CINVESTAV
Centro de Investigación y de Estudios Avanzados del I.P.N.
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# Controlabilidad en Redes de Petri continuas temporizadas

Tesis que presenta:

Carlos Renato Vázquez Topete

para obtener el grado de:

Maestro en Ciencias

en la especialidad:

Ingeniería Eléctrica

Directores de Tesis

Dr. José Javier Ruiz León
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CINVESTAV del IPN Unidad Guadalajara, Jalisco, Agosto de 2006.

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Por:

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# Resumen


Las redes de Petri constituyen un paradigma bien conocido y útil para modelar sistemas de eventos discretos. En algunos casos, es necesario adoptar un enfoque de enumeración de estados para verificar algunas propiedades de las redes de Petri. Desafortunadamente, para sistemas altamente marcados, el grafo de alcanzabilidad puede ser tan grande que muchas propiedades son muy complicadas de analizar. Este problema es conocido como el problema de explosión de estados. Sistemas que normalmente aparecen en la práctica, por ejemplo: procesos de manufactura reales, sistemas de telecomunicaciones, sistemas de tráfico, sistemas logísticos; dejan modelos de redes de Petri muy grandes. Por esto, se ha propuesto una técnica alternativa, llamada fluidificación, para poder analizar tales sistemas.

La fluidificación constituye una técnica para estudiar sistemas a través de un modelo continuo similar. Utilizando modelos continuos, se pueden utilizar más técnicas analíticas para el análisis de algunas propiedades de interés. En esta disertación, se consideran redes de Petri continuas temporizadas bajo semántica de servidores infinitos. La teoría de modelos completamente fluidificados se encuentra todavía en desarrollo, dado que es un área relativamente nueva. Por lo que es necesario enfocar más esfuerzos en la solución general de problemas importantes. Esta disertación provee el conocimiento teórico básico necesario para, eventualmente, obtener leyes de control efectivas para los sistemas de redes de Petri continuas temporizadas ($TCPN$).

En esta disertación, se estudian tipos generales de sistemas $TCPN$ con el fin de obtener condiciones necesarias de suficiencia y necesidad de alcanzabilidad y controlabilidad, y posteriormente se proponen algunas estructuras de leyes de control efectivas. Para esto, se introduce un concepto de controlabilidad que es una adaptación del concepto clásico de controlabilidad para sistemas lineales. Los sistemas $TCPN$ controlables son caracterizados y se resuelve el problema de alcanzabilidad para el caso en que todas las transiciones son controlables. Para el caso con transiciones incontrolables, se dan condiciones de suficiencia de controlabilidad sobre un conjunto de puntos de equilibrio y condiciones de necesidad de alcanzabilidad. También, se presentan dos estructuras de leyes de control para los casos: sin transiciones incontrolables, y con solo una transición controlable.

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# Summary

Petri Nets constitute a well-known paradigm useful to model discrete event systems. In some cases, an enumeration approach (state enumeration) has to be used in order to verify some properties of Petri nets. Unfortunately, for high marked systems, the reachability graph can be so large that many properties are very complex to analyze. This problem is known as the state explosion problem. Systems that appear normally in practice, for instance realistic manufacturing processes, telecommunications systems, traffic systems, logistic systems, leads to large Petri net models. So, in order to be able to analyze such systems, an alternative technique, named fluidification, has been proposed.

The fluidification constitutes a technique to study discrete systems through a similar but continuous model. Using fluid models more analytical techniques can be used for the analysis of some interesting properties. In this dissertation, timed continuous Petri net models with infinite server semantics are considered. The theory of fully fluidified models is still under development, since this area is relatively new. So, more efforts are needed for general solutions of important problems. This dissertation provides the basic theoretical knowledge needed to eventually obtain effective control laws for the timed continuous Petri net ($TCPN$) systems.

In this dissertation, general kinds of $TCPN$ systems are studied, in order to obtain sufficient and necessary conditions of reachability and controllability, and then some structures for effective control laws are proposed. For this, a concept of controllability is introduced as an appropriate adaptation of the linear system classical controllability concept, in this way, controllability is an structural property of the system. The controllable $TCPN$ systems are characterized and the reachability problem is solved for the case in which all transitions are controllable. For the case with uncontrolled transitions, sufficient conditions of controllability over a set of equilibrium points and necessary conditions of reachability are given. Also, two effective control laws are provided for both cases: without uncontrolled transitions, and with only one uncontrolled transition.

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# Contents

# Chapter

# Introduction

Petri Nets constitute a well-known paradigm useful to model discrete event systems. Although many researchers are investigating Petri nets properties from a standard point of view, in some cases, an enumeration approach (state enumeration) has to be used, in that case, the verification of some properties of Petri nets is performed from the knowledge of the Petri net reachability graph. Unfortunately, for high marked systems, the reachability graph can be so large that many properties are very complex to analyze. This problem is known as the state explosion problem. Systems that appear normally in practice, for instance realistic manufacturing processes, telecommunications systems, traffic systems, logistic systems, leads to large Petri net models. So, in order to be able to analyze such systems, an alternative technique, named fluidification, has been proposed.

The fluidification constitutes a technique to study discrete systems through a similar but continuous model. Using fluid models more analytical techniques can be used for the analysis of some interesting properties. However some modelling or analysis capabilities are missed during fluidification. In this way, the continuous model is considered as an approximation of the discrete one, and not properly as a model of the physical system. This technique has been applied to different paradigms. A comparison of those models can be seen in [5].

In Petri Nets, fluidification has been introduced from different perspectives. We will consider the approach studied by M. Silva, L. Recalde and coworkers [5], [9]. In this report, timed continuous Petri net models with infinite server semantics are considered. Based on this model, the firing count vector and the marking are fluidificated, in order to obtain the continuous model. The obtained continuous model is piecewise linear.

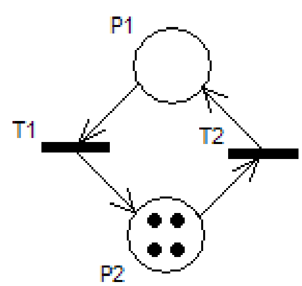


Figure 1.1 Example of a Petri net system.

In order to clarify the concept of fluidification, see the Petri net of figure 1.1. As a discrete Petri net, the marking can be changed in integer amounts. For example, say that $t_2$ is fired once, so the reached marking is $[1, 3]^T$. As a continuous Petri net, transitions can be fired in any enabled positive amount. Suppose that transition $t_2$ is fired in an amount of $0.2$, then the reached marking is $[0.2, 3.8]^T$. Finally, as a timed continuous Petri net, the transitions are not fired in certain amount, they are fired with certain speed. Considering infinite server semantics, say that transitions $t_1$ and $t_2$ are fired with an speed of 1 enabling degree by second, then the trajectory of the marking (the marking as a function of time) is that shown in figure 1.2.

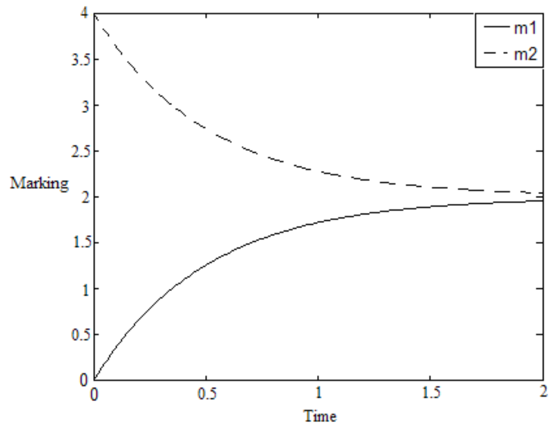


Figure 1.2 Marking evolution of the system of figure 1.1, considering it as a timed continuous Petri net.

There exist some interpretations of the marking in the continuous models. One of them, for timed continuous Petri nets, is that the normalized throughput of the transitions in the steady state of the continuous model approximate the average value of the normalized throughput of the transitions in the steady state of the original discrete system. The continuous system can be a good approximation of the discrete one when the tokens represent a large number of indistinguishable individuals/parts. For further details of this interpretation see [2].

The reader has to keep in mind that the theory of fully fluidified models (continuous models) is still under development, since this area is relatively new. So, more efforts are needed for general solutions of important problems. Now, we present some questions, which are mentioned in [9], that represent the most interesting problems to be solved for continuous Petri nets.

- Given a discrete Petri net system, the continuous model obtained from it is a good enough approximation?
- Which is the best firing semantic for a particular case?
- Given a timing semantic, when does a steady state exist?
- Once a good dynamic control is obtained for the continuous relaxation, how to come back to a "reasonable" design or control (scheduling) in the original setting?

Besides the problems involved in these questions, marking reachability, observation and control of continuous models deserve more efforts. Reachability in autonomous continuous Petri nets (non timed) has been studied by Julvez, Recalde and Silva in [6]. In that paper, reachability is introduced as the property of a marking to be reached from the initial marking, this marking can be reached in three different ways: with a finite firing sequence, with an infinite firing sequence, or just getting as close as desired to the marking with a finite firing sequence. The controllability for timed continuous Petri nets has been studied by Jiménez, Júlvez, Recalde and Silva [8]. They introduced a controllability definition as a property of markings, i.e., a marking is said to be controllable iff it is reachable and it is an equilibrium point (with a suitable bounded input). They characterized the set of "controllable markings" for join free Petri nets.

The main goal of this dissertation is to provide the basic theoretical knowledge needed to eventually obtain effective control laws for the timed continuous Petri net ($TCPN$) systems. The objectives are: to propose a structural controllability definition for $TCPN$ systems, to analyze and to provide necessary and sufficient conditions for controllability and reachability for general kinds of $TCPN$ systems, and finally to present control law structures that transfer the state from the initial state to the required state.

Although controllability and reachability have been studied by Jiménez, Júlvez, Recalde and Silva, the results obtained by them are not sufficient to compute effective control laws for general cases of timed continuous Petri net ($TCPN$) systems (they solved this problem for the case of join free Petri nets). So, in this dissertation, we study general kinds of $TCPN$ systems, in order to obtain sufficient and necessary conditions of reachability and controllability, and then we propose some structures for effective control laws.

The main contributions of this dissertation are:

- The characterization of the so called "state space".
- The introduction of the minimum order state equation.
- A definition of controllability for $TCPN$ systems as an adaptation of that for the linear continuous-time systems.
- The introduction of necessary and sufficient conditions of controllability and reachability for any kind of $TCPN$ systems, where all transitions are controllable.
- The introduction of sufficient conditions of controllability for any kind of $TCPN$ systems, where there are uncontrolled transitions.
- The introduction of necessary conditions of reachability for any kind of $TCPN$ systems, where there are uncontrolled transitions.
- An effective control law structure that transfers the marking from the initial marking to the required marking for any kind of $TCPN$, where all transitions are controllable.
- An effective control law structure that transfers the marking from the initial marking in $ES$ to the required marking in $ES$ for any kind of $TCPN$, where there is only one uncontrolled transition.

This report is organized as follows: In chapter 2, we introduce some basic concepts related to classic Petri nets, continuous Petri nets and timed continuous Petri nets under infinite server semantic. In the last section of this chapter we rewrite the state equation into a more useful form. In chapter 3, we present a brief discussion of the concept of state variable. Also, in this chapter, we present a characterization of the "state space", and finally we introduce the minimum order state equation. In chapter 4, a definition of controllability is introduced as an adaptation of the linear continuous-time classical controllability definition, in this way, controllability is a structural property of the system. For the case where all transitions are controllable, the controllable $TCPN$ systems are characterized, and the marking reachability problem is solved. For the case where there exist uncontrolled transitions, sufficient conditions of controllability over a set of equilibrium points are given. In chapter 5, two effective control law structures are proposed, one for the case where all transitions are controllable, and the other for the case where only one transition is uncontrolled. The conclusions and the future work are presented in chapter 6.

# Chapter

# Basic concepts on Petri nets, continuous Petri nets and timed continuous Petri nets

In the first three sections of this chapter basic definitions of classic Petri nets, continuous Petri nets and timed continuous Petri nets are presented. Also, the notation that will be used along this dissertation is introduced. These contents are mainly taken from references [4] and [7].

In the last section, a useful form of the state equation for $TCPN$ systems under infinite server semantics is proposed.

## 2.1 Petri Nets

In this section basic concepts on Petri nets are introduced. For further details see [4].

**Definition 2.1** *Nets, pre-sets, post-sets, subnets*

*A net $N$ is a 3-tuple $(P, T, F)$, where $P$ and $T$ are two finite and disjoint sets, and $F$ is a relation on $P \cup T$ such that $F \cap (P \times P) = F \cap (T \times T) = \varnothing$.*

*The elements of $P$ are called places, and are graphically depicted by circles. The elements of $T$ are called transitions, represented by boxes. $F$ is called the flow relation of the net, represented by arrows from places to transitions or from transitions to places. Often, the elements of $P \cup T$ are generically called nodes of $N$ or elements of $N$. The elements of $F$ are called arcs.*

*Given a node $x$ of $N$, the set ${}^\bullet x = \{y|(y, x) \in F\}$ is the pre-set of $x$ and the set $x^\bullet = \{y|(x, y) \in F\}$ is the post-set of $x$. The elements in the pre-set (post-set) of a place are its input (output) transitions. Similarly, the elements in the pre-set (post-set) of a transition are its input (output) places.*

*Given a set $X$ of nodes of $N$, define ${}^\bullet X = \cup_{x \in X} {}^\bullet x$ and $X^\bullet = \cup_{x \in X} x^\bullet$.*

*A triple $(P', T', F')$ is a subnet of $N$ if $P' \subseteq P$, $T' \subseteq T$ and $F' = F \cap ((P' \times T') \cup (T' \times P'))$.*

*If $X$ is a set of elements of $N$, then the triple $(P \cap X, T \cap X, F \cap (X \times X))$ is a subnet of $N$, called the subnet of $N$ generated by $X$.*

Figure 2.1 shows a Petri net model of some device, where:

$P = \{p_1, p_2, p_3, p_4, p_5\}$ is the set of places,

$T = \{t_1, t_2, t_3, t_4, t_5\}$ is the set of transitions, and

$F = \{(p_1, t_2), (p_2, t_1), (p_3, t_3), (p_4, t_4), (p_4, t_5), (p_5, t_2), (t_1, p_1), (t_2, p_2), (t_2, p_3), (t_3, p_4), (t_4, p_5), (t_5, p_3)\}$ is the flow relation.

Examples of pre- and post-sets are $t_2^\bullet = \{p_2, p_3\}$ and ${}^\bullet\{p_2, p_3\} = \{t_2, t_5\}$.

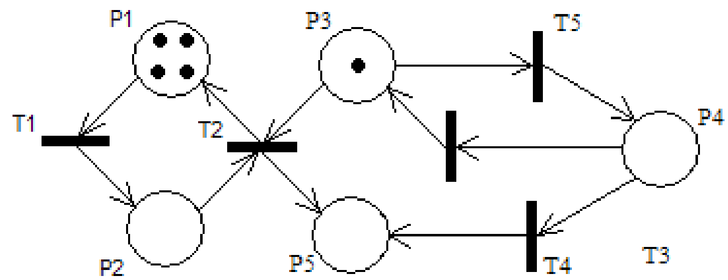


Figure 2.1 Example of a $PN$ system.

**Definition 2.2** *Paths, circuits*

*A path in a net $(P, T, F)$ is a nonempty sequence $x_1...x_k$ of nodes which satisfies $(x_1, x_2)$ ,..., $(x_{k-1}, x_k) \in F$. A path $x_1...x_k$ is said to lead from $x_1$ to $x_k$.*

*A path leading from a node $x$ to a node $y$ is a circuit if no element occurs more than once in it and $(y, x) \in F$. Observe that a sequence containing one element is a path but not a circuit, because for every node $x$, $(x, x) \notin F$.*

A net $(P, T, F)$ is called weakly connected (or just connected) if every two nodes $x$, $y$ satisfy $(x, y) \in (F \cup F^{-1})^*$. Where for any set $A$, $A^*$ is the reflexive and transitive closure of $A$.

$(P, T, F)$ is strongly connected if $\forall x, y \in P \cup T$, $(x, y) \in F^*$, i.e., for every two nodes $x$, $y$ there is a path leading from $x$ to $y$.

In the example of figure 2.1, $t_2p_2t_1p_1t_2p_3$ is a path and $p_3t_3p_4t_5$ is a circuit. The net is strongly connected.

Next definitions introduce markings and the occurrence rule (firing rule), which transform a net into a dynamic system.

**Definition 2.3** *Markings*

*A marking of a net $(P, T, F)$ is a mapping $m : P \to \{\mathbb{N} \cup 0\}$. A marking is represented by the vector $[m(p_1)...m(p_n)]^T)$, where $p_1, p_2, ..., p_n$ is an arbitrary fixed numeration of $P$.*

*A place $p$ is marked at a marking $m$ if $m(p) > 0$. A set of places $R$ is marked if some place of $R$ is marked.*

*The total number of tokens (marks) on a set $R$ is denoted by $m(R)$, i.e., $m(R)$ is the sum of all $m(p)$ for $p \in R$.*

*The null marking is the marking which maps every place to $0$.*

**Definition 2.4** *Arc weight*

*The arc weight is a function $w : F \to \mathbb{N}$, which associates a natural number to each arc.*

*When all arcs have weight equal to $1$, the net is called ordinary.*

In the graph, the weight of each arc is written near of it. When no weight is written at some arc, the weight of that arc is taken to be equal to $1$.

**Definition 2.5** *Occurrence rule*

*A marking $m$ enables a transition $t$ if for every place $p \in {}^{\bullet}t$, $m(p) \geq w(p,t)$. If $t$ is enabled at $m$, then it can occur, and its occurrence leads to the successor marking $m\prime$ (written $m \xrightarrow{t} m'$) which is defined for every place $p$ by*

$$m\prime(p) = \begin{cases} m(p) & \text{if } p \notin^{\bullet} t \text{ and } p \notin t^{\bullet} \\ m(p) - w(p,t) & \text{if } p \in^{\bullet} t \text{ and } p \notin t^{\bullet} \\ m(p) + w(t,p) & \text{if } p \notin^{\bullet} t \text{ and } p \in t^{\bullet} \\ m(p) - w(p,t) + w(t,p) & \text{if } p \in^{\bullet} t \text{ and } p \in t^{\bullet} \end{cases}$$

*($w(p_{in}, t)$ tokens are removed from the place $p_{in}$ in the pre-set of $t$ and $w(t, p_{out})$ tokens are added to the place $p_{out}$ in the post-set of $t$).*

*A marking $m$ is called dead if it enables no transition in the net.*

Graphically, a marking $m$ is represented by $m(p)$ tokens (black dots) or the number $m(p)$ in the place $p$. The marking of the net of figure 2.1 maps $p_1$ to $4$, $p_3$ to $1$ and all other place to $0$. Its vector representation is $\begin{bmatrix} 4 & 0 & 1 & 0 & 0 \end{bmatrix}^T$. The transition $t_3$ is enabled, and the marking reached after its occurrence is $\begin{bmatrix} 4 & 0 & 0 & 1 & 0 \end{bmatrix}^T$.

**Definition 2.6** *Occurrence sequences, reachable markings*

*Let $m$ be a marking of $N$. If $m \xrightarrow{t_1} m_1 \xrightarrow{t_2} ... \xrightarrow{t_n} m_n$ are transition occurrences then $\sigma = t_1 t_2 ... t_n$ is an occurrence sequence leading from $m$ to $m_n$ and it is written as $m \xrightarrow{\sigma} m_n$. This notion includes the empty sequence $\in$, i.e. $m \xrightarrow{\in} m$ for every marking $m$.*

*It is written $m \xrightarrow{*} m'$, when $m\prime$ is reachable from $m$, i.e. $m \xrightarrow{\sigma} m\prime$ for some occurrence sequence $\sigma$. The set of all markings reachable from $m$ is denoted by $RS(N, m)$.*

*If $m \xrightarrow{t_1} m_1 \xrightarrow{t_2} m_2 \xrightarrow{t_3}$ ... for an infinite sequence of transitions $\sigma = t_1 t_2 t_3 ...$ then $\sigma$ is an infinite occurrence sequence and it is written as $m \xrightarrow{\sigma}$.*

A sequence of transition $\sigma$ is enabled at a marking $m$ if $m \xrightarrow{\sigma} m\prime$ for some marking $m\prime$ (if $\sigma$ is finite) or $m \xrightarrow{\sigma}$ (if $\sigma$ is infinite).

**Definition 2.7** *$Pre$, $Post$ and Incidence matrices*

*Let $N$ be the net $(P, T, F)$. The $Pre$ matrix of order $|P| \times |T|$ is defined by*

$$Pre(p,t) = \begin{cases} 0 & \text{if } (p,t) \notin F \\ w(p,t) & \text{if } (p,t) \in F \end{cases}$$

*The $Post$ matrix of order $|P| \times |T|$ is defined by*

$$Post(p,t) = \begin{cases} 0 & \text{if } (t,p) \notin F \\ w(t,p) & \text{if } (t,p) \in F \end{cases}$$

*The incidence matrix denoted by $C$ is defined as:*

$$C = Pre - Post$$

Similarly to the vector representations of simple mappings, the matrix representation of the incidence matrix depends on enumerations of places and transitions.

The column vector $T$ of $C$ associated to a transition $t$ is denoted by $\mathbf{t}$. Similarly, the row vector $P$ associated to a place $p$ is denoted by $\mathbf{p}$.

The entry $C(p,t)$ corresponds to the change of the marking of the place $p$ caused by the occurrence of the transition $t$. Hence, if $t$ is enabled at a marking $m$ and $m \xrightarrow{t} m\prime$ then $m' = m + \mathbf{t}$. For a generalization of this equation to sequences of transitions the following definition is needed.

**Definition 2.8** *Parikh vectors of transition sequences*

*Let $(P,T,F)$ be a net and let $\sigma$ be a finite sequence of transitions. The Parikh vector $\overrightarrow{\sigma} : T \to \mathbb{N}$ of $\sigma$ maps every transition $t$ of $T$ to the number of occurrences of $t$ in $\sigma$.*

*The Parikh vector of the sequence $t_3t_5t_3t_4t_2$ is $\begin{bmatrix} 0 & 1 & 2 & 1 & 1 \end{bmatrix}^T$, while the Parikh vector of the sequence $t_1$ is $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$.*

Now, observe that for every transition $t$, $\mathbf{t} = C\overrightarrow{t}$. Therefore, if $m \xrightarrow{t} m\prime$, then $m\prime = m + Ct$ (where $m$ and $m\prime$ are taken as column vectors). For an arbitrary finite occurrences sequence $m \xrightarrow{\sigma} m\prime$, $m\prime = m + C\overrightarrow{\sigma}$, as shown in the following Lemma:

**Lemma 2.1** *Marking equation Lemma*

*For every finite sequence $m \xrightarrow{\sigma} m\prime$ of a net $N$ the following Marking Equation holds:*

$$m\prime = m + C\overrightarrow{\sigma}$$

The proof of this result is presented in [4].

A net is static - a special kind of graph- while a Petri net is dynamic and has a behavior.

**Definition 2.9** *Net systems, initial and reachable markings*

*A net system (or just a system) is a pair $(N, m_0)$ where $N$ is a connected net having at least one place and one transition, and $m_0$ is a marking of $N$ called the initial marking. A marking is called reachable in a system if it is reachable from the initial marking.*

Now formal definitions of some of the properties of Petri net systems are presented.

**Definition 2.10** *Liveness and related properties*

*A system is live if, for every reachable marking $m$ and every transition $t$, there exists a marking $m' \in RS(N,m)$ which enables $t$. If $(N, m_0)$ is a live system, then it is said that $m_0$ is a live marking of $N$.*

*A system is place-live if, for every reachable marking $m$ and every place $p$, there exists a marking $m\prime \in RS(N,m)$ which marks $p$.*

*A system is deadlock-free if every reachable marking enables at least one transition; in other words, if no dead marking can be reached from the initial marking.*

Loosely speaking, a system is live if every transition can always occur again.

Next, boundedness of systems is defined.

**Definition 2.11** *Bounded systems, bound of a place*

*A system is bounded if for every place* $p$ *there is a natural number* $b$ *such that* $m(p) \leq b$ *for every reachable marking* $m$*. If* $(N, m_0)$ *is a bounded system, it is said that* $m_0$ *is a bounded marking of* $N$*.*

*The bound of a place* $p$ *in a bounded system* $(N, m_0)$ *is defined as:*

$$max\{m(p)|m \in RS(N, m_0)\}$$

*A system is called* $b$*-bounded if no place has a bound greater than* $b$*.*

**Definition 2.12** *P-invariants (P-semiflows)*

*A P-invariant of a net* $N$ *is a rational-valued solution of the equation* $Y^T \cdot C = 0$*.*

**Proposition 2.1** *Fundamental property of P-invariants*

*Let* $(N, m_0)$ *be a system, and let* $I$ *be a P-invariant of* $N$*. If* $m_0 \xrightarrow{*} m'$*, then* $I \cdot m = I \cdot m_0$*.*

The proof of this result is presented in [4].

**Definition 2.13** *T-invariants (T-semiflows)*

*A T-invariant of a net* $N$ *is a rational-valued solution of the equation* $C \cdot X = 0$*.*

**Proposition 2.2** *Fundamental property of T-invariants*

*Let* $\sigma$ *be a finite sequence of transitions of a net* $N$ *which is enabled at a marking* $m$*. Then the Parikh vector* $\overrightarrow{\sigma}$ *is a T-invariant iff* $m \xrightarrow{\sigma} m$ *(i.e., iff the occurrence of* $\sigma$ *reproduces the marking* $m$*).*

P-systems are systems whose transitions have exactly one input place and one output place.

**Definition 2.14** *P-nets, P-systems*

*A net is a P-net if* $|{}^\bullet t| = 1 = |t^\bullet|$ *for every transition* $t$*.*

*A system* $(N, m_0)$ *is a P-system if* $N$ *is a P-net.*

The fundamental property of P-systems is that all reachable markings contain exactly the same number of tokens. In other words, the total number of tokens of the system remains invariant under the occurrence of transitions.

In T-systems places have exactly one input and one output transition.

**Definition 2.15** *T-nets, T-systems*

*A net is a T-net if* $|^\bullet p| = 1 = |p^\bullet|$ *for every place* $p$.

*A system* $(N, m_0)$ *is a T-system if* $N$ *is a T-net.*

The fundamental property of T-systems is that the token counts of circuits remain invariant under the occurrence of transitions.

Another kind of net and systems well studied is that of free-choice.

**Definition 2.16** *Free-choice nets, free-choice systems*

*A net* $N = (P, T, F)$ *is free-choice if* $(p, t) \in F$ *implies* $^\bullet t \times p^\bullet \subseteq F$ *for every place* $p$ *and every transition* $t$.

*A system* $(N, m_0)$ *is free-choice if its underlying net* $N$ *is free-choice.*

The fundamental property of a free-choice net is that if a marking enables some transition of $p^\bullet$, where $p$ is a place of the net, then it enables every transition of $p^\bullet$.

**Definition 2.17** *Siphons, proper siphons*

*A set* $R$ *of places of a net is a siphon if* $^\bullet R \subseteq R^\bullet$. *A siphon is called proper if it is not an empty set.*

Two important facts known about siphons are that: unmarked siphons remain unmarked, and live systems have no unmarked proper siphons.

**Definition 2.18** *Traps, proper traps*

*A set R of places of a net is a trap if* $R^\bullet \subseteq^\bullet R$. *A trap is called proper if it is not the empty set.*

Finally, a useful lemma, taken from [4], is presented. The proof in presented in the same reference.

**Lemma 2.2** *Every live and bounded system* $(N, m_0)$ *has a reachable marking* $m$ *and an occurrence sequence* $m \xrightarrow{\sigma} m$ *such that all transitions of* $N$ *occur in* $\sigma$.

## 2.2 Continuous Petri Nets

Loosely speaking, the fluidification or continuization is a procedure in which a continuous dynamic system is obtained from a discrete event one.

As it was mentioned in the introduction, the fluidification is one of the classical relaxations of $DES$ models. This relaxation can be applied to Petri Nets in order to deal with the so called state explosion problem. The computational gain is usually increased if dealing with highly populated systems, because in those cases the state explosion problem may become much more acute.

The firing logic of $PNs$ is of the type consumption/serves. Thus, continuization should be introduced through transitions, and extended to its neighborhood (input and output places). When not all transitions are continuized, the obtained model is said to be hybrid. If all the transitions are continuized the net is said to be continuous ($contPN$). This dissertation will focus only in continuous nets.

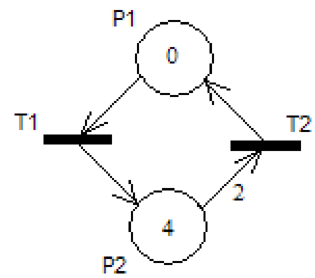


Figure 2.2 $ContPN$ system. Only transition $t_2$ is enabled to fire.

Unlike discrete $PN$, the amount in which a transition can be fired in $contPNs$ is not restricted to a natural number, actually, a transition $t$ is enabled at $m$ iff $\forall p \in^{\bullet} t$, $m[p] > 0$. Let us see the definition of the enabling degree of transitions.

**Definition 2.19** *Enabling degree*

*The enabling degree of $t$ is*

$$enab(t,m) = min_{p \in {}^{\bullet}t} \frac{m[p]}{Pre[p,t]}$$

The transition $t$ can fire in a certain amount $\alpha \in \mathbb{R}$, $0 \leq \alpha \leq enab(t,m)$ leading to a new marking $m\prime = m + \alpha C[P,t]$, where $C$ is the incidence matrix.

If $m$ is reachable from $m_0$ through a sequence $\sigma$, a fundamental equation can be written: $m = m_0 + C\sigma$, where $\sigma \in (\mathbb{R}^+ \cup \{0\})^{|T|}$ is the firing count vector.

Consider the next example.

**Example 2.1** *See the $contPN$ system of figure 2.2. The enabling degree of transition $t_2$ is $enab(t_2, m_0) = 2$, and the enabling degree of $t_1$ is $enab(t_1, m_0) = 0$, so $t_1$ cannot be fired. Suppose that transition $t_2$ is fired in an amount of $1.5$, so, after the firing the marking reached is $m = \begin{bmatrix} 1.5 & 1 \end{bmatrix}^T$.*

Next definitions are equivalents to those for discrete $PN$ systems.

**Definition 2.20** *Boundedness, liveness and lim-liveness on $contPNs$.*

*A $contPN$ is bounded when every place is bounded ($\forall p \in P$, $\exists b_p \in \mathbb{R}$ with $m[p] \leq b_p$ at every reachable marking $m$). It is live when every transition is live (it can ultimately occur from every reachable marking). Liveness may be extended to lim-live assuming that infinitely long sequence can be fired. A transition $t$ is non lim-live iff a sequence of successively reachable markings exists which converge to a marking such that none of its successors enables a transition $t$.*

**Definition 2.21** *Structural boundedness and structural liveness.*

*A net is structurally bounded when $(N, m_0)$ is bounded for every initial marking $m_0$ and is structurally live when a $m_0$ exists such that $(N, m_0)$ is live.*

**Definition 2.22** *P-semiflows and T-semiflows.*

*As in discrete $PNs$, left and right annulers of the incidence matrix $C$ are called P- and T- semiflows, respectively. The net $N$ is conservative iff $\exists y > 0$, $y \cdot C = 0$ and it is consistent iff $\exists x > 0$, $C \cdot x = 0$.*

If a $contPN$ is consistent and all transitions are fireable, then the (lim) reachable markings are solutions of the fundamental equation ($m = m_0 + C\sigma$, $m \geq 0$, $\sigma \geq 0$). Because of consistency, $\sigma \geq 0$ can be relaxed to $\sigma \in \mathbb{R}^{|T|}$, that is equivalent to $B^T \cdot m = B^T \cdot m_0$, $m \geq 0$ with $B^T$ a basis of P-semiflows. The set of all reachable markings at the limit is denoted by $lim - RS$. Like in discrete case, nets can be classified according to their structure.

## 2.3 Timed Continuous Petri Nets

Like in the discrete case, time can be associated to places, to transitions or to arcs in continuous $PNs$. A simple way to introduce time in discrete $PNs$ is to assume that all the transitions are timed with exponential probability distribution function (pdf). For the timing interpretation of continuous $PNs$ a first order (deterministic) approximation of the discrete case should be used (see [9]), assuming that the delays associated to the firing of transitions can be approximated by their mean values. For congested systems, this approximation is valid for any pdf, applying the central limit theorem.

There are some interesting properties of the timed continuous $PN$ systems that differ from that of others continuous models. In discrete $PN$, the places are essentially state variables, but redundancies may exist due to token conservation laws, this redundancies also appear in the timed continuous $PN$. The evolution of the timed continuous $PN$, as that of the discrete $PN$, takes place according to the information that each transition receives from its input places. The timed continuous $PN$ have only a flow of material that carries the information implicitly, and evolve according to information that, in standard uses, is local to each transition.

Now, basic definitions of timed continuous Petri nets are introduced.

**Definition 2.23** $TCPN$

*A timed $contPN$ or $TCPN = (N, \lambda)$ is the untimed $contPN$, $N$, together with a function $\lambda : T \rightarrow (\mathbb{R}^+)^{|T|}$, where $\lambda(t_i) = \lambda_i$ is the firing rate of transition $t_i$.*

**Definition 2.24** $TCPN$ *system*

*A $TCPN$ system is a tuple $\Sigma = (N, \lambda, m_0)$, where $(N, \lambda)$ is a $TCPN$ and $m_0$ is the initial marking of the net.*

Now, the fundamental equation depends on time $\tau$: $m(\tau) = m_0 + C \cdot \sigma(\tau)$. Deriving this equation with respect to time, the equation obtained is: $\dot{m}(\tau) = C \cdot \dot{\sigma}(\tau)$. Using the notation $f(\tau) = \dot{\sigma}(\tau)$ to represent the flow of the transition with respect of time, the fundamental equation becomes: $\dot{m}(\tau) = C \cdot f(\tau)$, which can be written in the short form:

$$\dot{m} = C \cdot f$$

but the dependence on time is considered.

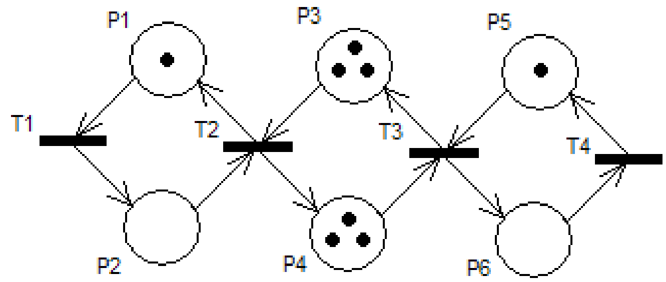


Figure 2.3 Example of a $TCPN$ system.

Depending on the flow definition, there are many firing semantics. Finite server (or constant speed) and infinite server (or variable speed) are the more frequently used. This dissertation is focused on infinite server semantics ($ISS$), with the flow of each transition defined by:

$$f_i = f[t_i] = \lambda[t_i] \min_{p_j \in {}^\bullet t_i} \left\{ \frac{m[p_j]}{Pre[p_j, t_i]} \right\}$$

Observe that the flow of transition $t$ is proportional to its enabling degree by means of the firing rate $\lambda(t_i) = \lambda_i$.

**Remark 2.1** *A $TCPN$ under infinite server semantics is a piecewise linear system due to the minimum operator that appears in the flow definition.*

**Example 2.2** *Consider the net of figure 2.3. The flows of the transitions are given by:*

$$\begin{cases} f_1 = \lambda[t_1] \cdot m[p_1] \\ f_2 = \lambda[t_2] \cdot \min(m[p_2], m[p_3]) \\ f_3 = \lambda[t_3] \cdot \min(m[p_4], m[p_5]) \\ f_4 = \lambda[t_4] \cdot m[p_6] \end{cases}$$

If $\boldsymbol{\lambda} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$, for example, then:

$$\begin{cases} \dot{m}[p_1] = f_2 - f_1 = m[p_2] - m[p_1] \\ \dot{m}[p_2] = f_1 - f_2 = m[p_1] - \min(m[p_2], m[p_3]) \\ \dot{m}[p_3] = f_3 - f_2 = \min(m[p_4], m[p_5]) - \min(m[p_2], m[p_3]) \\ \dot{m}[p_4] = f_2 - f_3 = \min(m[p_2], m[p_3]) - \min(m[p_4], m[p_5]) \\ \dot{m}[p_5] = f_4 - f_3 = m[p_6] - \min(m[p_4], m[p_5]) \\ \dot{m}[p_6] = f_3 - f_4 = \min(m[p_4], m[p_5]) - m[p_6] \end{cases}$$

Thus, nonlinearity appears due to synchronization ($|{}^\bullet t| > 1$). One linear system is defined by the set of arcs in Pre limiting the firing of the transitions.

**Definition 2.25** *Constraint on the dynamics of a transition*

*Let $\Sigma = (N, \lambda, m_0)$ be a $TCPN$ and $m$ a reachable marking. It will be said that the arc $(p, t)$ constraints the dynamic of $t$ at $m$ iff:*

$$f[t] = \lambda[t_i] \left\{ \frac{m[p]}{Pre[p, t]} \right\}$$

**Definition 2.26** *Configuration*

*A configuration of $\Sigma$ at $m$ is a set of $(p, t)$ arcs describing the effective flow of all the transitions.*

So, a configuration is a cover of $T$ by its inputs arcs. One possible representation of a given configuration is a matrix form, $\eth \in \{0, 1\}^{|P| \times |T|}$:

$$\eth[p_i, t_j] = \begin{cases} 1 & \text{if } p_j \text{ is limiting the flow of } t_i \\ 0 & \text{otherwise} \end{cases}$$

Obviously, $\eth \leq Pre$, even if the net is ordinary (i.e. all arcs have weight one). Each configuration defines an associated linear system.

**Example 2.3** *Let us consider the net of figure 2.3 with $\lambda = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$. As it was seen in previous example, this is a piecewise linear system. For the configuration $\{(p_1, t_1), (p_2, t_2), (p_5, t_3), (p_6, t_6)\}$, $m[p_2] \leq m[p_3]$ and $m[p_5] \leq m[p_4]$. Then the active linear system is:*

$$\begin{cases} \dot{m}[p_1] = m[p_2] - m[p_1] \\ \dot{m}[p_2] = m[p_1] - m[p_2] \\ \dot{m}[p_3] = m[p_5] - m[p_2] \\ \dot{m}[p_4] = m[p_2] - m[p_5] \\ \dot{m}[p_5] = m[p_6] - m[p_5] \\ \dot{m}[p_6] = m[p_5] - m[p_6] \end{cases}$$

or in matrix form:

$$\dot{m} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \cdot m \tag{2.1}$$

Now, let us consider the configuration $\{(p_1, t_1), (p_2, t_2), (p_4, t_3), (p_6, t_6)\}$. Then $m[p_5] \geq m[p_4]$ and$m[p_2] \leq m[p_3]$ and the linear system associated is:

$$\begin{cases} \overset{\bullet}{m}[p_1] = m[p_2] - m[p_1] \\ \overset{\bullet}{m}[p_2] = m[p_1] - m[p_2] \\ \overset{\bullet}{m}[p_3] = m[p_4] - m[p_2] \\ \overset{\bullet}{m}[p_4] = m[p_2] - m[p_4] \\ \overset{\bullet}{m}[p_5] = m[p_6] - m[p_4] \\ \overset{\bullet}{m}[p_6] = m[p_4] - m[p_6] \end{cases}$$

or in matrix form:

$$\overset{\bullet}{m} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \cdot m \tag{2.2}$$

Observe that, depending on the marking of the places, the evolution of the system will be given by one or other linear system. Equation (2.1) and (2.2) describe two of these different linear systems.

Any (reachable) marking defines a configuration. When the marking of several places are limiting the firing of the same transition, any of the associated linear systems can be used.

The number of minimal configurations (i.e. only one constraining arc per transition is taken) is bound by the net structure (i.e. it does not depend on the marking) and is equal to $\prod_{t_i \in T} |{}^\bullet t_i|$.

**Definition 2.27** *Matrix $H$*

*$H = [h_{i,j}]$ is $|T| \times |P|$ matrix, where*

$$h_{i,j} = \begin{cases} \frac{1}{Pre[i,j]} & \textit{if } Pre[j,i] > 0 \\ 0 & \textit{otherwise} \end{cases}$$

Observe that matrix $H$ is just the transposed of the matrix $Pre$ where the non null elements are not $Pre[p,t]$ but their inverses.

**Definition 2.28** *Matrix operator $\odot$*

*Let $R$, $W$ and $E$ be three matrices with identical dimensions. The matrix operator $\odot$ is defined as: $R = W \odot E$, where $r_{i,j} = w_{i,j} \cdot e_{i,j}$.*

**Definition 2.29** *Configuration operator*

*The configuration operator is the function $\Pi : RS(N, m_0) \to \mathbb{R}^{|T| \times |P|}$ such that:*

$$\Pi(m) = \partial(m) \odot H$$

*where $\partial(m)$ is the matrix representing the configuration associated to $m$.*

The configuration operator associated to every marking $m$ a matrix $|T| \times |P|$, such that each row $i = 1..|T|$ has only one non null element in the position $j$ that corresponds to the place $p_j$ that restricts the flow of transition $t_i$. The product $\Pi(m) \cdot m(\tau)$ is the enabling degree of each transition at time $\tau$, $enab(\tau)$.

**Definition 2.30** *Maximum firing rate matrix*

*The maximum firing rate matrix is denoted by:* $\Lambda = diag(\lambda_1, ..., \lambda_{|T|})$.

**Remark 2.2** *According to this notation, the flow vector and the fundamental equation are:*

$$\begin{aligned} f &= \Lambda \cdot \Pi(m) \cdot m \\ \dot{m} &= C \cdot \Lambda \cdot \Pi(m) \cdot m \end{aligned}$$

The only action that can be applied to a $TCPN$ system is to slow down their firing flow.

**Definition 2.31** *Controllable transition and uncontrolled transition*

*If the flow of a transition* $t$ *can be reduced or even stopped, it will be said that* $t$ *is a controllable transition, otherwise* $t$ *is an uncontrolled transition.*

The forced flow of a controllable transition $t_i$ becomes $f_i - u_i$, where $f_i$ is the flow of the unforced system (i.e. without control) and $u$ is the control action, with $0 \leq u_i \leq f_i$. The controlled flow vector is:

$$f = \Lambda \cdot \Pi(m) \cdot m - u$$

where $u_i = 0$ if $t_i$ is not a controllable transition. Thus the state equation of a controlled $TCPN$ system becomes:

$$\begin{cases} \dot{m} = C \cdot (\Lambda \cdot \Pi(m) \cdot m - u) \\ 0 \leq u \leq \Lambda \cdot \Pi(m) \cdot m \end{cases} \tag{2.3}$$

## 2.4 Rewriting the state equation

In order to obtain a simplified version of the state equation, the input vector $u$ is rewritten as:

$$u = I_u \Lambda \Pi(m) m \tag{2.4}$$

where $I_u = diag(I_{u_1}, I_{u_2}, ..., I_{u_p})$ and $I_{u_i} \in [0, ..., 1]$.

The meaning of $I_{ui}$ is the normalized breaking factor of transition $t_i$, in this case $0 \leq I_{u_i} \leq 1$. Substituting (2.4) into (2.3) results:

$$\dot{m} = C(I - I_u)\Lambda\Pi(m)m$$

where $I$ is the unit matrix.

Defining the matrix $I_c = I - I_u$, (notice that $I_{c_i} \in [0, ..., 1]$), the $TCPN$ state equation is rewritten as:

$$\overset{\bullet}{m} = CI_c\Lambda\Pi(m)m \tag{2.5}$$

The matrix $I_c$ is the new input and represents the actual percentage of transition firings. Notice that $I_c$ is a diagonal matrix and $0 \leq I_{c_i} \leq 1$.

# Chapter

# State variables and state space

The main topic in this dissertation is the study of controllability on $TCPN$ systems. As it was presented in section 2.2, a $TCPN$ is built through a procedure from a discrete $PN$.

Before starting the study of controllability, the concept of controllability for $TCPN$ systems must be clearly defined. Since $TCPN$ systems are continuous, the concept proposed in this dissertation is similar to that of continuous systems.

In continuous systems the definition of controllability is based on the concept of state, actually, the concept of state is basic in the theory of continuous systems, but, unfortunately, it differs from the concept of state for discrete event systems.

This is the main reason to review the definitions of state, state variable and state space of both continuous and discrete event systems, and to try to find the common underlaying idea of those definitions.

In the first section of this chapter, a brief discussion of the concepts of state and state variables is presented. In the second section, some results on reachability obtained from [6] are presented. In the last two sections, the admissible states set is defined and characterized, also a minimum order state equation, which is valid in this set, is obtained.

## 3.1 State and state variables

In this section, classic definitions of state, state variable and state space of linear continuous-time systems and discrete event systems are compared. These definitions are mainly taken from [16], [3] and [15].

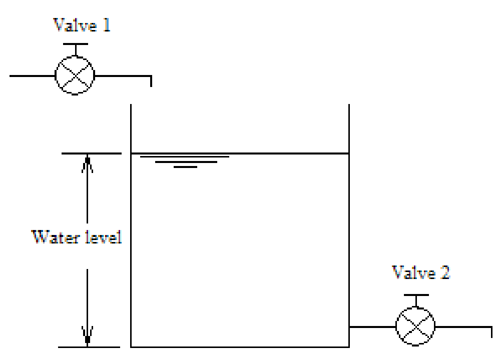


Figure 3.1 Water tank with an input flow and an output flow.

For discrete event systems, the states and state variables are usually defined directly from the physical system during the modelization, and once the states and the state variables are defined, the state space appears naturally. In other way, there is a formal definition of state through the Nerode equivalence relation.

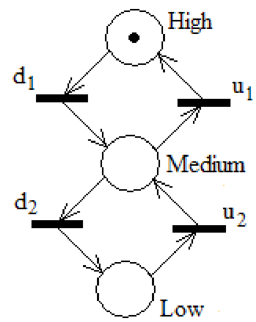


Figure 3.2 $PN$ system that models the physical system of figure 3.1.

In order to illustrate this, see the next example.

**Example 3.1** *Consider the physical system of figure 3.1.*

*In this system, the water level in the tank is the variable of interest. Three different levels or "states", named: high, medium and low, can be distinguished. The initial state is high. So, the state variable is the water level, and the state space is {high, medium, low}. The level can change from high to medium by the flow of the second valve, when this happen it is said that event* $d_1$ *occurs. Similarly, event* $d_2$ *occurs when level change from medium to low, event* $u_1$ *when level raise from medium to high, and event* $u_2$ *when level raise from low to medium.*

At this point, we are able to model this physical system into a $PN$ system, as shown in figure 3.2, but here we are interested in a formal definition of state and state variable, so, we will use a linguistic interpretation of the system.

The language of the system (the sequences of events that may happen in the system), denoted by $L$, includes words like: $\{\varepsilon, d_1u_1, d_1, d_1u_1d_1, d_1d_2, d_1u_1d_1d_2, d_1d_2u_2d_2, ...\}$. This language defines the states of the system through the Nerode relation.

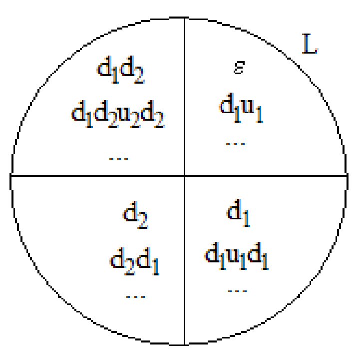


Figure 3.3 Partition of $\Sigma^*$ under Nerode equivalence relation.

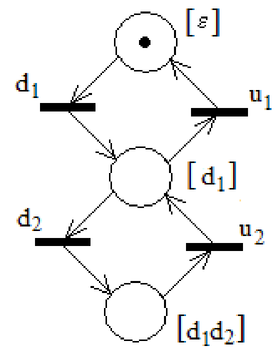


Figure 3.4 $PN$ system formally obtained.

**Definition 3.1** *Nerode equivalence relation*

*Let $L \subseteq \Sigma^*$ be an arbitrary language, where $\Sigma$ is its alphabet. The Nerode equivalence relation on $\Sigma^*$ with respect to $L$ is defined as follows.*

*For $s, t \in \Sigma^*$, $s \equiv_L t$ iff $\forall u \in \Sigma^*$, $su \in L$ iff $tu \in L$.*

In other words $s \equiv_L t$ iff $s$ and $t$ can be continued in exactly the same way to form a string of $L$.

Since this is an equivalence relation, it makes a partition of $\Sigma^*$ (see figure 3.3). Now a formal definition of state can be introduced.

**Definition 3.2** *State in discrete event systems.*

*An state of a discrete event system is an equivalence class or cluster of $\Sigma^*$ under the Nerode equivalence relation.*

Loosely speaking, the state variable is a function that takes values on the set of all the states (range of the state variable). Finally, considering only the states in which the words belong to the language, and the events, which makes a state change, as transitions, the model of figure 3.4 can be built .

Notice that this $PN$ system is equal to that of figure 3.2, but now the state and the state space (the range of the state variable for this example) are formally defined.

Loosely speaking, a state in a discrete event system is a set of the physical states (physical situations or conditions) for which the observer variables (output variables ) evolve in the same way.

Now, the state definition for continuous systems will be reviewed. The state of a continuous system at time instant $t$ should describe its behavior at that instant in some measurable way. In system theory, the term state has a much more precise meaning and constitutes the cornerstone of the modeling process and many analytical techniques.

See the next example.

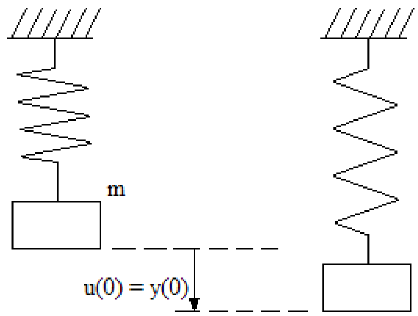


Figure 3.5 Example of a continuous system.

**Example 3.2** *Consider the system of figure 3.5. Suppose that at time $t = 0$ the mass is displaced from its rest position by an amount $u(0) = u_0 > 0$ and released it. Let the displacement at any time $t > 0$ be denoted by $y(t)$. It is known, from simple mechanics, that the motion of the mass defines an harmonic oscillation described by the second-order differential equation:*

$$m\ddot{y} = -ky \tag{3.1}$$

Now, suppose that the output $y(t)$ is observed at some time $t = t_1 \geq t_0$. Mathematically, from the equation (3.1), it is clear that it cannot be solved for $y(t_1 + \tau)$ with only one initial condition, i.e. $y(t_1)$; also information about the first derivative $\dot{y}(t_1)$ is needed .

Observe that together $y(t_1)$ and $\dot{y}(t_1)$ provide the information required which, along with full knowledge of the input function, allows to obtain a unique solution and hence the value of $y(t_1 + \tau)$. This leads to the well-known state definition for continuous time systems.

**Definition 3.3** *State and state variables in continuous time systems.*

*The state of a system at time $t_0$ is the information required at $t_0$ such that the output $y(t)$, for all $t \geq t_0$, is uniquely determined from this information and from $u(t)$, $t \geq t_0$.*

*Like the input $u(t)$ and the output $y(t)$, the state is also generally a vector, commonly denoted by $x(t)$. The components of this vector are called state variables.*

Notice that, according to previous definition, the state and the state variables are conceptually equivalents.

Now, let us introduce the term “state space”.

**Definition 3.4** *The state space in continuous time systems.*

*The state space of a system, usually denoted by $X$, is the set of all possible values that the state may take.*

In example 3.2, the state variables have a physical meaning, those are the position and the velocity of the mass. However, for a general case, sometimes the state variables have not a physical meaning. In fact, in some identification techniques only the number of state variables is proposed and the identification process determines the relation between those and the input and output variables, so the state variables are not physical variables. However, they are related to some physical variables so they are needed to model the dynamics of the physical system. The state variables act like dynamic memory elements in the dynamics of the physical system.

At this point, we can notice that the state definition of both continuous and discrete event systems are clearly different. So, now we propose the following definition of state variable, which can cover both previous definitions.

**Definition 3.5** *Concept of state variable.*

*An state variable is a function that captures a dimensional property, not necessary measurable, of the physical system as a value of a set named range of the state variable. The set of all the state variables must be sufficient to build a dynamic model of the physical system.*

It is easy to see that this concept agrees with the definition of state variable in continuous systems. In those, the dimensional properties can be physical variables such as position, velocity, temperature, pressure, etc., or physically meaningless variables, but even in this case there must exist something in the physical system related to the value assigned to this variable which is necessary for the dynamic behavior. The range of those variables is the set of real numbers.

So, in the continuous system, the "state" is the function named state variable.

For the example 3.1, which is modeled as a discrete event system, the dimensional property is the water level, and the range of it is the set $\{$high, medium, low$\}$. In discrete event systems, the state is a value that the state variable can take.

Now, we will focus in the transformation that the state variable suffers when a $DES$ model is fluidificated.

Notice that in $PN$ systems, the states, as defined in the $DES$ definition, are codified as given marking distributions, and the state variable is codified as the marking (as a function).

For the example of the water tank, the range of the state variable (or the states, according to $DES$ definitions) is equivalent to the set:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

which correspond to high, medium and low, respectively.

After fluidification, a continuous system is obtained in which a marking such that $m$ =[0.1, 0.8, 0.1]$^T$ may exist. The distribution of markings that can be generated by this continuous Petri net is shown in figure 3.6, where the places $p_1$, $p_2$ and $p_3$ correspond to those labeled as high, medium and low, respectively, in figure 3.2.

Now, the range of the state variable is isomorphic to $\mathbb{R}^2$, not to a finite set of markings like in $DES$ systems. Besides, in continuous systems the range of a state variable is $\mathbb{R}$. Notice that after the fluidification, the marking does not have a direct meaning of the physical situation of the plant, so it cannot be said that a set of state variables is codified into the marking of the $TCPN$ system because there does not exist a function that maps from the physical system to the marking in the $TCPN$ system. Therefore, it cannot be said that the marking of a set of places in the $TCPN$ system is a state variable.

Since the $TCPN$ is a continuous model, we will use the continuous linear systems theory concepts to study the controllability in this, therefore, we need that the state variables (as defined in continuous systems)

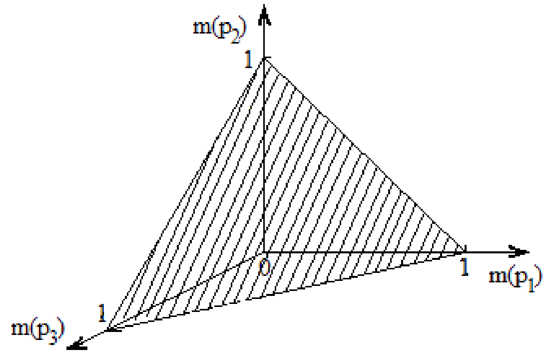


Figure 3.6 The shadowed triangle constitutes the set of markings that can be generated by the $contPN$.

be isomorphic to $\mathbb{R}$. So, for $TCPN$ systems, we will consider the marking of a place, which is isomorphic to $\mathbb{R}$, as a state variable, but we have to keep in mind that it is not, formally speaking, a state variable of the system.

## 3.2 Reachability in continuous Petri nets

In this section, some results on reachability obtained by Júlvez, Recalde and Silva [6] are presented . First, consider the case of untimed continuous Petri net.

The set of all reachable markings for a given system $(N, m_0)$ is denoted by $RS(N, m_0)$.

**Definition 3.6** *The set of all reachable markings*

*$RS(N, m_0)$ ={$m$| a finite fireable sequence $\sigma = \alpha_1 t_{\alpha_1} .. \alpha_k t_{\alpha_k}$ exists such that $m_0 \overset{\alpha_1 t_{\alpha_1}}{\rightarrow} m_1 \overset{\alpha_2 t_{\alpha_2}}{\rightarrow} m_2$ ... $\overset{\alpha_k t_{\alpha_k}}{\rightarrow} m_k = m$ where $t_{\alpha_i} \in T$ and $\alpha_i \in \mathbb{R}^+$}*

An interesting property of $RS(N, m_0)$ is that it is a convex set (see [10]). That is, if two markings $m_1$ and $m_2$ are reachable, then for any $\alpha \in [0, 1]$, $\alpha m_1 + (1 - \alpha) m_2$ is also a reachable marking.

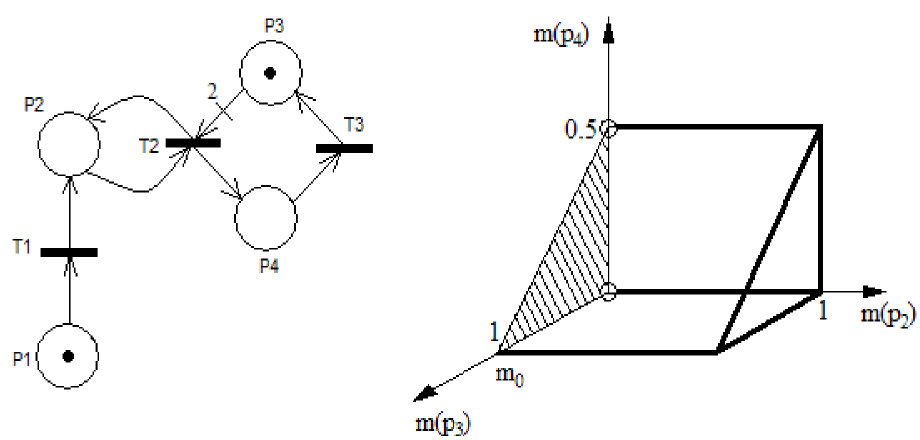


Figure 3.7 Autonomous continuous system and its lim-reachability space.

Consider the system in figure 3.7 with initial marking $m_0 = \begin{bmatrix} 0.5 & 0.5 & 0 & 0.5 \end{bmatrix}^T$. At this marking either transition $t_1$ or transition $t_3$ can be fired. The firing of $t_3$ in an amount of $0.5$ makes the system evolve to marking $\begin{bmatrix} 0.5 & 0.5 & 0.5 & 0 \end{bmatrix}^T$ from which $t_2$ can be fired in an amount of $0.25$ leading to marking $\begin{bmatrix} 0.5 & 0.5 & 0 & 0.25 \end{bmatrix}^T$. Now, the markings of places $p_1$, $p_2$ and $p_3$ are the same that those of the system at $m_0$, but the marking of $p_4$ is half of its marking at $m_0$. The continuous firing of transition $t_2$ and $t_3$ by its maximum enabling degree causes the elimination of half of the marking of $p_4$. Assume that it goes on firing transitions $t_2$ and $t_3$. Then, as the number of firings increases the marking of $p_4$ approaches $0$, value that will be reached only in the limit. The marking reached in the limit is $\begin{bmatrix} 0.5 & 0.5 & 0 & 0 \end{bmatrix}^T$. Now, the set of such markings will be defined, i.e. the markings that are reachable with a finite/infinite firing sequence:

**Definition 3.7** *The set of lim-reachable markings*

*Let $(N, m_0)$ be a continuous system. A marking $m \in (\mathbb{R}^+ \cup \{0\})^{|P|}$ is lim-reachable, iff a sequence of reachable markings $\{m_i\}_{i\geq 1}$ exists such that*

$$m_0 \xrightarrow{\sigma_1} m_1 \xrightarrow{\sigma_2} m_2 ... m_{i-1} \xrightarrow{\sigma_i} m_i ...$$

*and $lim_{i\to\infty} m_i = m$. The lim-reachable space is the set of lim-reachable markings, and it will be denoted $lim - RS(N, m_0)$.*

Consider again the system of figure 3.7. It is not necessary to represent the marking of place $p_1$ since $m_1 = 1 - m_2$. The set of lim-reachable markings is composed of the points inside the prism, the points in the non shadowed sides, the points in the thick edges and the points in the non circled vertices.

The set of reachable markings, $RS(N, m_0)$ is a subset of the set of lim-reachable markings, $lim - RS(N, m_0)$, and for some systems both sets are identical.

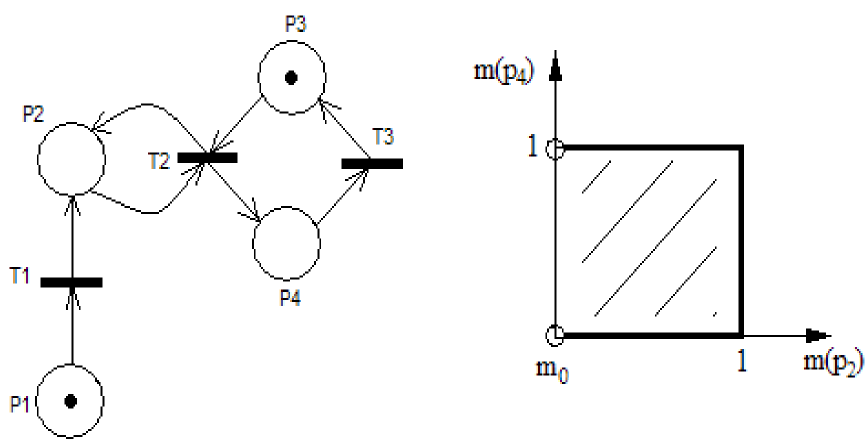


Figure 3.8 Autonomous continuous system and its reachability and lim-reachability spaces.

Both $RS(N, m_0)$ and $lim - RS(N, m_0)$ are not in general closed sets. Consider the system of figure 3.8. In this figure, the points on the segment going from $(0, 0)$ to $(0, 1)$ do neither belong to $RS(N, m_0)$ nor to $lim - RS(N, m_0)$. Nevertheless, any point on the right of this segment belong to both sets.

**Definition 3.8** *Closure of a set*

*For a given set A, the closure of A is equal to the points in A plus those points which are infinitely close to points in A, but are not contained in A.*

The set $\delta$-reachable markings will be written as $\delta - RS(N, m_0)$ and accounts for those markings to which the system can get as closed as desired firing a finite sequence. Formally:

**Definition 3.9** *The set of $\delta$-reachable markings*

*$\delta - RS(N, m_0)$ is the closure of $RS(N, m_0)$ : $\delta - RS(N, m_0) = \{m|$ for every $\varepsilon > 0$ a marking $m' \in RS(N, m_0)$ exists such $|m' - m| < \varepsilon\}$.*

Since the closure of $RS(N, m_0)$ is equal to the closure of $lim - RS(N, m_0)$, $\delta - RS(N, m_0)$ is also equal to the set of markings to which the system can get as close as desired firing an infinite sequence. $RS(N, m_0)$ and $lim - RS(N, m_0)$ are, therefore, subsets of $\delta - RS(N, m_0)$.

Therefore, until now three different kinds of reachability concepts have been defined:

- Markings that are reachable with a finite firing sequence, $RS(N, m_0)$.

- Markings to which the system converges, eventually, with an infinitely long sequence, $lim - RS(N, m_0)$.

- Markings to which the system can get as close as desired with a finite sequence, $\delta - RS(N, m_0)$.

These reachability spaces can be fully characterized using, among other elements, the state equation. Moreover, it is decidable whether a marking is reachable according to each concept. Furthermore, there is an inclusion relationship among the sets of markings : $RS(N, m_0) \subseteq lim - RS(N, m_0) \subseteq \delta - RS(N, m_0)$. The only difference among these sets are in the border points of the spaces (i.e., the convex hull).

Full characterization of each reachability space can be seen in [6].

For $TCPN$ systems, consider the following definition of reachability.

**Definition 3.10** *Reachability for $TCPN$ systems.*

*Given a $TCPN$ system $\langle N, m_0 \rangle$, the set of all reachable markings ($RS_t$) is defined as $RS_t(N, m_0) = \{m_f | \exists u(\tau)$ suitable bounded such that $m_0 \xrightarrow{u} m_f$ in finite time$\}$.*

A marking that belongs to $RS_t(N, m_0)$ is said to be reachable. Like in the untimed case, the sets $lim - RS_t$ and $\delta - RS_t$ are defined.

When all transitions are controllable, there is an important result about reachability introduced in [7].

**Proposition 3.1** *Equivalence of lim-reachable sets of timed and untimed $contPNs$*

*Given a $TCPN$ system, if all transitions are controllable, then all the reachable markings of the untimed $contPN$ can be reached in the timed model, maybe at the limit. ($lim - RS_t = lim - RS$).*

In the general case in which there exist uncontrolled transitions the reachability spaces presented in this section are not characterized. Even deciding whether a given marking $m$ is reachable or not is a difficult task.

In the next chapters, the notation $RS(N, m_0)$ will be used for the reachable set of the timed continuous systems.

## 3.3 Admissible states set and reachable states set

As it was presented in previous section, the characterization of the reachability space is a difficult task because it strongly depends on the initial marking. So, the reachability and controllability problems will be studied with a different approach in this dissertation.

In this way, we will propose a set of markings to study if that set is reachable from the initial marking, and if the system is controllable on it.

As it was presented in section 3.1 we consider the marking of each place as a state variable. Then the range of a state variable is a subset of $\mathbb{R}$. Considering that the state space of a continuous system is the cartesian product of the ranges of the state variables, and that the markings of every place are defined as positives, we introduce next definition:

**Definition 3.11** *Structural admissible states set*

*Let $N$ be a $TCPN$ . The structural admissible states set is defined as* $SASS(N) = \{R^{+} \cup \{0\}\}^{|P|}$

Given a general $TCPN$ system $(N, m_0)$, not always all markings in $SASS(N)$ belong to the state space of that system, as it can be seen in the system of figure 3.9. However, all reachable markings belong to $SASS(N)$. (i.e., $lim - RS(N, m_0) \subseteq SASS(N)$). Actually, when $N$ is conservative, i.e. it has P-semiflows, there exists a static relation between markings of the places which belong to the same P-semiflow. It causes that the $lim - RS(N, m_0)$ be an invariant subset of $SASS(N)$. In order to characterize this invariant set, we introduce next definitions.

**Definition 3.12** *Relation $\beta$*

*Let $N$ be a $TCPN$ . Let $B$ be the base of the left annuller of the incidence matrix $C$. The relation $\beta : SASS(N) \rightarrow SASS(N)$, is defined as:*

$$m_1 \beta m_2 \textit{ iff } B^T m_1 = B^T m_2,\ \forall m_1, m_2 \in SASS(N)$$

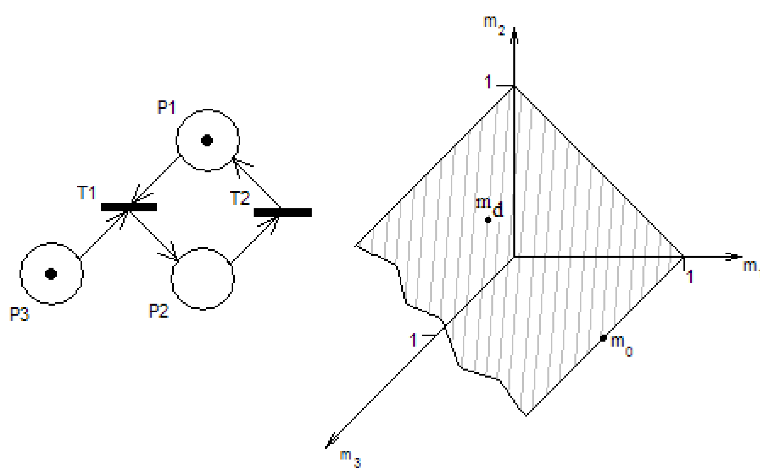


Figure 3.9 A $TCPN$ system, its $SASS(N)$ and its $Class(m_0)$.

Notice that $\beta$ is an equivalence relation so it makes a partition of $SASS(N)$.

**Definition 3.13** *System admissible states set*

*Let $\langle N, m_0 \rangle$ be a $TCPN$ system. The system admissible states set is the equivalent class of the initial marking $Class(m_0)$ under $\beta$.*

The $Class(m_0)$ set is not equivalent to the sets of all reachable markings $RS(N, m_0)$, $lim - RS(N, m_0)$, or $\delta - RS(N, m_0)$ defined by Júlvez, Recalde and Silva [6].

In order to illustrate previous definitions see the figure 3.9, in this example, $SASS(N) = \{R^+ \cup \{0\}\}^3$; the shadowed surface corresponds to $Class(m_0)$. Notice that $m_d$ belongs to $Class(m_0)$ but is not reachable from $m_0$ in finite or infinite time, i.e. $m_d \notin lim - RS(N, m_0)$.

Since every reachable marking of the $TCPN$ system $(N, m_0)$ must fulfill that $B^T m = B^T m_0$ (because the P-semiflows) and $Class(m_0)$ is the greatest set of nonnegative markings that fulfills this condition, then $lim - RS(N, m_0) \subseteq Class(m_0)$.

So, we have defined the set $Class(m_0)$ which includes the set $lim - RS(N, m_0)$. Notice that $Class(m_0)$ is easier to characterize than $lim - RS(N, m_0)$. In next chapter, we will study when either $Class(m_0)$ or a subset of $Class(m_0)$ (which will be subsequently defined) is reachable and the system is controllable on it.

## 3.4 The minimum order state equation

Consider a conservative $TCPN$ system. Let $\{m_1^i, m_2^i, ..., m_q^i\}$ be the set of the markings that belong to the $i - th\ P - semiflow$, therefore:

$$m_1^i + m_2^i + ... + m_q^i = K, K \in \mathbb{N} \tag{3.2}$$

Deriving previous equation, the following equation is obtained:

$$\overset{\bullet}{m}_1^i + \overset{\bullet}{m}_2^i + ... + \overset{\bullet}{m}_q^i = 0$$

Thus, the marking dynamics can be computed using $q - 1$ places and the conservative marking law imposed by the $i - th\ P - semiflow$.

In order to obtain a $TCPN$ minimum state equation, it is needed to eliminate the linearly dependent rows of the incidence matrix $C$, such that the rank of $C$ is preserved. Let $m_m$ be the state of the $TCPN$ minimum state equation, then $m_m(\tau)$ is a projection of $m(\tau)$, i.e.:

$$m_m = Pm \tag{3.3}$$

where $P$ is a projection matrix; $P$ is, in general, not invertible. In order to obtain $m(\tau)$ from $m_m(\tau)$ the following equation is used:

$$\begin{bmatrix} P \\ B^T \end{bmatrix}^{-1} \begin{bmatrix} m_m(\tau) \\ B^T m_0 \end{bmatrix} = m(\tau) \tag{3.4}$$

So, there is a bijection between $m(\tau)$ and $m_m(\tau)$. Notice that $B^T m_0$ is constant and contains the information of the P-semiflows. Now, let define the function $G$ such that:

$$G(m_m(\tau)) = \Pi(m)m(\tau) \tag{3.5}$$

Finally, the $TCPN$ minimum state equation is written as:

$$\overset{\bullet}{m}_m = C_m I_c \Lambda G(m_m) \tag{3.6}$$

where $m_m(\tau)$ is the minimum state vector. This equation does not represent a minimum model of the net because the $P - Semiflows$ are also needed to compute the whole $TCPN$ marking.

Matrices $I_c$ and $\Lambda$ are the previously defined ones, while $C_m = PC$.

**Definition 3.14** *Minimum order Class of equivalence.*

*Let* $\langle N, m_0\rangle$ *be a* $TCPN$ *system. The minimum order Class of equivalence of* $m_0$ *is defined as* $Class_m(m_0) = \{m_m | m_m = Pm, m \in Class(m_0)\}$.

**Proposition 3.2** *Characterization of the interior of* $Class_m(m_0)$.

*Let* $m = [m_1, m_2, ..., m_{|P|}]^T \in Class(m_0)$ *be a marking.* $\forall i\ m_i \neq 0$ *iff* $m_m$ *is an interior point of* $Class_m(m_0)$.

**Proposition 3.3** *Equivalence of solutions of the state equation and the minimum state equation.*

*An input* $u$ *transfers the state* $m$ *from* $m_0 \in Class(m_0)$ *to* $m_1 \in Class(m_0)$ *at time* $t_1$ *iff* $u$ *transfers the state* $m_m$ *from* $m_{m_0}$ *to* $m_{m_1}$ *at time* $t_1$. *Where* $m_{m_0} = P\ m_0$ *and* $m_{m_1} = Pm_1$ .

**Proof** Let $\langle N, m_0\rangle$ be a $TCPN$ system. Consider the state equation of the system as the equation (2.5), and its minimum order state equation as the equation (3.6). Let $m_{m_0}$ be the minimum initial marking. Now, suppose that the input $Ic$ is applied to both the state equation and the minimum order state equation, then the marking reached by the state equation at time $t_1$ fulfills with

$$m_1(t_1) = m_0 + C\Lambda \int_0^{t_1} I_c \Pi(m) m dt \tag{3.7}$$

and the marking reached by the minimum order state equation at the same time fulfills with

$$m_{m1}(t_1) = m_{m_0} + C_m \Lambda \int_0^{t_1} I_c G(m_m) dt. \tag{3.8}$$

For the necessity, premultiplying the equation (3.7) by the projection matrix $P$, and according to equations ((3.3)) and ((3.5)), the next equation is obtained.

$$Pm_1 = m_{m_0} + C_m \Lambda \int_{t_0}^{t_1} I_c G(m_m) dt$$

Comparing to equation (3.8), then $m_{m_1} = Pm_1$.

For the sufficiency, follow the same reasoning and the fact that $m_0$ and $m_1$ can be obtained from $m_{m_0}$ and $m_{m_1}$ with the equation (3.4). ■

# Chapter

# Controllability

For classical linear systems controllability has been thoroughly studied. Although $TCPN$ systems are continuous systems, the classical linear systems definition of controllability cannot be applied to $TCPN$ systems because the required hypothesis are not fulfilled, i.e. the input should be unbounded and the state space should be $\mathbb{R}^{|P|}$.

However, an interpretation of the controllability of $TCPN$ systems under that definition is first presented in this chapter, before introducing a new controllability definition. This interpretation is taken from [7].

The linear system controllability classical definition is the following.

**Definition 4.1** *Controllability for linear continuous-time systems.*

*An state equation is fully controllable if there exists an input such that for any two states $x_1$ and $x_2$ of the state space, it is possible to transfer the state from $x_1$ to $x_2$ in finite time. Otherwise the state equation is uncontrollable.*

Notice that the reachable markings of a $TCPN$ system does not form a state space (vector space) and the input of $TCPN$ systems must be positive and bounded. Contrary to linear continuous-time systems in which the state space is a vector space and it does not exist any constraint imposed to the input.

In system theory, a very well-known controllability criterion exists which allows to verify whether a continuous linear system is controllable or not, for this, let us introduce the controllability matrix:

**Definition 4.2** *Controllability matrix.*

*Given a linear system $\overset{\bullet}{x}(\tau) = A \cdot x(\tau) + B \cdot u(\tau)$, the controllability matrix is defined as:*

$$\mathbb{C} = [B, ..., A^k B, ..., A^{n-1} B]$$

Then, next proposition gives sufficient and necessary conditions to controllability in linear continuous systems:

**Proposition 4.1** *Controllability of a linear continuous-time system.*

*A linear continuous-time system $\overset{\bullet}{x}(\tau) = A \cdot x(\tau) + B \cdot u(\tau)$ is completely controllable iff the controllability matrix $\mathbb{C}$ has full rank. If $\mathbb{C}$ is not a full rank matrix then the system has only $rank(\mathbb{C})$ controllable state variables.*

For $TCPN$ systems, every $\Pi(m)$ leads to a linear and time-invariant dynamic system with controllability matrix $\mathbb{C}(m)$. Considering the state equation as in (2.3), the controllability matrix is:

$$\mathbb{C}(m) = [C, ..., (C \cdot \Lambda \cdot \Pi(m))^{n-1} \cdot C]$$

**Proposition 4.2** *Equivalence of spaces generated for $\mathbb{C}(m)$ and $C$.*

*If all transitions are controllable, $\forall m$, the space generated by the columns of $\mathbb{C}(m)$ and $C$ are equal. Thus $rank(\mathbb{C}(m)) = rank(C) = |P| - dim(B)$.*

**Proof** Observe that $(C \cdot \Lambda \cdot \Pi(m))^{n-1} \cdot C = C \cdot (C \cdot \Lambda \cdot \Pi(m))^{n-1}$. Thus, $rank(C) = rank(\mathbb{C})$. ■

Notice that $\mathbb{C}(m)$ depends on $\Pi(m)$, but the space generated by its columns is always the same, just that one defined by that of matrix $C$. Thus is something that can be easily expected because all transitions have been assumed to be controllable.

Nets with at least one P-semiflow are non controllable in the classical sense of dynamic system for any firing rate $\lambda$ and any initial marking $m_0$. P-semiflows based token conservation laws make $|P| - rank(\mathbb{C})$ places linearly-redundant. As it was presented in section 3.3, this token conservation laws causes that the reachable space be an invariant subset of $SASS(N)$ of dimension $rank(C)$. The difference between the dimension of the space generated by $C$ and the number of the states variables $|P|$ corresponds to the $|P| - rank(\mathbb{C})$ zero valued poles of the $TCPN$ system, described in [7]. This zero valued poles, which also are uncontrollable, are eliminated in the minimum order state equation.

In the next section we propose a definition of controllability for $TCPN$ systems as an adaptation of the classical linear continuous-time systems controllability definition. In the second section of this chapter we study the controllability in $TCPN$ systems where all transitions are controllable. Finally, in the last section we study the controllability for the case with uncontrolled transitions.

## 4.1 Definitions

Now, we propose a definition of controllability which is an adaptation of that for linear continuous systems.

**Definition 4.3** *Fully controllability with bounded input $BIFC$.*

*Let $\langle N, m_0 \rangle$ be a $TCPN$ system. $\langle N, m_0 \rangle$ is fully controllable with bounded input ($BIFC$) if there is an input such that for any two markings $m_1, m_2 \in Class(m_0)$, it is possible to transfer the marking from $m_1$ to $m_2$ and the input fulfills that $0 \leq u_i \leq [\Lambda\Pi(m)m]_i$ along the trajectory.*

This controllability definition can be restricted to a set of states.

**Definition 4.4** *Controllability with bounded input $BIC$.*

*Let $\langle N, m_0 \rangle$ be a $TCPN$ system. The $TCPN$ system is controllable with bounded input ($BIC$) over $S \subseteq Class(m_0)$ if there is an input such that for any two states $m_1, m_2 \in S$ it is possible to transfer the state from $m_1$ to $m_2$ and the input fulfills that $0 \leq u_i \leq [\Lambda\Pi(m)m]_i$ along the trajectory.*

As we demonstrated in section 3.4, there exists an equivalence between the solutions of both the state equation and the minimum state equation. Now, next proposition shows the equivalence of controllability for both equations.

**Proposition 4.3** *Equivalence of controllability for the state equation and the minimum state equation.*

*Let $\langle N, m_0\rangle$ be a $TCPN$ system. Consider the state equation of the system as equation* (2.5)*, and its minimum state equation as* (3.6) *with initial condition $m_{m_0} = Pm_0$. The system $\langle N, m_0\rangle$ is $BIFC$ iff its minimum state equation is fully controllable over $Class_m(m_0)$ and $I_{c_i} \in [0, 1]$ along the trajectory.*

**Proof** (Sufficiency). Let $m_1$ and $m_2$ be any two markings that belong to $Class(m_0)$. Let $m_{m_1}$ and $m_{m_2}$ be two markings that belong to $Class_m(m_0)$ such that $m_{m_1} = Pm_1$ and $m_{m_2} = Pm_2$. By hypothesis, the minimum state equation is fully controllable over $Class_m(m_0)$ so there is an input $u$ that transfers $m_1$ to $m_2$ and $0 \le u \le \Lambda G(m_m)$, i.e. $0 \le u \le \Lambda\Pi(m)m$. According to proposition 3.3, the same input $u$ transfers the state from $m_0$ to $m_1$, therefore the system is $BIFC$. The necessity follows from a similar reasoning. ■

Next definition introduces an important concept for the study of continuous systems, which will be very useful for the study of controllability in case of existing uncontrolled transitions.

**Definition 4.5** *Equilibrium points.*

*Let $(N, m_0)$ be a $TCPN$ system. Let $m_d \in RS(N, m_0)$ and $0 \le ud \le \Lambda \cdot \Pi(m_d) \cdot m_d$. Then $(m_d, u_d)$ is an equilibrium point if $\dot{m}_d(u_d) = 0$.*

An equilibrium point represents a state in which the system can be maintained using the defined control action. Given an initial marking $m_0$ and a required marking $m_d$, one control problem is to reach and maintain $m_d$. From definition, a marking $m_d$ is an equilibrium marking if $C \cdot (\Lambda \cdot \Pi(m_d) \cdot m_d - u_d) = 0$. Therefore, the flow of a controlled $TCPN$ in the equilibrium marking $m_d$, with $u_d$ as input, is a T-semiflow.

A broad study of equilibrium points in $TCPN$ systems can be found in [7].

## 4.2 The case of all transitions as controllable

In this section we will study the controllability of $TCPN$ systems, according to the definitions of the previous section, for the case in which all transitions are controllable.

Next theorem gives sufficient and necessary conditions to verify whether a $TCPN$ system is $BIC$ over the interior of $Class(m_0)$ or not.

**Theorem 4.1** *Controllability over the interior of $Class(m_0)$.*

*Let $\langle N, m_0\rangle$ be a $TCPN$ system. Consider the minimum state equation of the net as in equation* (3.6)*, and let $n$ be the order of the minimum state equation. Let $S$ be the set of all interior points of $Class_m(m_0)$. The system $\langle N, m_0\rangle$ is $BIC$ over $S$ iff $\forall d \in R^n$ $\exists v \in \{R^+ \cup \{0\}\}^{|T|}$ such that $C_m v = d$.*

**Proof** According to proposition 3.2, $\forall m_m \in S$ its elements are non zero. (Sufficiency) Let $d$ be any vector in $R^n$, by hypothesis $\exists v \in \{R^+ \cup \{0\}\}^{|T|}$ such that $C_m v = d$. The vector $G(m_m)$ can be written as:

$$G(m_m) = \Pi(m)m = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_{|T|} \end{bmatrix} \tag{4.1}$$

By hypothesis $\forall i,\ m_i \neq 0$, then $\pi_i \neq 0$. From definition $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_{|T|})$ and $I_c = diag(I_{c_1}, I_{c_2}, ..., I_{c|T|})$, so the column vector $[I_c \Lambda G(m_m)]$ can be written as:

$$I_c \Lambda G(m_m) = \begin{bmatrix} I_{c1}\lambda_1\pi_1 \\ I_{c2}\lambda_2\pi_2 \\ \vdots \\ I_{c|T|}\lambda_{|T|}\pi_{|T|} \end{bmatrix} \tag{4.2}$$

Notice that it is always possible to independently change all the elements of the vector $[I_c \Lambda G(m_m)]$ through $I_c$, so there exists a factor $\alpha \in R^+ - \{0\}$ and an input $I_c$ with $I_{ci} \in [0, 1]$ such that:

$$\alpha v = I_c \Lambda G(m_m)$$

Applying to the minimum state equation:

$$\dot{m}_m = C_m \alpha v$$

and by sufficiency hypothesis:

$$\dot{m}_m = \alpha d$$

Therefore it is always possible to direct the field vector in all $m_m \in Class_m(m_0)$ to any desired direction $d$, and then to follow any trajectory in $Class_m(m_0)$, and due the convexity of $Class_m(m_0)$, there is a trajectory from $m_{m_0}$ to any $m_{m_d} \in Class_m(m_0)$. Finally, the minimum state equation is fully controllable, besides $I_c \in [0, 1]$, then the system $\langle N, m_0 \rangle$ is $BIFC$.

(Necessity). For the following reasoning, refer to the figure 4.1.

By hypothesis, $\exists d \in R^n$ such that $\forall v \in \{R^+ \cup \{0\}\}^{|T|}$, $C_m v \neq d$ (i.e. the vector $d$ is not a positive linear combination of the columns of $C_m$ ).

Suppose that $d$ is such that all the elements of $C_m^T d$ are not positive, (if it is not the case, a new vector $d'$, such that all the elements of $C_m^T d'$ are non positive, can be found from $d$ eliminating its positive components in the directions of the columns of $C_m$).

Let $q$ be an interior point of $Class_m(m_0)$, let $s$ be the perpendicular plane to the direction of $d$ that passes through $q$, then $s$ divides $Class_m(m_0)$ in two regions, named $\Omega^+$ and $\Omega^-$, where $(d + q) \in \Omega^+$. Then:

$$\forall p \in \Omega^+,\ f^T \cdot d > 0 \text{ where } f = p - q$$

It means that there is a positive component of $f$ in the direction of $d$, then $f$ is not a positive linear combination of the columns of $C_m$, so:

$$\forall v \in R^{+|T|},\ C_m v \neq f$$

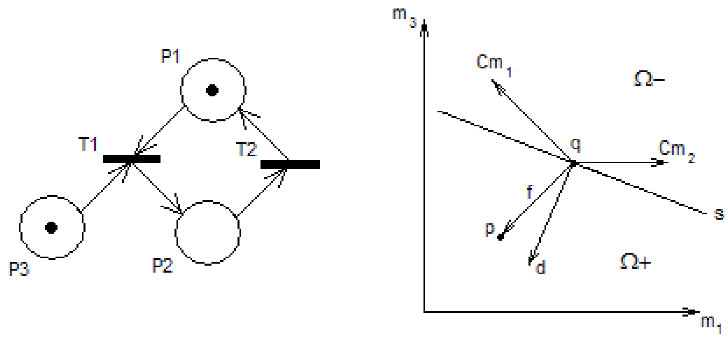


Figure 4.1 A $TCPN$ system and its $Class(m_0)$ set.

Since $I_c \Lambda G(m_m) \in R^{+|T|}$, then:

$$C_m I_c \Lambda G(m_m) = \overset{\bullet}{m}_m \neq \alpha f \text{ where } \alpha \in R^+ - \{0\},\, I_{c_i} \in [0,1]$$

Therefore it is not possible to direct the field vector $\overset{\bullet}{m}_m$in $s$ to any point $p \in \Omega^+$ (i.e. it is not possible to cross $s$ to $\Omega^+$) then $\Omega^+$ is not reachable from $q$, and the system $\langle N, m_0 \rangle$ is not $BIFC$. ■

Next theorem provides a condition of controllability easier to test than the condition which is required in the theorem 4.1.

**Theorem 4.2** *Equivalent condition of controllability.*

*$\forall d \in R^n$, $\exists v \in \{R^+ \cup \{0\}\}^{|T|}$ such that $C_m v = d$ iff $\exists k \in ker_d(C_m)$, $k \in R^{+|T|}$, where $ker_d(C_m)$ is the right annuller of $C_m$.*

**Proof** (Sufficiency). By hypothesis $\exists k \in ker_d(C_m)$, $k \in R^{+|T|}$. Let $C_{mI}$ be a matrix built with the first $n$ linearly independents columns of $C_m$, then $C_{mI}$ is not singular, therefore:

$$\forall d \in R^n \; \exists v \in R^n \text{ such that } C_{mI} v = d$$

Now, let $w$ be a column vector of order $|T|$, such that $w_i = v_i$ if the $i$ column of $C_m$ is in $C_{mI}$, and $w_i = 0$ otherwise. Then $C_m w = d$. Let $w_{\min}$ be the minimum element of $w$. In case of $w_{\min} < 0$, there is a scalar $\alpha \in R^+$ such that all the elements of the vector $x = w - \alpha w_{\min} k$ are nonnegative, and $C_m x = d$; in other case $w_{\min} \geq 0$, then $w \in \{R^+ \cup \{0\}\}^{|T|}$.

(Necessity). Suppose that $\forall k \in \ker_d(C_m)$, $k \notin R^{+|T|}$. Let be $v \in R^{-|T|}$ , i.e. all the elements of $v$ are negative.

Let $d = C_m v$. Since $\forall w \in R^{|T|}$ such that $C_m w = d$ it happens that $w = v + k$, $k \in \ker_d(C_m)$, but $k \notin R^{+|T|}$ and $v \in R^{-|T|}$, then $w \notin R^{+|T|}$. Therefore, there is $d$ such that, $\forall v$ that fulfills $C_m v = d$, it happens that $v \notin \{R^+ \cup \{0\}\}^{|T|}$. ■

An useful consequence from the proof of the theorem 4.1 is introduced in the next theorem which provides necessary and sufficient conditions for reachability.

**Theorem 4.3** *Reachability.*

*Let $\langle N, m_0 \rangle$ be a $TCPN$ system. Consider the minimum state equation of the net as in equation* (3.6)*, and let $n$ be the order of the minimum state equation. Let $S \subseteq Class_m(m_0)$ be a convex set such that $\forall m_m \in S$ its elements are nonzero. The marking $m_{md} \in S$ is reachable from $m_{m_0} \in S$ iff $\exists v \in \{R^+ \cup \{0\}\}^{|T|}$ such that $C_m v = (m_{md} - m_{m_0})$.*

**Proof** (Sufficiency) Let $v \in \{R^+ \cup \{0\}\}^{|T|}$ such that $C_m v = (m_{md} - m_{m_0})$. Consider the column vector $[I_c \Lambda G(m_m)]$ as in equation (4.2), then it is always possible to independently change all the elements of the vector $[I_c \Lambda G(m_m)]$ through $I_c$, so there is a factor $\alpha \in R^+ - \{0\}$ and an input $I_c$ with $I_{ci} \in [0,1]$ such that:

$$\alpha v = I_c \Lambda G(m_m)$$

Applying to the minimum state equation:

$$\overset{\bullet}{m}_m = C_m \alpha v$$

and by hypothesis:

$$\overset{\bullet}{m}_m = \alpha(m_{md} - m_{m_0})$$

Therefore it is always possible to direct the field vector in all $m_m \in S$ (including $L = \{m_m | m_m = \gamma m_{m_0} + (1-\gamma) m_{md}, \gamma \in [0,1]\}$) to the direction $(m_{md} - m_{m_0})$, and due the convexity of $S$, to reach $m_{md}$ through $L$. (Necessity). Follow the same reasoning as the proof of the theorem 4.1, with $d = (m_{md} - m_{m_0})$. ■

Notice that the previous theorem provides conditions of reachability whenever the system is $BIFC$ or not.

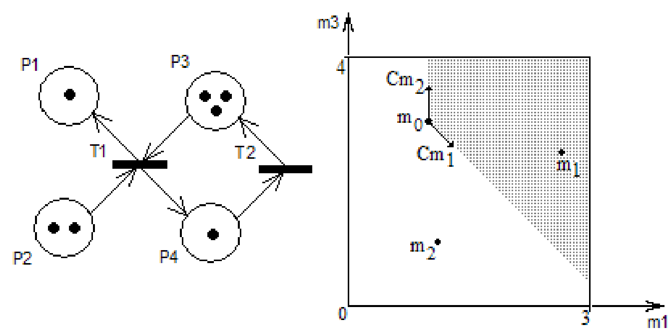


Figure 4.2 A $TCPN$ system and its $Class_m(m_0)$. The marking $m_1$ is reachable from $m_0$, but the marking $m_2$ is not.

The controllability and reachability can be understood from a graphical point of view. Consider the system of the figure 4.2. The columns of the matrix $C_m$ are $C_{m1}$ and $C_{m2}$. Notice that the vector $(m_1 - m_0)$ is a positive linear combination of $C_{m1}$ and $C_{m2}$.and since the vector field is a positive linear combination of the columns of $C_m$, then $m_1$ is reachable from $m_0$, and therefore all the points in the shadowed area compose the set of all reachable markings. The vector $(m_2 - m_0)$ is not a positive linear combination of the columns of $C_m$, then $m_2$ is not reachable from $m_0$.

Now, consider the system of figure 4.3. The net of this system is similar to that of figure 4.2 but it has another transition. In this system, the vector $(m_2 - m_0)$ is a positive linear combination of the columns of $C_m$, actually all vectors in $R^2$ can be considered as a positive linear combination of the columns of $C_m$, so, this is a $BIFC$ system.

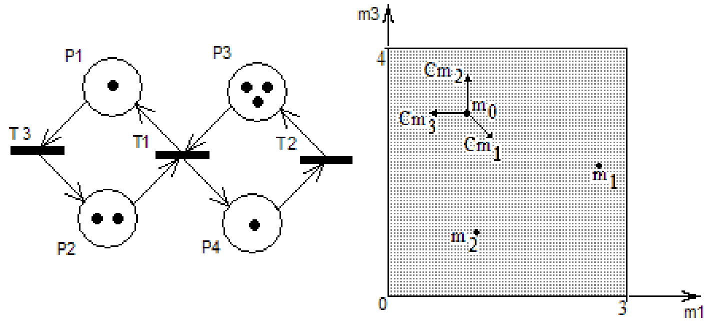


Figure 4.3 A $TCPN$system. The columns of the matrix $C_m$ cover all $Class_m(m_0)$, and then this system is $BIFC$.

Next theorem study the possibility of transferring the marking from a border point to an interior point of $Class_m(m_0)$.

**Theorem 4.4** *Controllability at border points.*

*Let $\langle N, m_0 \rangle$ be a $TCPN$ system, such that it is live and bounded as discrete. Let $m_0$ be a marking with null elements. An input, such that $I_c$ is invertible, transfers the state from $m_0$ to some $m_f$,where $m_f$ has not null elements.*

**Proof** Consider a place $p_i$ without tokens at time $\tau$, so $p_i$ cannot lose tokens. When an input such that $I_c$ is invertible is applied, then for any transition $t_j$, $[\Lambda I_c \pi(m) m(\tau)]_j = 0$ iff there is an input place to transition $t_j$ without tokens. In the same way $p_i$ cannot win tokens iff there exist unmarked input places to all the input transitions to $p_i$, i.e.:

$$\forall m(\tau)_i = 0,\ \dot{m}(\tau)_i = 0 \text{ iff } \forall t_k^\bullet = p_i, \exists p_r = {}^\bullet t_k \text{ such that } m(\tau)_r = 0.$$

If a place $p_i$ has not tokens at time $\tau$ and remains without tokens for future time, then there exists an input place to $p_i$ which remains without tokens for all time. Now, for this new place it should comply the same rule. Therefore, $p_i$ belongs to an initially unmarked siphon, but since the system is live as discrete there is not such siphon.

Therefore, a control law such that $I_c$ is invertible should give tokens to the unmarked places, so the state will be transferred to some $m_f$ which has not null elements. ■

Although liveness of the discrete system does not imply liveness of the continuous system, previous theorem is sustained by the liveness of the discrete system. Notice that for the proof, liveness of the $TCPN$ system is not required, only the property that it does not exist initially unmarked siphon in the system is required, which follows from the liveness of the discrete system.

Theorems 4.1, 4.2 and 3.2 establish that a $TCPN$ system is $BIC$ over the set of all the interior points of $Class_m(m_0)$ iff $\exists k \in \ker_d(C_m)$ such that $k \in R^{+|T|}$. Even when $Class_m(m_0)$ is not open, it is possible to asymptotically transfer the state to an $m_d$ not interior, following an interior points trajectory. By theorem4.4, if the system is live as discrete, then it is always possible to transfer the marking from a border point to an interior point of $Class(m_0)$. So, we conclude that a $TCPN$ system, which is live as discrete, is $BIFC$ iff $\exists k \in \ker_d(C_m)$ such that $k \in R^{+|T|}$.

Notice that the theorem 4.1 gives a structural test of controllability. This structural sense is explored in the next proposition.

**Proposition 4.4** *The controllability is an structural property for live systems.*

*Let $N$ be a $TCPN$. Then the system $\langle N, m_0\rangle$, which is live as discrete, is $BIFC$ over $Class(m_0)$ iff the system $\langle N, m_1\rangle$, which is live as discrete, is $BIFC$ over $Class(m_1)$; where $m_0, m_1 \in SASS(N)$ and $Class(m_0) \neq Class(m_1)$.*

**Proof** Let the system $\langle N, m_0\rangle$ be $BIFC$, then, according to the theorem 4.1, $\forall d \in R^n\ \exists v \in \{R^+ \cup \{0\}\}^{|T|}$ such that $C_m v = d$, so the system $\langle N, m_1\rangle$ fulfills the conditions of the theorem 4.1. Therefore it is $BIC$ over the interior of $Class(m_1)$. Finally, since the system $\langle N, m_1\rangle$ is live as discrete, according to the theorem4.4, for any marking in the frontier of $Class(m_1)$ there exists an input that transfers the state to the interior, therefore, the system $\langle N, m_1\rangle$ is $BIFC$. ■

Finally next theorem is presented.

**Theorem 4.5** *Controllability in live and bounded Petri nets.*

*Let $(N, m_0)$ be a live and bounded discrete Petri net system. Then the respective $TCPN$ is $BIFC$.*

**Proof** From lemma 2.2 it is known that there exists an occurrence sequence $\sigma$ for the discrete Petri net such that $\sigma$ contains all transitions of $N$, and that $m \xrightarrow{\sigma} m$ for some reachable marking $m$. Consider the Parikh vector of $\sigma$ as $\overrightarrow{\sigma}$, then all elements of $\overrightarrow{\sigma}$ are positives. Now, considering the marking equation, then:

$$m = m + C\overrightarrow{\sigma}$$

So, $\overrightarrow{\sigma}$ is in the right kernel of the incidence matrix, and according to theorems 4.2 and 4.1 the continuous system is $BIC$ over $Class(m_0)$. Now, since the system is live and bounded as discrete, then theorem4.4 can be applied, so the continuous system is $BIFC$. ■

## 4.3 The case of uncontrolled transitions

In this section the controllability of $TCPN$ systems is studied for the case with uncontrolled transitions. The controllability in this case has been explored by Jiménez, Júlvez, Recalde and Silva [8]. They introduced a controllability definition as a property of markings, i.e., a marking is said controllable iff it is reachable and it is an equilibrium point (with a suitable bounded input). They characterized the set of "controllable markings" for join free Petri nets.

In this dissertation, the controllability is studied according to the definitions previously presented in this chapter. In this section, a definition of the equilibrium points set is introduced and next, the controllability is studied on this set for a general kind of net. Since this set is defined from the structure, then the controllability proposed in this dissertation is an structural property of the system, not a property of markings.

Along this section, both approaches are compared. Remember that, according to definitions of section 2.3, for any uncontrolled transition $t_i$, the input $u$ is such that $u(t_i) = 0$ and so $I_c(t_i) = 1$.

An important definition, which was introduced in [8] by Jiménez, Júlvez, Recalde and Silva, is that of the controllability space, which is shown next:

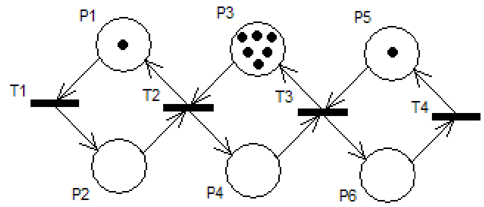


Figure 4.4 A $TCPN$ system. Consider transition $t_4$ as the only uncontrolled transition.

**Definition 4.6** *Controllability Space $CS$.*

*Given an initial marking $m_0$ and a set of controlled transitions $Tc \subseteq T$, the Controllability Space ($CS$) is defined as the set of all the controllable markings, i.e., $CS = \{m_f | \exists u(\tau)$ such that $m_0 \xrightarrow{u} m_f$ and $\overset{\bullet}{m}_f(u) = 0\}$.*

An inconvenience with this definition is that $CS$ is defined as a function of $m_0$, not from the structure. The $CS$ constitutes the set of markings that can be equilibrium markings given $Tc$ and that can be reached from $m_0$. In order to define this concept independently of the initial marking, the next definition is proposed.

**Definition 4.7** *Equilibrium set $ES$.*

*Let $\langle N, m_0 \rangle$ be a $TCPN$ system. Given the set of controlled transitions $Tc \subseteq T$, the Equilibrium Set is defined as $ES = \{m \in Class(m_0) | \exists u$ bounded with $u_i = 0, \forall t_i \notin T_c$ and $\overset{\bullet}{m}(u) = 0\}$.*

For the cases studied in [8], the equilibrium set and the controllability space are the same, but for a general case they are not equivalent.

In order to illustrate the difference between $ES$ and $CS$, consider the net of the figure 4.4 with $t_4$ as the only controllable transition, and let $m_0 = [0.5, 0.5, 4, 2, 0.5, 0.5]^T$ be the initial marking. The $Class_m(m_0)$ is shown in figure 4.5. The bold line inside corresponds to $ES$. Since $t_4$ is the only controllable transition, the marking $m_d = [0.5, 0.5, 3, 3, 0.5, 0.5]^T$ is not reachable from $m_0$ even when both belong to $ES$, so $m_d$ doesn't belong to $CS$. In this case, only the markings in the segment $[m_0, m_q]$ belong to $CS$.

Next definition introduce different subsets of $ES$.

**Definition 4.8** *The sets $S_i$, $S_i^{int}$ and $S_i^+$.*

*Let $\langle N, m_0 \rangle$ be a $TCPN$ system. Let $T_c$ be the set of controlled transitions. The set of all equilibrium markings with the same configuration $\Pi_i$ is defined as $S_i = \{m \in ES | \Pi(m) = \Pi_i\}$. The interior of $S_i$ is given by $S_i^{int} = \{m \in S_i | (m, I_c^q)$ is an equilibrium point and $I_{ci}^q \in (0,1), \forall t_i \in T_c\}$. In the same way, the subset of $S_i$, in which all equilibrium inputs are positives, is defined as $S_i^+ = \{m \in S_i | (m, I_c^q)$ is an equilibrium point and $I_{ci}^q \in (0,1], \forall t_i \in T_c\}$. Notice that $S_i^{int} \subset S_i^+ \subset S_i$.*

Another important result obtained by Jiménez, Júlvez, Recalde and Silva [8] is the convexity of $CS$ for join free nets. The generalization of this result is introduced in the next proposition.

**Proposition 4.5** *Convexity of the sets $S_i$, $S_i^{int}$ and $S_i^+$.*

*Let $\langle N, m_0 \rangle$ be a $TCPN$ system, and $T_c$ be the set of controlled transitions. If for a given configuration $\Pi_i$, $S_i^{int}$ is not null, then $S_i$, $S_i^+$ and $S_i^{int}$ are convex sets.*

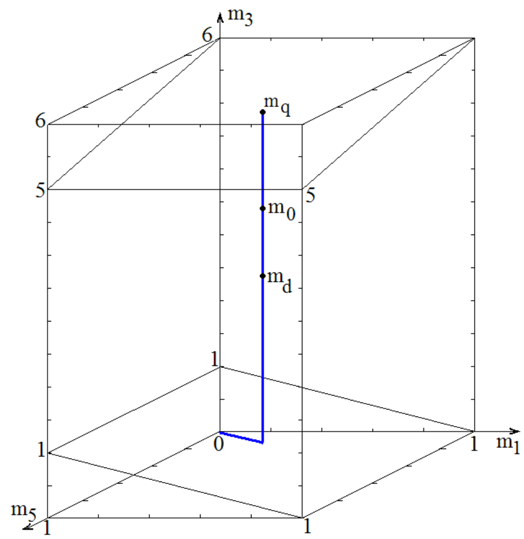


Figure 4.5 The $Class_m(m_0)$ of the system of the figure 5.2. The bold line at the center of the cube is the $ES$, where $t_4$ is the only controlled transition.

The proof follows directly from the linearity of the flow and it is the same presented by Jiménez, Júlvez, Recalde and Silva in [8].

The projection of these sets over $Class_m(m_0)$ are defined in the next definition.

**Definition 4.9** *The sets $S_{mi}$, $S^+_{mi}$ and $S^{int}_{mi}$.*

*The projection of the set $S_i$ over $Class_m(m_0)$ is defined as $S_{mi} = \{m_m | m_m = Pm, m \in S_i\}$. In the same way, the projection of the set $S^+_i$ over $Class_m(m_0)$ is $S^+_{mi} = \{m_m | m_m = Pm, m \in S^+_i\}$ and the projection of $S^{int}_i$ is $S^{int}_{mi} = \{m_m | m_m = Pm, m \in S^{int}_i\}$.*

Since the projection is a linear operator, the sets $S_{mi}$, $S^+_{mi}$ and $S^{int}_{mi}$ are convex too.

Next definitions are useful to explore the controllability.

**Definition 4.10** *The input transfer matrix $C_{mc}$.*

*Let $\langle N, m_0\rangle$ be a $TCPN$ system. Let $T_c = \{t_{c_1}, t_{c_2}, ..., t_{c_{|Tc|}}\}$ be the set of controllable transitions, and define the controllable projection matrix as $O_c = \left[\begin{array}{cccc} e_{c_1} & e_{c_2} & \dots & e_{c_{|Tc|}} \end{array}\right]$, where $e_j$ is the $j-th$ column vector of the unity matrix of order $|T|$. Then, the input transfer matrix $C_{mc}$ is defined as $C_{mc} = C_m O_c$.*

**Definition 4.11** *The local constant flow vector $A_i$ and the local flow matrix $J_i$.*

*Consider a configuration $\Pi(m) = \Pi_i$, where $m \in Class(m_0)$. The local constant flow vector $A_i$ and the local flow matrix $J_i$ related to $\Pi_i$ are defined such that $\forall m \in \{m \in Class(m_0) | \Pi(m) = \Pi_i\}$, it fulfills that $\Pi(m)m = \Pi_i m = A_i + J_i m_m$, where $m_m = Pm$.*

Now, we introduce the next theorem, which gives sufficient conditions to controllability in $S^+_i$.

**Theorem 4.6** *Local controllability with bounded input.*

*Let $\langle N, m_0 \rangle$ be a $TCPN$ system, where the minimum initial marking $m_{m_0}$ belongs to some $S_{mi}^+$. Define $I_{nc}$ as a diagonal matrix where $I_{nci} = 1, \forall t_i \notin T_c$, and $I_{nci} = 0, \forall t_i \in T_c$ The system is $BIC$ over $S_i^+$ if*

$$\exists k \in R^{+|T_c|(z+1)} \text{ such that } k \in ker_d([C_{mc}, (C_m I_{nc} \Lambda J_i) C_{mc}, ..., (C_m I_{nc} \Lambda J_i)^z C_{mc}]) \text{ for some } z \in \mathbb{N}$$

**Proof** Consider the minimum state equation as in equation (3.6). Define $I_{con}$ such that $I_c = I_{nc} + I_{con}$. Let $I'_{con}$ and $I''_{con}$ be two matrices such that $I_{con} = I'_{con} + I''_{con}$, then

$$I_c = I_{nc} + I'_{con} + I''_{con} \tag{4.3}$$

Considering previous definitions, the state equation can be rewritten as:

$$\dot{m}_m = C_m I_{nc} \Lambda A_i + C_m I_{nc} \Lambda J_i m_m + C_m I'_{con} \Lambda G(m_m) + C_m I''_{con} \Lambda G(m_m)$$

Which is valid in $S_{mi}^+$. Now, consider the equilibrium point $(m_{mq}, I_c^q)$, where $m_{mq} \in S_{mi}^+$, and define $I_{con}^q$ such that

$$I_c^q = I_{nc} + I_{con}^q \tag{4.4}$$

Let $I'_{con}$ be calculated such that

$$I'_{con} \Lambda G(m_m) = I_{con}^q \Lambda G(m_{mq}) \tag{4.5}$$

Then, the state equation is rewritten as:

$$\dot{m}_m = C_m I_{nc} \Lambda A_i + C_m I_{nc} \Lambda J_i (m_m - m_{mq}) + C_m I_{nc} \Lambda J_i m_{mq} + C_m I_{con}^q \Lambda G(m_{mq}) + C_m I''_{con} \Lambda G(m_m)$$

Notice that

$$C_m I_{nc} \Lambda A_i + C_m I_{nc} \Lambda J_i m_{mq} + C_m I_{con}^q \Lambda G(m_{mq}) = C_m I_c^q \Lambda G(m_{mq}) = 0$$

and that $C_m I''_{con} \Lambda G(m_m) = C_{mc} u_2$, where the new input $u_2$ is defined as:

$$u_2 = O_c^T I''_{con} \Lambda G(m_m) \tag{4.6}$$

and

$$O_c O_c^T I''_{con} = I''_{con}$$

So, substituting in the previous equation:

$$\dot{m}_m = C_m I_{nc} \Lambda J_i (m_m - m_{mq}) + C_{mc} u_2$$

Define a new variable $\mu = m_m - m_{mq}$, then $\overset{\bullet}{\mu} = \overset{\bullet}{m}_m$, that is:

$$\overset{\bullet}{\mu} = C_m I_{nc} \Lambda J_i \mu + C_{mc} u_2$$

The solution of this state equation is given by:

$$\mu(\tau) = e^{C_m I_{nc} \Lambda J_i \tau} \mu(0) + \int_0^\tau e^{C_m I_{nc} \Lambda J_i \zeta} C_{mc} u_2(\tau - \zeta) d\zeta$$

But, considering $m_m(0) = m_{mq}$, then $\mu(0) = 0$. Developing previous equation, then:

$$\begin{aligned} \mu(\tau) &= e^{C_m I_{nc} \Lambda J_i \tau} \mu(0) + \int_0^\tau e^{C_m I_{nc} \Lambda J_i \zeta} C_{mc} u_2(\tau - \zeta) d\zeta \\ m_m(\tau) - m_{mq} &= \int_0^\tau e^{C_m I_{nc} \Lambda J_i \zeta} C_{mc} u_2(\tau - \zeta) d\zeta \\ &= \int_0^\tau \left[ I + (C_m I_{nc} \Lambda J_i)\zeta + (C_m I_{nc} \Lambda J_i)^2 \frac{\zeta^2}{2!} + \ldots \right] C_{mc} u_2(\tau - \zeta) d\zeta \end{aligned}$$

Finally, taking out the constant elements from the integral and arranging the equation, the next equation is obtained:

$$m_m(\tau) - m_{mq} = \left[ \begin{array}{cccc} C_{mc} & (C_m I_{nc} \Lambda J_i) C_{mc} & (C_m I_{nc} \Lambda J_i)^2 C_{mc} & \ldots \end{array} \right] \left[ \begin{array}{c} \int_0^\tau u_2(\tau - \zeta) d\zeta \\ \int_0^\tau \zeta u_2(\tau - \zeta) d\zeta \\ \int_0^\tau \frac{\zeta^2}{2!} u_2(\tau - \zeta) d\zeta \\ \vdots \end{array} \right] \quad (4.7)$$

Notice that if the input were unbounded and the matrix $[C_{mc}, (C_m I_{nc} \Lambda J_i) C_{mc}, (C_m I_{nc} \Lambda J_i)^2 C_{mc}, ...]$ were a full rank matrix, any marking of $Class_m(m_0)$ would be reachable from $m_{mq}$, but in this case the input is already bounded. So, in order to investigate the reachability from $m_{mq}$ it is necessary to analyze the boundedness in the input.

Now, consider any controllable transition $t_i \in T_c$. According to equation (4.4), $I_{ci}^q = I_{coni}^q$, due to the fact that $I_{nci} = 0$. From equation (4.5), $I_{coni}^1 G_i(m_m) = I_{coni}^q G_i(m_{mq})$.

So, according to these equations:

$$I_{coni}^1 = I_{ci}^q \frac{G_i(m_{mq})}{G_i(m_m)}$$

Substituting in (4.3):

$$I_{coni}^2 = I_{ci} - I_{ci}^q \frac{G_i(m_{mq})}{G_i(m_m)}$$

Since $I_{ci} \in [0, 1]$ then:

$$I_{coni}^2 \in \left[ -I_{ci}^q \frac{G_i(m_{mq})}{G_i(m_m)}, 1 - I_{ci}^q \frac{G_i(m_{mq})}{G_i(m_m)} \right] \quad (4.8)$$

Notice that for all $m_{mq} \in S_{mi}^+$ the corresponding equilibrium input $I_c^q$ is such that $\forall t_i \in T_c$, $I_{ci}^q \in (0, 1]$.

Consider the case in which $I_{ci}^q \in (0, 1)$. Since $G(\bullet)$ is a linear function, there exists a small enough neighborhood of $m_{mq}$ named $V(m_{mq})$ such that for all $m_m \in V(m_{mq})$, $I_{ci}^q \frac{G_i(m_{mq})}{G_i(m_m)} < 1$. Then, $I_{coni}^2$ can be done either positive or negative.

In case that $I_{ci}^{q} = 1$, then, according to equation (4.8), $I_{coni}^{2}$ can be settled as a negative value, and as small in magnitude as desired, just considering a small enough neighborhood.

Therefore, $\forall m_{mq} \in S_i^+$ there exists a neighborhood $V(m_{mq})$ of $m_{mq}$ where $I_{coni}^{2}$ can be settled as a negative value, and as smaller in magnitude as desired. Now, since $u_2 = O_c^T I''_{con} \Lambda G(m_m)$, then the elements of $u_2$ can be settled also as a negative value, and independently as small in magnitude as desired. Notice that the negative bound of the input $u_2$ is determined by the equilibrium marking, not by the current marking.

Finally, since the elements of the right side vector of equation (4.7) are linearly independent functions of $u_2$, then the elements of this vector can be settled as a negative value, and independently as small in magnitude as desired.

Now, by hypothesis and the theorem 4.2, $\forall m_{md} \in Class_m(m_0)$ there exists $v \in R^{+|T_c|(z+1)}$ such that

$$-(m_{md} - m_{mq}) = [C_{mc}, (C_m I_{nc} \Lambda J_i) C_{mc}, (C_m I_{nc} \Lambda J_i)^2 C_{mc}, ..., (C_m I_{nc} \Lambda J_i)^z C_{mc}] v, \text{ where } z > n.$$

And according to the Calley-Hamilton's theorem, there is a vector $w$, where all its elements are negative, such that

$$(m_{md} - m_{mq}) = [C_{mc}, (C_m I_{nc} \Lambda J_i) C_{mc}, (C_m I_{nc} \Lambda J_i)^2 C_{mc}, ...] w \tag{4.9}$$

Notice that it is always possible to find a positive scalar $\alpha$ and an input $I''_{con}$, bounded by (4.8), such that

$$w = \alpha \left[ \begin{array}{cccc} \int_0^\tau u_2(\tau - \zeta) d\zeta & \int_0^\tau \zeta u_2(\tau - \zeta) d\zeta & \int_0^\tau \frac{\zeta^2}{2!} u_2(\tau - \zeta) d\zeta & ... \end{array} \right]^T$$

Then, substituting $w$ in (4.9) we have:

$$\alpha \left( m_{md} - m_{mq} \right) = \left[ \begin{array}{cccc} C_{mc} & (C_m I_{nc} \Lambda J_i) C_{mc} & (C_m I_{nc} \Lambda J_i)^2 C_{mc} & ... \end{array} \right] \left[ \begin{array}{c} \int_0^\tau u_2(\tau - \zeta) d\zeta \\ \int_0^\tau \zeta u_2(\tau - \zeta) d\zeta \\ \int_0^\tau \frac{\zeta^2}{2!} u_2(\tau - \zeta) d\zeta \\ \vdots \end{array} \right]$$

Comparing this equation with the equation (4.7), we conclude that the marking $m_{mq} + \alpha \left( m_{md} - m_{mq} \right)$ is reachable from $m_{mq}$, and since it is valid for any $m_{md} \in Class_m(m_0)$, then there exists a reachable neighborhood of $m_{mq}$.

This result is also easy to see from equation (4.7), just consider that the hypothesis and the Calley-Hamilton's theorem implicate that all directions in $Class_m(m_0)$ can be covered with an input such that all its elements are negative, and by the equation (4.8) an input, such that all its elements are negative, can be always applied, at least for a small neighborhood of $m_{mq}$.

Finally, since $S_{mi}^+$ is a convex set, and $\forall m_{mq} \in S_{mi}^+$ there exists a reachable neighborhood from $m_{mq}$, which includes another markings of $S_{mi}^+$, then the system is $BIC$ over $S_i^+$. ■

Next theorem provides a relaxed sufficient condition to controllability in $S_i^{int}$.

**Theorem 4.7** *Controllability over $S_i^{int}$.*

*Let $\langle N, m_0\rangle$ be a $TCPN$ system, where the minimum initial marking $m_{m_0}$ belongs to $S_{mi}^{int}$. The system is $BIC$ over $S_i^{int}$ if the controllability matrix $Cont(C_m I_{nc}\Lambda J_i, C_{mc})$ defined as*

$$Cont(C_m I_{nc}\Lambda J_i, C_{mc}) = [C_{mc}, (C_m I_{nc}\Lambda J_i)C_{mc}, (C_m I_{nc}\Lambda J_i)^2 C_{mc}, ..., (C_m I_{nc}\Lambda J_i)^{n-1} C_{mc}]$$

*has full rank.*

**Proof** Consider the proof of the theorem 4.6. Notice that for all $m_{mq} \in S_{mi}^{int}$ the corresponding equilibrium input $I_c^q$ is such that $\forall t_i \in T_c$, $I_{ci}^q \in (0,1)$. In this case, since $G(\bullet)$ is a linear function, there exists an enough small neighborhood of $m_{mq}$ named $V(m_{mq})$ such that for all $m_m \in V(m_{mq})$, $I_{ci}^q \frac{G_i(m_{mq})}{G_i(m_m)} < 1$. Then, according to the equation (4.8) $I_{coni}^2$ can be settled as either a positive or a negative value. Since $u_2 = O_c^T I''_{con}\Lambda G(m_m)$ and the elements of the right side vector of equation (4.7) are linearly independent, then the elements of this vector can be settled independently as either a positive or a negative value.

According to the hypothesis and the Calley-Hamilton's theorem, $\forall m_{md} \in Class_m(m_0)$ there is a vector $w$, such that

$$(m_{md} - m_{mq}) = [C_{mc}, (C_m I_{nc}\Lambda J_i)C_{mc}, (C_m I_{nc}\Lambda J_i)^2 C_{mc}, ...]w \tag{4.10}$$

Notice that it is always possible to find a positive scalar $\alpha$ and an input $I''_{con}$, bounded by (4.8), such that

$$w = \alpha \begin{bmatrix} \int_0^\tau u_2(\tau-\zeta)d\zeta & \int_0^\tau \zeta u_2(\tau-\zeta)d\zeta & \int_0^\tau \frac{\zeta^2}{2!} u_2(\tau-\zeta)d\zeta & ... \end{bmatrix}^T$$

Then, substituting $w$ in (4.10) we have:

$$\alpha\,(m_{md} - m_{mq}) = \begin{bmatrix} C_{mc} & (C_m I_{nc}\Lambda J_i)C_{mc} & (C_m I_{nc}\Lambda J_i)^2 C_{mc} & ... \end{bmatrix} \begin{bmatrix} \int_0^\tau u_2(\tau-\zeta)d\zeta \\ \int_0^\tau \zeta u_2(\tau-\zeta)d\zeta \\ \int_0^\tau \frac{\zeta^2}{2!} u_2(\tau-\zeta)d\zeta \\ \vdots \end{bmatrix}$$

Comparing this equation with the equation (4.7), then we conclude that the marking $m_{mq} + \alpha\,(m_{md} - m_{mq})$ is reachable from $m_{mq}$, and since it is valid for any $m_{md} \in Class_m(m_0)$, then there exists a reachable neighborhood of $m_{mq}$.

Finally, since $S_{mi}^{int}$ is a convex set, and $\forall m_{mq} \in S_{mi}^{int}$ there exists a reachable neighborhood from $m_{mq}$, which includes other markings of $S_{mi}^{int}$, then the system is $BIC$ over $S_i^{int}$. ■

Next theorem provides a necessary condition to reachability from the initial marking to another marking, where both belong to the same configuration.

**Theorem 4.8** *Reachability.*

*Let $\langle N, m_0\rangle$ be a $TCPN$ system, where the minimum initial marking $m_{m_0}$ belongs to $S_{mi}$. Define the set of all markings with the configuration $\Pi_i$ as $S_i^\Pi = \{m \in Class(m_0) | \Pi(m) = \Pi_i\}$. Consider a marking $m_d \in S_i^\Pi$, and let $m_{md} = Pm_d$.*

*If $m_d$ is reachable from $m_0$ through a trajectory in $S_i^\Pi$, then the vector $(m_{md} - m_{m_0})$ is in the range of the controllability matrix.*

**Proof** The proof follows by contradiction. Suppose that the vector $[m_{md} - m_{m_0}]$ is not in the image of the controllability matrix, then, due to the Calley-Hamilton's theorem, the vector is not in the image of the matrix $\begin{bmatrix} C_{mc} & (C_m I_{nc} \Lambda J_i) C_{mc} & (C_m I_{nc} \Lambda J_i)^2 C_{mc} & \ldots \end{bmatrix}$. Finally, according to the equation (4.7), there does not exists an input $u_2$, bounded or not, such that $m_{md}$ be reachable from $m_{m_0}$. ■

The next example illustrate the use of previous theorems.

**Example 4.1** *Consider the system of the figure 4.6, where the minimum marking is* $m_m = \begin{bmatrix} m_1 & m_3 \end{bmatrix}^T$. *In this example, the transition* $t_3$ *is the only uncontrolled transition, and the structure of the system is given by the following matrices:*

$$C_m = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \; \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; \Pi(m) = \Pi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; \forall m \in Class(m_0) | m_2 < m_3$$

The upper shadowed triangle in the figure 4.6 correspond to $ES$. The matrices defined in the previous theorems are:

$$A_1 = \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix} \; J_1 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \; I_{nc} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; C_{mc} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$C_m I_{nc} \Lambda J_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \; Cont(C_m I_{nc} \Lambda J_1, C_{mc}) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

According to the theorem 4.7, the system is $BIC$ over the set $S_i^{int}$ if the controllability matrix has full rank. For this example, considering $\Pi_i = \Pi_1$, the shadowed area in the interior of the triangle in figure 4.6 is equivalent to $S_{mi}^{int}$. Since for this example the matrix $Cont(C_m I_{nc} \Lambda J_i, C_{mc})$ has full rank, the system is $BIC$ over the shadowed area.

Now, the set $S_{mi}^+$, where $\Pi_i = \Pi_1$, includes the shadowed area in the interior of the triangle and the edges $e_1$ and $e_2$ in figure 4.6.

The controllability $S_i^+$ can be checked using the theorem 4.6. The matrix to be checked with $z = 1$ is:

$$[C_{mc}, (C_m I_{nc} \Lambda J_i) C_{mc}] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Since $k = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T \in ker_d([C_{mc}, (C_m I_{nc} \Lambda J_i) C_{mc}]) \cap R^{+4}$, the system is $BIC$ over $S_i^+$.

Next theorem establishes a sufficient condition of controllability in a subset of $ES$, valid when there is only one uncontrolled transition. Although previous theorems can be applied for this case, the proof of next theorem will be subsequently useful when the structure of a control law is defined.

**Theorem 4.9** *Controllability in case of* $|T - Tc| = 1$.

*Let* $\langle N, m_0 \rangle$ *be a* $TCPN$ *system, such that it is live and bounded as discrete. Suppose that* $\exists k \in \ker_d(C_m)$ *such that* $k \in R^{+|T|}$. *Let* $Tc$ *be the set of controllable transitions such that* $|T - Tc| = 1$.

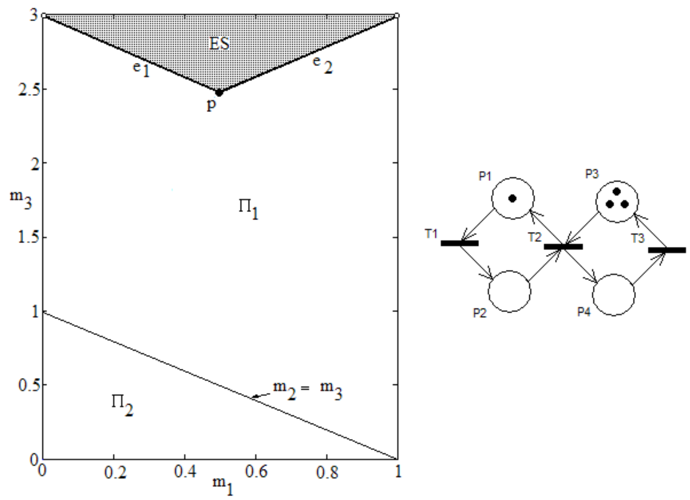


Figure 4.6 A $TCPN$ system and its $Class_m(m_0)$ set. The transtition $t_3$ is uncontrolled, and the transitions $t_1$ and $t_2$ are controllable. The shadowed area is the corresponding $ES$.

*Let $S$ be a connected subset of $ES$ such that all markings in $S$ have the same configuration, then the system is $BIC$ over $S$.*

**Proof** Let $m_1$ be an interior marking of $S$, and $m_{m1} = Pm_1$. Since $m_1 \in S$ there is an input $I_c^0$ such that $C_m I_c^0 \Lambda G(m_{m1}) = 0$, with $I_{ci}^0 = 1$, where $\{t_i\} \notin Tc$. Since $m_1$ is an interior point of $S$ then $I_{cj}^0 \in (0,1), \forall t_j \in Tc$.

Let $m_2$ be a marking such that $m_2 \in Class(m_1)$ and $m_{m2} = Pm_2$. Since $m_1 \in S$, there is a vector $k \in \ker_d(C_m)$, $k \in \{R^+ \cup \{0\}\}^{|T|}$ such that $I_c^0 \Lambda G(m_{m1}) = \alpha_2 k$.

Now, let $v_0 \in R^{|T|}$ be a particular solution of $C_m v = (m_{m2} - m_{m1})$, such that $v_{0i} = 0$ where $t_i \notin Tc$, (notice that it is always possible to find such $v_0$ because $C_m$ has a right kernel), then $v = \alpha_1 v_0 + \alpha_2 k, \alpha_1, \alpha_2 \in R^+$, is a solution of $C_m v = \alpha(m_{m2} - m_{m1}), \alpha \in R^+$.

As $I_c^0$ is such that $I_{cj}^0 \in (0,1), \forall t_j \in Tc$, then it is always possible to find $\alpha_1 \in R^+$, such that $0 \leq v_i \leq [\Lambda G(m_{m1})]_i$, and therefore $I_c^1$ such that $C_m I_c^1 \Lambda G(m_{m1}) = \alpha(m_{m2} - m_{m1})$ where $\alpha \in R^+$, $I_{ci}^1 \in [0,1]$, and $I_{cj}^1 = 1 \; \forall tj \notin Tc$.

So, it is always possible to point the field vector in all the interior markings of $S$ to any $m \in Class(m_0)$. Because $S$ is a connected set, there exists a trajectory that connects any $m_1$ and $m_2$ in $S$. Thus the system is $BIC$ over $S$. ■

# Chapter

# Control laws

The main goal of this chapter is to provide effective control laws for $TCPN$ systems, i.e. suitable bounded control laws that transfer the marking from the initial marking to the required equilibrium marking.

In the first section of this chapter, the behavior of the controlled $TCPN$ system, when a classical feedback state control law is applied, is studied.

After that, in the following two sections, we will introduce two effective control laws: one for the case in which all transitions are controllable, and the other for the case in which there is only one uncontrolled transition.

Finally, in the last section, we will propose a control law scheme for the case in which there are several uncontrolled transitions.

Along this chapter some stability concepts will be used. For a proper introduction to those concepts see [11].

## 5.1 Classical feedback state control law

As it is well known, the classical feedback state control law for a linear continuous-time system $\overset{\bullet}{x} = A \cdot x + B \cdot u$, is a linear state function as:

$$u = k \cdot x$$

where the constant matrix $k$ is chosen such that the matrix $(A - B \cdot k)$ has all its eigenvalues as negative.

Now consider a $TCPN$ system $(N, m_0)$ in which all transitions are controllable. Let $m_d$ be a desired equilibrium marking.

Consider a feedback state control law as:

$$u_{fb} = K_1(m) \cdot m + K_2(m) \cdot m_d \tag{5.1}$$

According to equation (2.3), this control law must fulfill that:

$$0 \leq K_1(m) \cdot m + K_2(m) \cdot m_d \leq \Lambda \cdot \Pi(m) \cdot m \tag{5.2}$$

Functions $K_1(\bullet)$ and $K_2(\bullet)$ can be defined as constants by configurations, i.e.:

$$
\begin{aligned}
K_1(m_1) &= K_1(m_2) \text{ iff } \Pi(m_1) = \Pi(m_2) \\
K_2(m_1) &= K_2(m_2) \text{ iff } \Pi(m_1) = \Pi(m_2)
\end{aligned}
$$

So, if this control law is applied to the state equation (2.3), the following equation of the closed-loop system is obtained.

$$\overset{\bullet}{m} = C(\Lambda \cdot \Pi(m) \cdot m - K_1(m) \cdot m - K_2(m) \cdot m_d) \tag{5.3}$$

Then for any configuration $\Pi_i$ a closed-loop system equation can be written as:

$$\overset{\bullet}{m} = C(\Lambda \cdot \Pi_i - K_{1i}) \cdot m - K_{2i} \cdot m_d$$

Now, consider the error vector as:

$$e = m_d - m$$

So, the dynamic behavior of the error in the configuration $\Pi_i$ is characterized by the equation:

$$\overset{\bullet}{e} = -C(\Lambda \cdot \Pi_i - K_{1i}) \cdot m + K_{2i} \cdot m_d$$

This equation is equivalent to:

$$\overset{\bullet}{e} = C(\Lambda \cdot \Pi_i - K_{1i}) \cdot e + C(K_{2i} - \Lambda \cdot \Pi_i + K_{1i}) \cdot m_d$$

Finally, choosing $K_1(\bullet)$ such that the matrix $C(\Lambda \cdot \Pi_i - K_{1i})$ has all its eigenvalues as negative, and $K2(\bullet)$ such that $K_{2i} = \Lambda \cdot \Pi_i - K_{1i}$, the marking $m_d$ is the unique asymptotically stable equilibrium point in the closed-loop system.

This control law has two important problems. The first one is that the input of a $TCPN$ is bounded, so, we cannot be sure that the control law proposed fulfills the bound of equation (5.2) for all markings along the trajectory.

The second problem is that the "state space" of a $TCPN$ system is bounded (at least it is positive). So, the solution of (5.3) may try to transfer the marking outside of $Class(m_0)$, which is not possible.

In order to illustrate this second problem, consider the figure .5.1. In this figure the "state space" of a $TCPN$ controlled system, in which the feedback state control law previously described is applied, is shown.

The circle in figure 5.1 corresponds to the Lyapunov surface that passes through $m_0$. So, the points inside the circle constitutes the region of attraction of the closed-loop system. Even when the trajectory $tr$ is inside the region of attraction, it includes the point $m_f$ which does not belong to $Class(m_0)$. Therefore, the

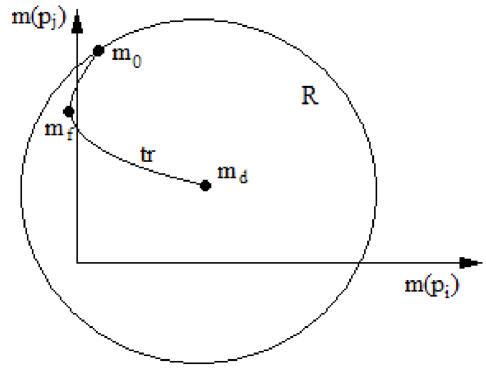


Figure 5.1 Region of attraction of a closed-loop system with a feedback state control law.

closed-loop system cannot generate such trajectory.

Actually, the feedback state control law can block the system in a border point of $Class(m_0)$.

Finally we conclude that this control approach is not always effective.

## 5.2 The case of all transitions as controllable

Through this section, only $TCPN$ systems such that all transitions are controllable will be considered.

Let $\langle N, m_0 \rangle$ be a live, bounded and $BIFC$ $TCPN$ system. Let $m_{m_d} \in Class_m(m_0)$ be a desired minimum marking, and let $m_{m_0} = Pm_0$ be the minimum initial marking such that $m_{m_0}$ is an interior point of $Class_m(m_0)$.

The error vector is defined as:

$$e_m = (m_{md} - m_m)$$

Since $\langle N, m_0 \rangle$ is $BIFC$, there is a vector $v \in R^{+|T|}$ such that $C_m v = e_m$. Let $m_m$ be an interior point of $Class_m(m_0)$, then there is always a function $\alpha : R^{+|T|} \times R^{+|T|} \longrightarrow R^+$ and an input $Ic$ (with $Ic_i \in [0, 1]$), such that $\alpha(v, \Lambda G(m_m))v = I_c \Lambda G(m_m)$.

Substituting previous equality into the minimum state equation leads:

$$\dot{m}_m = \alpha(v, \Lambda G(m_m))v$$

Considering that $\alpha(v, \Lambda G(m_m))$ is a scalar function and substituting the error vector equality, then:

$$\dot{m}_m = \alpha(v, \Lambda G(m_m))e_m$$

So, the field vector has the error vector direction.

Consider the next function as a Lyapunov candidate function

$$V = e_m^T e_m \tag{5.4}$$

Derivating the Lyapunov function:

$$\overset{\bullet}{V} = 2\overset{\bullet}{e}{}_m^T e_m = -2\overset{\bullet}{m}{}_m^T e_m = -2\alpha(v, \Lambda G(m_m))e_m^T e_m = -2\alpha(v, \Lambda G(m_m))V \tag{5.5}$$

Since $\alpha(v, \Lambda G(m_m)) \in R^+$, then $\overset{\bullet}{V}$ is negative defined. Therefore $e_m$ is asymptotically stable. This control law transfers the marking from $m_{m_0}$ to $m_{md}$ following a linear trajectory.

Now, let $e_{m_0}$ be the error vector in $m_{m_0}$, and let $v_0$ be such that:

$$C_m v_0 = e_{m_0}, v_0 \in R^{+|T|}.$$

By controllability hypothesis, there are solutions for $\alpha$ and $I_c$ in the equation

$$\alpha(v_0, \Lambda G(m_m))v_0 = I_c \Lambda G(m_m). \tag{5.6}$$

Consider the elements of $\Lambda G(m_m)$ and $v$ as:

$$\Lambda G(m_m) = \begin{bmatrix} \lambda_1\pi_1 \\ \lambda_2\pi_2 \\ \vdots \\ \lambda_{|T|}\pi_{|T|} \end{bmatrix}, \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{|T|} \end{bmatrix}$$

If $m_{m_d}$ is an interior point of $Class_m(m_0)$, then $\exists \mu \in R^+$ such that $\forall i, \lambda_i\pi_i > \mu$, and if $Ic$ is such that $Ic_j = 1$ (this is always possible) then $Ic_j\lambda_j\pi_j > \mu$, and, according to the equality of equation (5.6), $\alpha(v_0, \Lambda G(m_m)) > \mu/v_{0j}$ where $\mu/v_{0j} \in R^+$.

Then:

$$\| \overset{\bullet}{m}_m \| = \alpha(v_0, \Lambda G(m_m)) \| e_{m0} \| > (\mu/v_{0j}) \| e_{m0} \|$$

Consider the Lyapunov candidate function as in equation (5.4), then its derivative is:

$$\overset{\bullet}{V} < -(2\mu/v_{0j}) \| e_{m0} \| \sqrt[2]{V} \tag{5.7}$$

The above inequality implies that the state $m_{md}$ is reached in finite time.

In order to calculate $I_c$ it is necessary to solve the equation (5.6). Suppose that $\alpha$ is defined as:

$$\alpha(v, \Lambda G(m_m)) = \frac{1}{max(v_1/(\lambda_1\pi_1), v_2/(\lambda_2\pi_2), ..., v_{|T|}/(\lambda_{|T|}\pi_{|T|}))} \tag{5.8}$$

where $max(\cdot)$ is the greater element of the argument. Then $I_c$ is equal to:

$$I_c = \frac{1}{max(v_1/(\lambda_1\pi_1), v_2/(\lambda_2\pi_2), ..., v_{|T|}/(\lambda_{|T|}\pi_{|T|}))} diag(v_1/(\lambda_1\pi_1), v_2/(\lambda_2\pi_2), ..., v_{|T|}/(\lambda_{|T|}\pi_{|T|})) \tag{5.9}$$

Notice that $I_{ci} \in [0, 1]\ \forall i \in \{1, 2, ..., |T|\}$,and $max(I_{c1}, I_{c2}, ..., I_{c|T|}) = 1$ whenever $v \neq 0$.

Finally the original input $u$ can be calculated as:

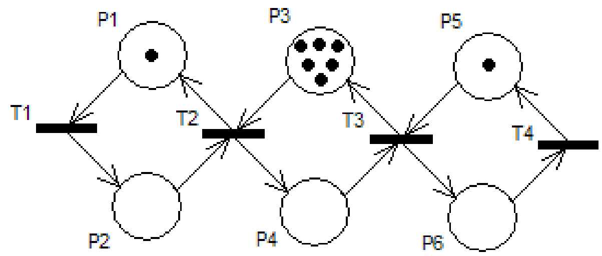


Figure 5.2 The $TCPN$ system for the example 5.1.

$$u = I_u \Lambda \Pi(m) m = (I - I_c) \Lambda \Pi(m) m \tag{5.10}$$

**Example 5.1** *Let $N$ be the net of the figure 5.2, which structure is represented by the next matrices:*

$$C = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \Lambda = diag(1,1,1,1)$$

The configuration matrix is given by the next rules:

$$\begin{aligned} \Pi(m) &= \Pi_1, \text{ if } m_2 < m_3 \text{ and } m_4 < m_5 \\ \Pi(m) &= \Pi_2, \text{ if } m_3 < m_2 \text{ and } m_4 < m_5 \\ \Pi(m) &= \Pi_3, \text{ if } m_2 < m_3 \text{ and } m_5 < m_4 \\ \Pi(m) &= \Pi_4, \text{ if } m_3 < m_2 \text{ and } m_5 < m_4 \end{aligned}$$

where:

$$\Pi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \Pi_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Pi_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \Pi_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that $N$ is live and bounded, and has 3 $P - semiflows$, therefore, the order of the minimum state equation is 3. Consider the minimum state vector as $m_m = [m_1, m_3, m_5]^T$, then:

$$C_m = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The right annuller of $C_m$ is $\ker_d(C_m) = \gamma[1,1,1,1]^T$, and since $k = [1,1,1,1]^T \in R^{+|T|} \cap \ker_d(C_m)$, then the system $\langle N, m_0 \rangle$ is $BIFC$.

The initial marking is $m_0 = [1,0,6,0,1,0]^T$ and let $m_d = [1,0,3,3,0,1]^T$ be the required marking. The corresponding minimum order markings are: $m_{m_0} = [1,6,1]^T$ and $= [1,3,0]^T$. Notice that both markings are not interior points of $Class_m(m_0)$.

In this example, three steps control law is applied. In the first step, a control law such that $Ic$ is invertible is applied, so the marking is transferred from $m_{m_0}$ to an interior point of $Classm(m_0)$. In second step, the control law obtained with the equation (5.9) is applied, but the central marking $m_{m_c} = [0.5, 3, 0.5]^T$ of $Class_m(m_0)$ is considered as the required marking instead $m_{m_d}$, thus the central marking is reached in finite time (because it is an interior point). In the third step, the same control law is applied in order to reach the original required marking $m_{m_d}$.

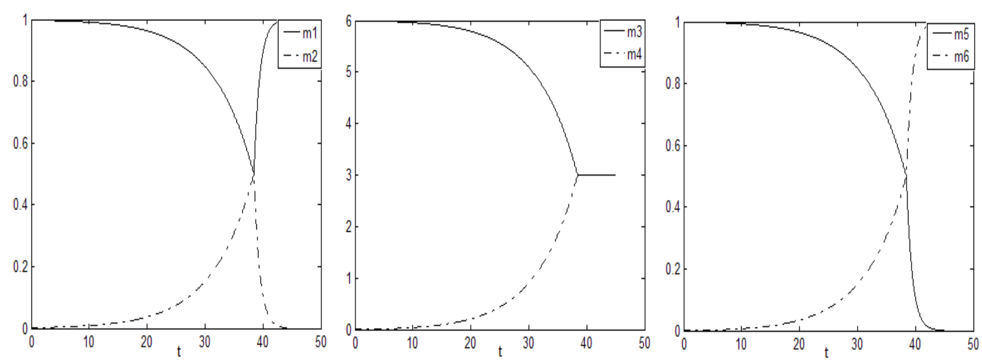


Figure 5.3 The marking evolution of the net of the example 5.1. The central marking is reached at $38s$, after that, the marking asymptotically goes to the desired marking.

Even when the second step is not necessary, it is very useful because the flow through the transitions

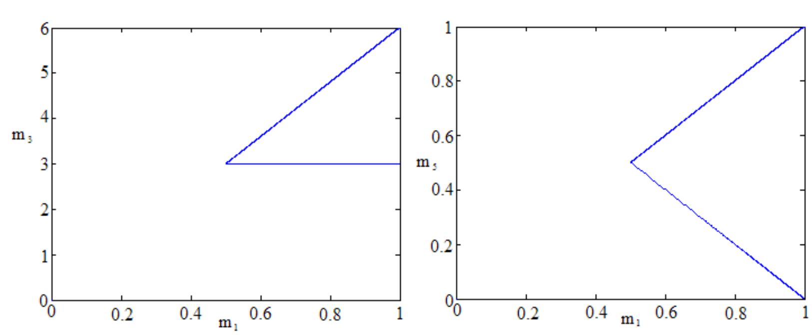


Figure 5.4 The trajectory of the marking in the $Class_m(m_0)$ for the example 5.1. The figure at left is a projection of $Class_m(m_0)$ in the $m_{m1}$ and the $m_{m3}$ axes. The figure at rigth is the projection in the $m_{m1}$ and the $m_{m5}$ axes.

decreases considerably in the markings near to the border of $Class_m(m_0)$ causing a very slow movement of the state. The simulation results are shown in figures 5.4 and 5.3.

The linear trajectories of the steps 2 and 3 of the control law can be observed in the figure 5.4. Notice that the flow through the transitions (proportional to the derivative of the markings observed in the figure 5.3) is larger in the central marking than in the markings near to the border. The central marking is reached in finite time (38 seconds) and after that, the state asymptotically goes to the required marking.

This control law is not efficient, i.e. the trajectory followed is not the fastest, but it is effective. This example was simulated in Simulink of MatLab. The m-files are shown in the appendix.

## 5.3 The case of only one uncontrolled transition

Consider a $BIFC\ TCPN$ system described by $(N, m_0)$. Let $t_i \in T$ be the only uncontrolled transition, then $Tc = T - \{t_i\}$, and $ES$ the equilibrium set as defined in section 4.3.

Let $S$ be a set defined as $S = \{m_m \in Class_m(m_0) | m_m = Pm$ where $m \in ES$ and $\Pi(m) = \Pi_i\}$. Consider the minimum initial marking $m_{m_0}$ such that $m_{m_0}$ is an interior point of $S$, and let $m_{md} \in S$ be the minimum required marking.

Define the error vector as:

$$e_m = (m_{md} - m_m)$$

Since $\langle N, m_0 \rangle$ is $BIFC$, for all $m_m$ interior point of $S$ there should exist an input $I_c^0$ such that $C_m I_c^0 \Lambda G(m_m) = 0$, with $I_{ci}^0 = 1$, and $I_{cj}^0 \in (0,1), \forall t_j \in Tc$.

Let $C_{mI}$ be a matrix built with any $n$ linearly independents columns of $C_m$ except $C_{mi}$, then $C_{mI}$ is not singular, therefore:

$$\forall d \in R^n\ \exists v \in R^n \text{ such that } C_{mI} v = d$$

Let $m_m$ be an interior point of $S$. Define $v_0^p$ such that:

$$C_{mI} v_0^p = (m_{md} - m_m)$$

Now, let $v_0$ be a column vector of order $|T|$ such that:

$$v_{0j} = \begin{cases} v_{0j}^p & \text{if the } j-th \text{ column of } C_m \text{ is in } C_{mI} \\ 0 & \text{otherwise} \end{cases}$$

For all $m_m \in S$ there is a column vector $k \in \ker_d(C_m) \cap \{R^+ \cup \{0\}\}^{|T|}$ and a scalar $\alpha_2 \in R^+$ such that $Ic\Lambda G(m_m) = \alpha_2 k$. Let $\alpha_2$ be such that $[\Lambda G(m_m)]_i = \alpha_2 k_i$.

Then, the vector $v$ defined as $v = \alpha_1 v_0 + \alpha_2 k$, where $\alpha_1 \in R^+$, is such that:

$$C_m v = \alpha_1 (m_{md} - m_m)$$

and

$$v_i = [\Lambda G(m_m)]_i.$$

Now, it is necessary to find $\alpha_1 \in R^+$ such that $0 \leq v_j \leq [\Lambda G(m_m)]_j$ (it is always possible to find $\alpha_1$, because $I^0_{cj} \in (0,1), \forall t_j \in Tc$). Then, define the following vector:

$$minv_0 = min\left(\frac{v_{01}}{k_1}, \frac{v_{02}}{k_2}, ..., \frac{v_{0|T|}}{k_{|T|}}\right)$$

If $minv_0 < 0$ then a valid value for $\alpha_1$ is $\frac{-\alpha_2}{minv_0}$, otherwise calculate the $fmax$ vector as:

$$fmax_j = \begin{cases} \frac{[\Lambda G(m_m)]_j - \alpha_2 k_j}{v_{0j}} & \text{for } v_{0j} > 0 \\ \infty & \text{other case} \end{cases}$$

and let $\alpha_1$be equal to $min(fmax)$. Finally, the matrix $Ic$ can be calculated as:

$$Ic_j = \frac{v_j}{[\Lambda G(m_m)]_j}$$

The original input $u$ can be calculated using the equation (5.10).

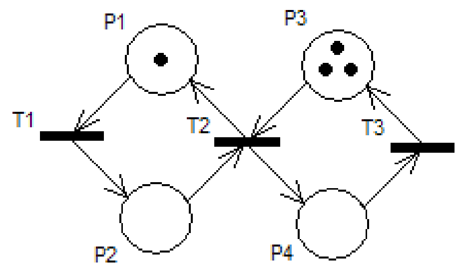


Figure 5.5 The $TCPN$ system for the example 5.2. If all transitions were controllable, the net would be $BIFC$.

**Example 5.2** *Let $\langle N, m_0\rangle$ be the $TCPN$ system of the figure 5.5.*

The structure of the net is given by the next matrices:

$$C = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \Lambda = diag(1,1,1,1)$$

The configuration matrix is given by next rules:

$$\begin{aligned} \Pi(m) &= \Pi_1, \text{ if } m_2 < m_3 \\ \Pi(m) &= \Pi_2, \text{ if } m_3 < m_2 \end{aligned}$$

where

$$\Pi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \Pi_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that the net is live and bounded, and has 2 $P-semiflows$ , therefore, the order of the minimum state equation is 2. Consider the minimum state vector as $m_m = [m_1, m_3]^T$, then:

$$C_m = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

The right annuller of $C_m$ is $\ker_d(C_m) = \gamma[1,1,1]^T$, and since $k = [1,1,1]^T \in R^{+|T|} \cap \ker_d(C_m)$, then the system $\langle N, m_0\rangle$ is $BIFC$ in the interior points of $Class_m(m_0)$.

Let $t_3$ be the only uncontrollable transition. In this case the equilibrium set $ES$ is represented by the upper triangle in figure 5.6.

Let $m_0 = [0.6; 0.4; 2.6; 0.4]^T$ be the initial marking and $m_d = [0.3; 0.7; 2.8; 0.2]^T$ the required marking, then the corresponding minimum order markings are: $m_{m_0} = [0.6, 2.6]^T$ and $m_{m_d} = [0.3, 2.8]^T$. Notice

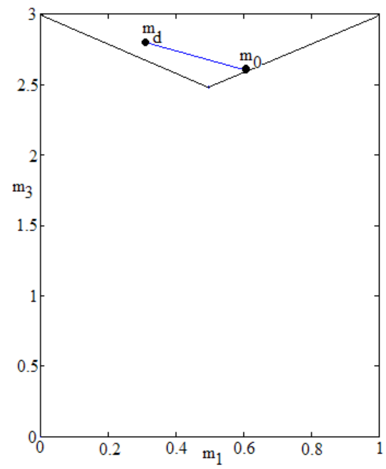


Figure 5.6 The trajectory of the marking in the set $Class_m(m_0)$, for the example 5.2. The $ES$ is composed of all points inside the upper triangle.

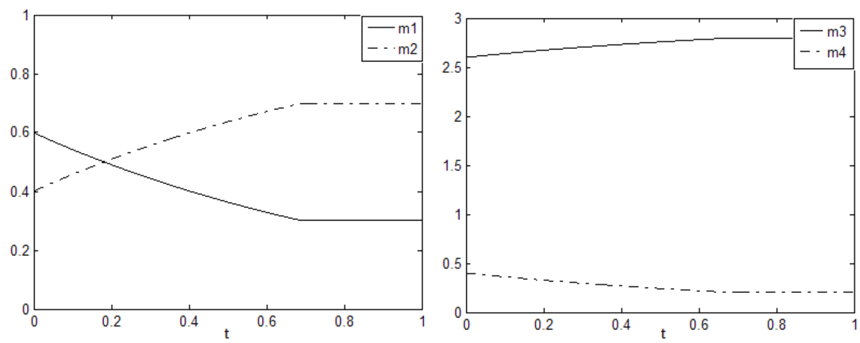


Figure 5.7 Evolution of the marking of example 5.2.

that both markings belongs to $ES$. Applying the control law described above the results are shown in figures 5.6 and 5.7.

Notice that the trajectory draws a line, and the required marking is reached in finite time.

Even when this control law is not efficient, it is effective, i.e. the trajectory followed is not the fastest but the required marking is reached in finite time. This example was simulated in Simulink of MatLab. The m-files are shown in the appendix.

## 5.4 The case of several uncontrolled transitions

In this section we will loosely propose an scheme of a control law for a $TCPN$ system in which there are several uncontrolled transitions.

Consider a $TCPN$ system $(N, m_0)$ where the set of uncontrolled transitions is not null, i.e. $|T| - |Tc| \neq 0$. Suppose that $m_0$ is an equilibrium marking and define $\Pi i$ as its configuration, i.e. $\Pi_i = \Pi(m_0)$. Consider the equilibrium set $ES$ and the set $S_i^{int}$ as defined in section 4.3. Let $m_d$ be a desired equilibrium marking such that $m_d \in S_i^{int}$.

Finally, suppose that the system fulfills the conditions of controllability over $S_i^{int}$ established in theorem 4.7,. so the matrix:

$$Cont(C_m I_{nc} \Lambda J_i, C_{mc}) = [C_{mc}, (C_m I_{nc} \Lambda J_i) C_{mc}, (C_m I_{nc} \Lambda J_i)^2 C_{mc}, ..., (C_m I_{nc} \Lambda J_i)^{n-1} C_{mc}]$$

has full rank.

Now, consider a control law as defined in equation (4.3), i.e.$I_c = I_{nc} + I'_{con} + I''_{con}$, where $I'_{con}$ can be easily calculated to fulfill with equation (4.5). Also, consider the new input $u_2$ as in equation (4.6), which is a function of $I''_{con}$.

Defining a new variable $\mu = m_m - m_{mq}$, the state equation can be transformed to:

$$\overset{\bullet}{\mu} = C_m I_{nc} \Lambda J_i \mu + C_{mc} u_2$$

So, according to the condition of controllability, there exist a matrix $K$ such that

$$u_2 = K\mu$$

where the closed-loop transfer matrix $(C_m I_{nc} \Lambda J_i - C_{mc} K)$ has all its eigenvalues as negative. Therefore the desired point $m_d$ is the unique asymptotically stable equilibrium point in $S_i^{int}$ in the closed-loop system.

It can be noticed that this is a feedback state control law as that described in section 5.1. So, it has the same problems described in that section, but, under the conditions required to the system in this section, these problems can be avoided.

For this, consider a set of $q$ markings $\{m_{f1}, m_{f2}, ..., m_{fq}\}$ that belong to $S_i^{int}$ as in figure 5.8.

In this example a four state feedback state control law is considered.

In the first step the marking $m_{f1}$ is considered as the required marking, since $m_{f1} \in S_i^{int}$ the controllability condition remains, so, a control law as it is described in this section can be applied to the system. Notice that the region of attraction of $m_{f1}$, named $R_{f1}$, is included in $S_i^{int}$, so the second problem of the feedback state control laws is avoided. Now, according to the controllability hypothesis, there exists a neighborhood of $m_0$ in which our control law makes $m_{f1}$be the unique asymptotically stable equilibrium point, so, $m_{f1}$ is defined closed enough to $m_0$ in order to $R_{f1}$ be in that neighborhood. In this way, the marking $m_{f1}$ can be reached through this control law.

Once the marking is closed enough to $m_{f1}$ the second step of this control law is applied, for this the required marking is $m_{f2}$. In this way, in the third step the required marking is $m_{f3}$, and it is applied when the marking is closed enough to $m_{f2}$.

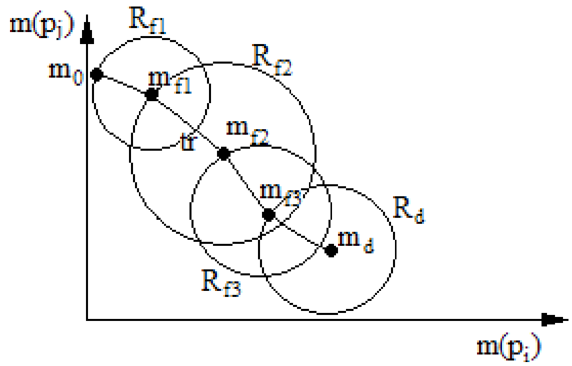


Figure 5.8 State space of a closed-loop system with a four steps feedback state control law.

Finally, in the four step the required marking is $m_d$, and is applied when the marking is closed enough to $m_{f3}$.

So, with this control law scheme the marking can reach any required marking $m_d$, whenever the conditions of this section are fulfilled.

However, this control law scheme has two major difficulties. The first one is to define the set of markings $\{m_{f1}, m_{f2}, ..., m_{fq}\}$ such that the regions of attraction are included in $S_i^{int}$. The second one is to choose the eigenvalues for the closed-loop transfer matrix and also the markings $\{m_{f1}, m_{f2}, ..., m_{fq}\}$ such that the respective input is properly bounded.

Due to those difficulties, this control law scheme is not applied to an example in this dissertation. So, the control law for the case with several uncontrolled transitions is still an open problem.

# Chapter

# Conclusions

This dissertation deals with a new technique in the theory of Petri nets. For this reason, many unsolved problems and many open questions were found. The answers for some of this questions result essential for a basic understanding of the studied model. In this report, we tried to unify the previously known results with our results. So, with this dissertation, the reader is able to introduce himself in the study of $TCPN$ systems.

The main advantage of the results obtained by us with respect to that previously known is that our results can be applied to different kinds of Petri nets. The contributions of this dissertation are following presented.

- A brief discussion of the concept of state variable was presented. In this, the $TCPN$ systems are finally considered as a parallel model of the original Petri net system, and not as a proper model of the physical system.
- The so called “state space” was characterized.
- A definition of controllability for $TCPN$ systems was introduced as an adaptation of that defined for linear continuous-time systems. The reason for that, is that $TCPN$ systems are more alike linear systems than discrete event systems.
- For the case where all transitions are controllable, sufficient and necessary conditions of controllability and reachability, which are easy to test, were given. The hypothesis for those theorems does not impose heavy constraints for its application. Therefore, for this case, the problems of controllability and reachability have been solved.
- For the case where there are uncontrolled transitions, the problem of controllability is more complex. Even that, sufficient conditions of controllability over the set of equilibrium points were found.
- In reachability, for the case with uncontrolled transitions and where the initial marking has the same configuration that the required marking, a necessary condition was found. However, the problem of finding necessary and sufficient conditions of reachability for a general case is still open.
- A control law structure for the case where all transitions are controllable was proposed. The effectiveness of this control law structure was demonstrated through a Lyapunov function (it makes the system reach the required marking). Although it is not an optimal control law, it can be easily modified, in order to make the marking follow a desired optimal trajectory.
- A second control law structure was proposed for the case in which there exists only one uncontrolled transition. This control law structure is also effective but limited because the initial marking must be in the same equilibrium set and configuration that the required marking. So, the problem of finding control law structures for the case with several uncontrolled transitions is still open.

## 6.1 Future work

In order to apply and to extend the results obtained in this dissertation for general $TCPN$ systems, the following problems need to be solved in the future:

- It is necessary to obtain a reachability theorem which gives necessary and sufficient conditions for the case with uncontrolled transitions. Such theorem must consider the case where the initial marking and the required marking does not belong to the equilibrium set, and have different configurations. This problem is very difficult to solve, due to the hybrid nature of the timed fluidified model.
- In order to apply the theorems 4.6 and 4.7, an easy characterization of the equilibrium set is required.
- In order to easily apply the results of section 4.2, an algorithm to test if a given incidence matrix fulfills the condition of theorem 4.2 and an algorithm to test if a given $TCPN$ system with a required marking

fulfill the condition of theorem 4.3 are needed. This second algorithm could be obtained analyzing the projections of the error vector to the columns of the incidence matrix.

- Considering that all transitions are controllable, it is necessary to find the optimal trajectory in the state space, from the initial marking to the required marking, as a function of the marking.
- Considering the case where there are uncontrolled transitions, it is also necessary to synthesize a general control law which could be applied even when the initial marking does not belong to the equilibrium set and has a different configuration of that of the initial marking. This problem is very difficult to solve, because it implies a wide study of stability of $TCPN$ systems, which does not exist yet.

As it was mentioned in the introduction, this theory is still new, so, there are another many unsolved problems. At this moment, we consider that the main problems, not only for the controllability study but also for the general understanding of the $TCPN$ theory, are those enunciated next:

- How should the steady states of the $TCPN$ system be interpreted in the original $PN$ system?
- Given an effective control law for the $TCPN$ system, how can a firing policy, which could be applied to the original $PN$ system with the expecting results, be obtained?

These two questions have to be solved in order to apply the whole $TCPN$ theory. Therefore, we consider that all efforts in future works should be focused to find the answer of these two questions.

# Appendix

# Computation of $A_i$ and $J_i$

The following procedure allows to calculate the local constant flow vector $A_i$ and the local flow matrix $J_i$, described in section 4.3.

Consider the vector $[\Pi_i m]$ as in equation (4.1).

Then $\forall j \in \{1, 2, ..., |T|\}$ do the next procedure:

Define $k$ and $\alpha$ such that $[\Pi_i m]_j = \alpha_k m_k$.

Consider the projection matrix $P$.

If $\exists l$ such that $P_{[l,k]} \neq 0$ then define $J_{i[j,l]} = \alpha_k$ and $A_{i[j]} = 0$.

If $\nexists l$ such that $P_{[l,k]} \neq 0$ then there exists a conservative marking law such that $m_k + m_{p_1} + m_{p_2} + ... + m_{p_r} = C_k$, where $C_k$ is a constant value, so $\alpha_k m_k = \alpha_k C_k - \alpha_k m_{p_1} - \alpha_k m_{p_2} - ... - \alpha_k m_{p_r}$ and $\forall p_i \in \{p_1, p_2, ..., p_r\}$ $\exists l_i$ such that $P_{[l_i,p_i]} \neq 0$. Then define $J_{i[j,l_i]} = -\alpha_k$ and $A_{i[j]} = \alpha_k C_k$.

Other elements in $J_i$ and $A_i$ are defined as zero.

# Appendix

## Simulation of the control law for the case of all transitions as controllable

Here the model and the m-files of MatLab-Simulink for the simulated example of section 5.2 are presented.

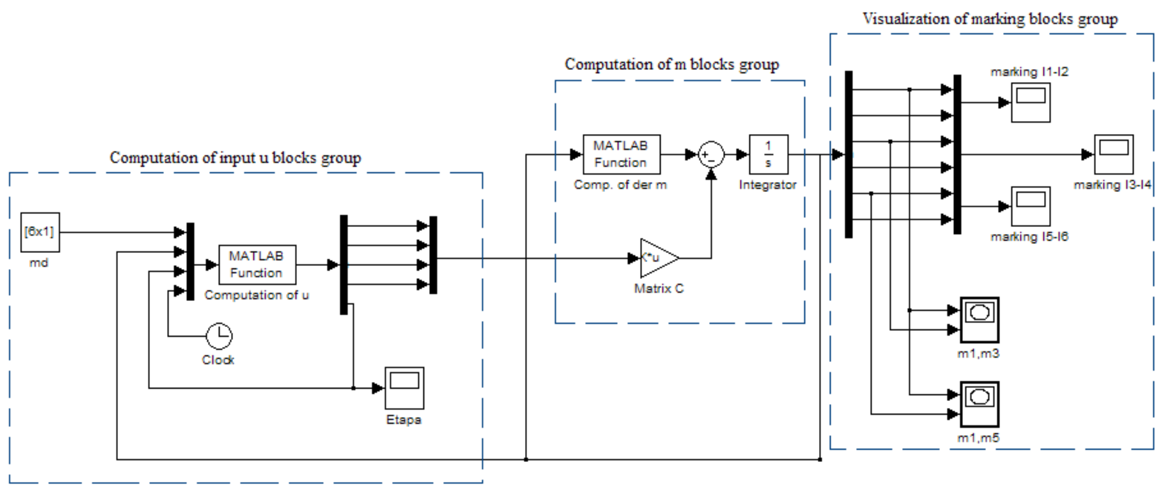


Figure B.1 Model of simulation.

Previous figure shows the model for the simulation, with its respective block groups. The apparent complexity of the computation of input $u$ blocks group is due to the three steps control law. The clock is needed only to set the first step at zero time.

Now, the “Comp. of der m” m-file is presented. This block corresponds to the computation of:

$$derx = C\Lambda\Pi(m)m$$

```
function derx=RPF(u)

if u(2)<=u(3) & u(4)<=u(5)

    dx1=-u(1)+u(2);

    dx2=u(1)-u(2);

    dx3=-u(2)+u(4);

    dx4=u(2)-u(4);
```

```
    dx5=-u(4)+u(6);
    dx6=u(4)-u(6);
elseif u(3)<u(2) & u(4)<=u(5)
    dx1=-u(1)+u(3);
    dx2=u(1)-u(3);
    dx3=-u(3)+u(4);
    dx4=u(3)-u(4);
    dx5=-u(4)+u(6);
    dx6=u(4)-u(6);
elseif u(2)<=u(3) & u(5)<u(4)
    dx1=-u(1)+u(2);
    dx2=u(1)-u(2);
    dx3=-u(2)+u(5);
    dx4=u(2)-u(5);
    dx5=-u(5)+u(6);
    dx6=u(5)-u(6);
elseif u(3)<u(2) & u(5)<u(4)
    dx1=-u(1)+u(3);
    dx2=u(1)-u(3);
    dx3=-u(3)+u(5);
    dx4=u(3)-u(5);
    dx5=-u(5)+u(6);
    dx6=u(5)-u(6);
end;
derx=[dx1;dx2;dx3;dx4;dx5;dx6];
```

The block “Computation of u” has the next m-file.

```
function SalU=RetroF(u)
%cm is the minimum incidence matrix.
cm=[-1 1 0;0 -1 1;0 0 -1];
%ker is the right kernel of matrix cm.
ker=[1;1;1;1];
```

```
m=[u(7);u(8);u(9);u(10);u(11);u(12)];
md=[u(1);u(2);u(3);u(4);u(5);u(6)];
%etapa is a discrete variable. 2 means that the control will take the
%marking to the center of the state space, after that, etapa changes its
%value to 3, then the control will take the marking to md.
%Tiempo is the simulation real time, is used only to reset the value of
%etapa as 2. Notice that for etapa 1 we only need to generate an input
%such that Ic be invertible, but, for this example, etapa 2 generates such
%input, that's why we don't set etapa as 1.
etapa=u(13);
tiempo=u(14);
%mr is the minimum marking. mdr is the desired minimum marking.
mr=[u(7);u(9);u(11)];
mdr=[u(1);u(3);u(5)];
%Now, we calculate the vector lp=ΛΠ(m).
if m(2)<=m(3) & m(4)<=m(5)
    lp1=[1 0 0 0 0 0];
    lp2=[0 1 0 0 0 0];
    lp3=[0 0 0 1 0 0];
    lp4=[0 0 0 0 0 1];
elseif m(2)>m(3) & m(4)<=m(5)
    lp1=[1 0 0 0 0 0];
    lp2=[0 0 1 0 0 0];
    lp3=[0 0 0 1 0 0];
    lp4=[0 0 0 0 0 1];
elseif m(2)<=m(3) & m(4)>m(5)
    lp1=[1 0 0 0 0 0];
    lp2=[0 1 0 0 0 0];
    lp3=[0 0 0 0 1 0];
    lp4=[0 0 0 0 0 1];
elseif m(2)>m(3) & m(4)>m(5)
    lp1=[1 0 0 0 0 0];
    lp2=[0 0 1 0 0 0];
    lp3=[0 0 0 0 1 0];
```

```
        lp4=[0 0 0 0 0 1];
    end;
    lp=[lp1;lp2;lp3;lp4];
    lpm=lp*m;
    vic=[0;0;0;0];
    %ep is a value which indicates how close will be the marking of the center
    %to change the step from 2 to 3.
    ep=0.002;
    %Now, we calculate the error vector, and a solution v for the
    %equation (Cr*v=e), such that all elements in v be positives.
    %In step 2, the required marking is the center of the state space
    %[0.5,0.5,3,3,0.5,0.5], in step 3 the required marking is md.
    if tiempo==0
        etapa=2;
    end;
    if ((mr(1)<(.5+ep))&(mr(1)>(0.5-ep)))&((mr(2)<(3+3*ep))&(mr(2)>(3-3*ep)))&((mr(3)<(.5+ep))&(mr(3)>(0.5-
ep)))
        etapa=3;
    end;
    if etapa==2
        e=[0.5;3;0.5]-mr;
    elseif etapa==3
        e=mdr-mr;
    end;
    v=inv(cm)*e;
    v=[v;0];
    vmin=min(v);
    if vmin<0
        v=v-2*vmin*ker;
    end;
    %Now, we calculate the corresponding values of Ic, named ic's, as a column
    %vector vic, where 0<ici<1, v=ic*lp*m.
    for i=1:4
        if lpm(i)>0
```

```
        vic(i)=v(i)/(lpm(i));
    elseif lpm(i)==0
        vic(i)=1;
    end;
end;
%At this point, vic has the desired direction. Now we multiply it by a
%factor such that the maximum element of vic be 1.
icmax=max(vic);
if icmax>0
    vic=(1/icmax)*vic;
end;
%Now, we transform the column vector vic into the matricial form Ic.
ic1=[1 0 0 0]*vic(1);
ic2=[0 1 0 0]*vic(2);
ic3=[0 0 1 0]*vic(3);
ic4=[0 0 0 1]*vic(4);
ic=[ic1;ic2;ic3;ic4];
%Finally, we calculate the input u.
SalU=[(eye(4)-ic)*lpm;etapa];
```

# Appendix

# Simulation of the control law for the case of one uncontrolled transition

The model and the m-files of MatLab-Simulink for the simulated example of section 5.3 are presented.

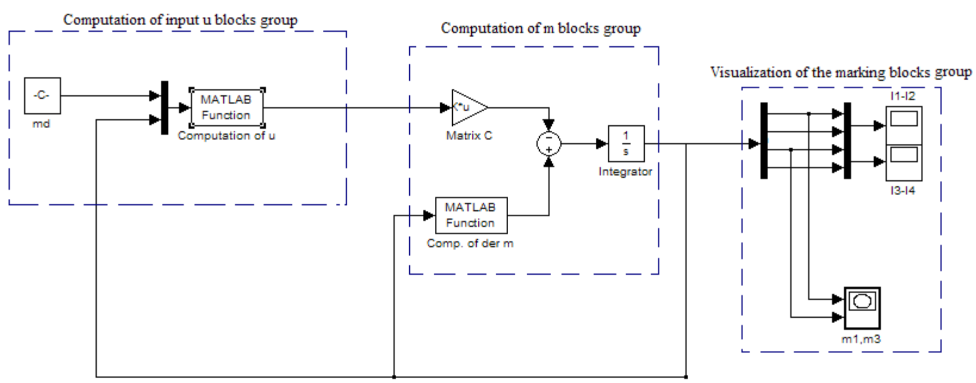


Figure C.1 Model of simulation.

Previous figure shows the model for the simulation, with its respective block groups.

The “Comp. of der m” m-file is following presented. This block corresponds to the computation of:

$$derx = C\Lambda\Pi(m)m$$

```
function derx=RPFal(u)
if u(2)<=u(3)
    dx1=-u(1)+u(2);
    dx2=u(1)-u(2);
    dx3=-u(2)+u(4);
    dx4=u(2)-u(4);
elseif u(3)<u(2)
    dx1=-u(1)+u(3);
    dx2=u(1)-u(3);
    dx3=-u(3)+u(4);
```

```
    dx4=u(3)-u(4);
end;
derx=[dx1;dx2;dx3;dx4];
```

The block “Computation of u” has the next m-file.

```
function SalU=RetroF2(u)
%cm is the minimum incidence matrix.
cm=[-1 1;0 -1];
%ker is the right kernel of cm.
ker=[1;1;1];
m=[u(5);u(6);u(7);u(8)];
md=[u(1);u(2);u(3);u(4)];
%mr is the minimum marking. mdr is the minimum required marking.
mr=[u(5);u(7)];
mdr=[u(1);u(3)];
%Calculate the vector lp=ΛΠ(m).
if m(2)<=m(3)
lp1=[1 0 0 0];
lp2=[0 1 0 0];
lp3=[0 0 0 1];
elseif m(2)>m(3)
lp1=[1 0 0 0];
lp2=[0 0 1 0];
lp3=[0 0 0 1];
end;
lp=[lp1;lp2;lp3];
lpm=lp*m;
vic=[0;0;0];
% We calculate the error vector and an initial solution vo such that
% cm*co=e.
e=mdr-mr;
vo=inv(cm)*e;
```

```
vo=[vo;0];
%Now, we calculate the kernel factor f, and the initial solution factor fp.
f=lpm(3);
vomin=min(vo);
if vomin<0
    fp=-f/vomin;
else
    for i=1:3
        if vo(i)==0
            pmax(i)=10000;
        elseif vo(i)>0
            pmax(i)=(lpm(i)-f)/vo(i);
        end;
    end;
    fp=min(pmax);
    if fp>1
        fp=1;
    end;
end;
%Now, we find the particular solution v.
v=fp*vo+f*ker;
%Now, we calculate the elements of Ic, named ic's, as a vector vic.
for i=1:3
    if lpm(i)>0
        vic(i)=v(i)/(lpm(i));
    elseif lpm(i)==0
        vic(i)=1;
    end;
end;
%We consider the uncontrolled transitions and the bound.
for i=1:3
    if vic(i)<0
        vic(i)=0;
    elseif vic(i)>1
```

```
        vic(i)=1;
    end;
end;
vic(3)=1;
%We transform the input from the vector vic to its matricial form Ic.
ic1=[1 0 0]*vic(1);
ic2=[0 1 0]*vic(2);
ic3=[0 0 1]*vic(3);
ic=[ic1;ic2;ic3];
%Finally, we calculate the input u.
SalU=[(eye(3)-ic)*lpm];
```

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