so, comparing this equation with (3), it can be concluded that $\mathbf{m}^r$ is reachable from $\mathbf{m}^q$. Then, there exists a neighborhood of $\mathbf{m}^q$ in which all equilibrium markings are reachable from $\mathbf{m}^q$. Finally, since $E_i^* \cap int(\Re_i)$ is a convex set and $\forall \mathbf{m}^q \in E_i^* \cap int(\Re_i)$ there exists such reachable neighborhood, in which $\mathbf{m}^q$ is an interior point, then the set $E_i^* \cap int(\Re_i)$ is covered by these reachable neighborhoods and the system is BIC over $E_i^* \cap int(\Re_i)$.

On the other hand, notice that at any marking in $\Re_i$ the input is such that $\mathbf{u}^* \geq \mathbf{0}$. So, if the condition asked for the theorem is not fulfilled, then $\exists \mathbf{m}^r$ for which $\nexists \mathbf{b}^r \geq \mathbf{0}$ that fulfills previous equation, and according to equation (3), there does not exist a marking trajectory from $\mathbf{m}^q$ to $\mathbf{m}^r$ in $\Re_i$ in which $\mathbf{u}^* \geq \mathbf{0}$, therefore, $\mathbf{m}^r$ is not reachable from $\mathbf{m}^q$, considering all the marking trajectories in $\Re_i$ and so, the system is not BIC over E_i^*.

Now, let us demonstrate that controllability over $E_i^* \cap int(\Re_i)$ implies controllability over E_i^+. For this, suppose that the system is controllable over $E_i^* \cap int(\Re_i)$. Consider any marking $\mathbf{m}^r \in E_i^+ - (E_i^* \cap int(\Re_i))$ and the markings $\mathbf{m}^q, \mathbf{m}^{r\prime}, \mathbf{m}^{q\prime} \in E_i^* \cap int(\Re_i)$, such that $(\mathbf{m}^r - \mathbf{m}^q) = \alpha(\mathbf{m}^{r\prime} - \mathbf{m}^{q\prime})$ with $\alpha \in \mathbb{R}_{>0}$.

Since the system is BIC over $E_i^* \cap int(\Re_i)$, then $\exists \mathbf{u}'$ that transfers the state from $\mathbf{m}^{r\prime}$ to $\mathbf{m}^{q\prime}$. So, by linearly, an input $\mathbf{u}$ such that $(\mathbf{u} - \mathbf{u}^q) = \alpha(\mathbf{u}' - \mathbf{u}^{q\prime})$ transfers the state from $\mathbf{m}^r$ to $\mathbf{m}^q$ (this is easy to prove by using equation (2)). Actually, $(\mathbf{m}(\tau) - \mathbf{m}^q) = \alpha(\mathbf{m}'(\tau) - \mathbf{m}^{q\prime})$, where $\mathbf{m}(\tau) \ (\mathbf{m}'(\tau))$ is the marking at time τ if $\mathbf{m_0} = \mathbf{m}^q \ (\mathbf{m_0} = \mathbf{m}^{q\prime})$ and $\mathbf{u} \ (\mathbf{u}')$ is applied. So, choosing a suitable trajectory for $\mathbf{m}'(\tau)$ in $E_i^* \cap int(\Re_i)$, we can make $\mathbf{m}(\tau)$ stay always inside $\Re_i$. Then, $\mathbf{m}^q$ is reachable from $\mathbf{m}^r$ if $\mathbf{u}$ is suitably bounded.

Analyzing the boundedness of the input, since $\mathbf{0} \leq \mathbf{u}' \leq \boldsymbol{\Lambda}\boldsymbol{\Pi}_i\mathbf{m}'$ then $-\alpha \mathbf{u}^{q\prime} \leq \alpha(\mathbf{u}' - \mathbf{u}^{q\prime}) \leq \alpha(\boldsymbol{\Lambda}\boldsymbol{\Pi}_i\mathbf{m}' - \mathbf{u}^{q\prime})$. Now, substituting $\mathbf{u}'$ and $\mathbf{m}'$ and arranging the terms, we obtain: $(\mathbf{u}^q - \alpha \mathbf{u}^{q\prime}) \leq \mathbf{u} \leq \boldsymbol{\Lambda}\boldsymbol{\Pi}_i\mathbf{m} - (\mathbf{w}^q - \alpha \mathbf{w}^{q\prime})$. Since $\mathbf{w}^q > \mathbf{0}$ and $(\mathbf{u}_j^{q\prime} > 0 \Rightarrow \mathbf{u}_j^q > 0)$, then $\exists \alpha > 0$ small enough such that $\mathbf{u}^q \geq \alpha \mathbf{u}^{q\prime}$

and $\mathbf{w}^q \geq \alpha \mathbf{w}^{q\prime}$, so, $\mathbf{0} \leq \mathbf{u} \leq \mathbf{\Lambda}\mathbf{\Pi}_i\mathbf{m}$, i.e. $\mathbf{u}$ is suitably bounded (by definition of $\mathbf{u}$, $\forall t_j \in T_{nc}$ $\mathbf{u}_j = 0$).

Therefore, for any marking $\mathbf{m}^r \in E_i^+ - (E_i^* \cap int(\Re_i))$ there exists a marking $\mathbf{m}^q \in E_i^* \cap int(\Re_i)$ reachable from $\mathbf{m}^r$. Besides, notice that the points of $E_i^+ - (E_i^* \cap int(\Re_i))$ are limit points of $E_i^* \cap int(\Re_i)$, so, every marking in $E_i^+ - (E_i^* \cap int(\Re_i))$ can be reached (at least in infinite time) from any marking in $E_i^* \cap int(\Re_i)$, following a trajectory in $E_i^* \cap int(\Re_i)$. So, the system is controllable over E_i^+ if it is controllable over $E_i^* \cap int(\Re_i)$.

Finally, by definition, if the system is not controllable over $E_i^* \cap int(\Re_i)$, considering all marking trajectories in $\Re_i$, then it is not BIC over E_i^+. $\square$

Proof (of Proposition 4). Let m be the number of columns of $\mathbf{B}$.

(Sufficiency) Suppose that there exists $\mathbf{X}_{2n-1} \leq \mathbf{0}$ such that $Cont^{2n-1}(\mathbf{A}, \mathbf{B}) \cdot \mathbf{X}_{2n-1} = \mathbf{Y}$. So, consider k as $2n - 1$.

(Necessity) First, let us introduce some useful equalities.

According to the Calley-Hamilton's theorem $\mathbf{I}\alpha_0 + \mathbf{A}\alpha_1 + ... + \mathbf{A}^n\alpha_n = \mathbf{0}$ where $\{\alpha_0, \alpha_1, ..., \alpha_n\}$ are the coefficients of the characteristic polynomial of $\mathbf{A}$. Without lost of generality, suppose that $\alpha_n = 1$. Now, post-multiplying this equation by $\mathbf{B}$ and arranging the terms, we obtain

$$\begin{bmatrix} \mathbf{B} \ \mathbf{AB} \cdots \ \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{I}\alpha_0 \\ \mathbf{I}\alpha_1 \\ \vdots \\ \mathbf{I}\alpha_{n-1} \end{bmatrix} = -\mathbf{A}^n\mathbf{B}$$

where $\mathbf{I}$ is the unity matrix of order m (the number of columns of $\mathbf{B}$).

Define $\hat{\mathbf{a}} = \begin{bmatrix} \mathbf{I}\alpha_0 \ \mathbf{I}\alpha_1 \ ... \ \mathbf{I}\alpha_{n-1} \end{bmatrix}^T$, and let $\mathbf{0}_{km \times m}$ be a null matrix of order $km \times m$. So, according to the previous equation

$$-\mathbf{A}^{n+k}\mathbf{B} = \mathbf{A}^k \begin{bmatrix} \mathbf{B} \ \mathbf{AB} \ ... \ \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \hat{\mathbf{a}} = Cont^{n+k-1} \begin{bmatrix} \mathbf{0}_{km \times m} \\ \hat{\mathbf{a}} \end{bmatrix} \tag{4}$$

Now, every solution $\mathbf{X}_k$ for

$$Cont^k(\mathbf{A}, \mathbf{B}) \cdot \mathbf{X}_k = \mathbf{Y} \tag{5}$$

has the form: $\mathbf{X}_k = \mathbf{X}_k^p + \mathbf{ker}\left\{Cont^k(\mathbf{A}, \mathbf{B})\right\}\gamma$, where $\mathbf{X}_k^p$ is a particular solution and $\mathbf{ker}\left\{Cont^k(\mathbf{A}, \mathbf{B})\right\}\gamma$ is an element in the kernel of $Cont^k(\mathbf{A}, \mathbf{B})$. In this way, it is easy to prove, using equation (4), that any solution $\mathbf{X}_k$ has the form

$$\mathbf{X}_k = \begin{bmatrix} \mathbf{X}_{n-1} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{I}\alpha_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{I}\alpha_1 & \mathbf{I}\alpha_0 & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{I}\alpha_{n-1} & \mathbf{I}\alpha_{n-2} & & \mathbf{0} \\ -\mathbf{I} & \mathbf{I}\alpha_{n-1} & \cdots & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \cdots & \mathbf{I}\alpha_0 \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \gamma_n \\ \vdots \\ \gamma_k \end{bmatrix}$$

where $\mathbf{X}_{n-1}$ is a solution of equation (5) for $k = n - 1$. So, the first term of this equation is a particular solution, while the second is an element of $\mathbf{ker}(Cont^k(\mathbf{A}, \mathbf{B}))$. It can be separated into two simultaneous equations:

$$\mathbf{X}_k^1 = [\mathbf{X}_{n-1}] + \begin{bmatrix} \mathbf{I}\alpha_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{I}\alpha_1 & \mathbf{I}\alpha_0 & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{I}\alpha_{n-1} & \mathbf{I}\alpha_{n-2} & \cdots & \mathbf{I}\alpha_0 \end{bmatrix} \begin{bmatrix} \gamma_n \\ \vdots \\ \gamma_{2n-1} \end{bmatrix} \tag{6}$$

$$\mathbf{X}_k^2 = \begin{bmatrix} -\mathbf{I} & \mathbf{I}\alpha_{n-1} & \cdots & \mathbf{I}\alpha_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{I}\alpha_{n-1} & \cdots & \mathbf{I}\alpha_0 & \cdots & \vdots \\ \vdots & & & & & & \vdots \\ \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \mathbf{0} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \gamma_n \\ \vdots \\ \gamma_k \end{bmatrix} \tag{7}$$

where $\mathbf{X}_k = \begin{bmatrix} \mathbf{X}_k^1 & \mathbf{X}_k^2 \end{bmatrix}^T$. Therefore, for any solution $\mathbf{X}_k$ of equation (5) there must exist $\mathbf{X}_{n-1}$ and $\begin{bmatrix} \gamma_n & \dots & \gamma_k \end{bmatrix}^T$ that fulfill equations (6) and (7).

Now, suppose that $\exists \mathbf{X}_k$ that fulfills equation (5) for $k = 2n - 1$, but $\mathbf{X}_k \nleq \mathbf{0}$. (it is easy to demonstrate, via Calley-Hamilton's theorem, that if there does not exist such solution, then there does not exist any solution $\mathbf{X}_k$ for any index $k \neq 2n-1$). Proceeding by contradiction, suppose also that for some $k > 2n-1$ there exists $\mathbf{X}_k' \leq \mathbf{0}$ that fulfills equation (5) (if there exists a solution for $k < 2n-1$, then there also exists a solution for any $k \geq 2n-1$, so, we only consider the case in which there exist solution for $k > 2n-1$). So, there exist $\mathbf{X}_{n-1}'$ and $\begin{bmatrix} \gamma_n' & \cdots & \gamma_k' \end{bmatrix}^T$ that fulfill equations (6) and (7). Moreover, according to equation (7), it is easy to see, considering that $\hat{\mathbf{a}} \geq \mathbf{0}$, $\mathbf{X}_k' \leq \mathbf{0}$ and following a backward substitution reasoning, that $\begin{bmatrix} \gamma_n' & \cdots & \gamma_k' \end{bmatrix}^T \geq \mathbf{0}$. But, by the contradiction hypothesis, there do not exist $[\mathbf{X}_{n-1}]$ and $\begin{bmatrix} \gamma_n & \cdots & \gamma_{2n-1} \end{bmatrix}^T$ such that

$$[\mathbf{X}_{n-1}] + \begin{bmatrix} \mathbf{I}\alpha_0 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{I}\alpha_1 & \mathbf{I}\alpha_0 & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{I}\alpha_{n-1} & \mathbf{I}\alpha_{n-2} & \cdots & \mathbf{I}\alpha_0 \end{bmatrix} \begin{bmatrix} \gamma_n \\ \vdots \\ \gamma_{2n-1} \end{bmatrix} \leq \mathbf{0}$$

$$\begin{bmatrix} -\mathbf{I} & \mathbf{I}\alpha_{n-1} & \cdots & \mathbf{I}\alpha_1 \\ \mathbf{0} & -\mathbf{I} & \cdots & \mathbf{I}\alpha_2 \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \cdots & \cdots & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \gamma_n \\ \vdots \\ \gamma_{2n-1} \end{bmatrix} \leq \mathbf{0}$$

This is a contradiction, since the previous values of $\mathbf{X}_{n-1}'$ and $\begin{bmatrix} \gamma_n' & \cdots & \gamma_{2n-1}' \end{bmatrix}^T \geq \mathbf{0}$ clearly fulfill these last inequalities. $\qquad \square$