# Hybrid Petri nets as an approximation to Markovian Petri nets ⋆

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**Abstract:** A Markovian Petri net is a stochastic discrete event system frequently used for analysis and performance evaluation purposes. In the past, the fluidification has been proposed as a relaxation technique for avoiding the "state explosion problem". Following the same approach, in this paper a hybrid Petri net model is defined as a *partial* relaxation of an original Markovian Petri net. It is shown through a simple example that such partial relaxation can be worse than a full relaxation (given by a fully continuous Petri net). Therefore, the rest of the paper is devoted to deal with the approximation of a Markovian Petri net system by means of a hybrid model.

*Keywords:* Petri-Nets, Hybrid, Markov models, Approximate analysis.

## 1. INTRODUCTION

Different authors have proposed hybrid models based on Petri nets ($PN$). David and Alla introduced fluid and hybrid Petri nets with constant and variable speeds, for which they have explored their modeling capabilities (Alla & David (1998)). Following a different approach, Trivedi and his group introduced the so called Fluid Stochastic Petri Nets (Trivedi & Kulkarni (1993)): stochastically timed hybrid models defined for performance evaluation purposes. Alternative approaches are provided by Valette (Valette et al. (1998)) and Demongodin (Demongodin & Koussoulas (1998)), by adding continuous modeling capabilities to $PN$'s. For instance, in the Valette's approach, differential equations associated to places are introduced (as a generalization to hybrid automata).

In this paper, an approach similar to that introduced by Alla and David is considered. In (Silva & Recalde (2004); Jiménez et al. (2004)) fluid and hybrid Petri nets are introduced as a relaxation of an original discrete $PN$, rather than considering them as models *per se*. In this way, the analysis of the relaxed version of the system can provide information about the original one, but avoiding the so called "state explosion problem", which frequently appears in discrete event systems. Moreover, in the relaxed model it appears an interesting advantage: techniques from both $PN$'s and Control theories can be applied for analysis and design purposes (Vázquez, Ramírez, Recalde & Silva (2008); Mahulea et al. (2008)).

Under such approach, the approximation of Markovian Petri nets $MPN$ (i.e., stochastic Petri nets under exponential services assumption) by the corresponding fully continuous $PN$ was studied in (Vázquez, Recalde & Silva (2008)). In the present paper, those results are extended to partially relaxed models. The goal is to provide sufficiency rules for guaranteing the approximation of a $MPN$ system by the corresponding hybrid relaxation.

This paper is structured as follows: In Section 2 some basic concepts on continuous and Markovian Petri nets are introduced. After that, results related to the approximation of timed continuous Petri nets to Markovian Petri nets are recalled in Section 3. The hybrid Petri net model under study is introduced in Section 4. The approximation to $MPN$ by the hybrid relaxation is analyzed in Section 5. Finally, sufficient conditions for the approximation are provided as conclusions in Section 6.

## 2. BASIC CONCEPTS ON CONTINUOUS AND MARKOVIAN PETRI NETS

We assume that the reader is familiar with $PN$'s ( see for instance Silva (1993)). The structure $\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post}\rangle$ of *continuous Petri nets* ($CPN$) is the same as the structure of discrete PNs. That is, $P$ is a finite set of places, $T$ is a finite set of transitions with $P \cap T = \emptyset$, **Pre** and **Post** are $|P| \times |T|$ sized, natural valued, *pre- and post- incidence matrices*. We assume that $\mathcal{N}$ is connected and that every place has a successor, i.e., $|p^\bullet| \geq 1$. The usual PN system, $\langle \mathcal{N}, \mathbf{M}_0\rangle$ with $\mathbf{M}_0 \in \mathbb{N}^{|P|}$, will be said to be *discrete* so as to distinguish it from a *continuous* PN system $\langle \mathcal{N}, \mathbf{m}_0\rangle$, in which $\mathbf{m}_0 \in \mathbb{R}^{|P|}_{\geq 0}$. In the following, the marking of a $CPN$ will be denoted in lower case **m**, while the marking of the corresponding *discrete* one will be denoted in upper case **M.** The main difference between both formalisms is in the evolution rule, since in *continuous* PNs firing is not restricted to be done in integer amounts. As a consequence the marking is not forced to be integer. More precisely, a transition $t$ is *enabled*

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at $\mathbf{m}$ iff for every $p \in^\bullet t$, $\mathbf{m}(p) > 0$, and its *enabling degree* is $enab(t, \mathbf{m}) = \min_{p\in^\bullet t}\{\mathbf{m}(p)/\mathbf{Pre}(p,t)\}$. The firing of $t$ in a certain amount $\alpha \leq enab(t, \mathbf{m})$ leads to a new marking $\mathbf{m}' = \mathbf{m} + \alpha \cdot \mathbf{C}$, where $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$ is the token-flow matrix. As in discrete systems, right and left integer annullers of the token flow matrix are called *T-* and *P-flows*, respectively. When they are non-negative, they are called *T-* and *P-semiflows*. If there exists $\mathbf{y} > \mathbf{0}$ such that $\mathbf{y}\cdot\mathbf{C} = \mathbf{0}$, the net is said to be *conservative*, and if there exists $\mathbf{x} > \mathbf{0}$ such that $\mathbf{C}\cdot\mathbf{x} = \mathbf{0}$ the net is said to be *consistent*. Here, we consider net systems whose initial marking marks all *P-semiflows*.

A *Markovian Stochastic Petri Net* system *(MPN)* is a *discrete* system in which the transitions fire at independent exponentially distributed random time delays. Then, the firing time of each transition is characterized by its firing rate. In this way, a $MPN$ is a tuple $\langle \mathcal{N}, \mathbf{M}_0, \boldsymbol{\lambda}\rangle$, where $\boldsymbol{\lambda} \in \mathbb{R}^{|T|}_{>0}$ represents the transition rates. Transitions (like stations in queueing networks) are the meeting points of clients and servers. In this paper, we will assume infinite server semantics for all transitions. Then, the time to fire a transition $t_i$, at a given marking $\mathbf{M}$, is an exponentially distributed random variable with parameter $\lambda_i \cdot Enab(t_i, \mathbf{M})$, where the integer enabling degree is $Enab(t_i, \mathbf{M}) = \min_{p\in^\bullet t_i}\{\lfloor \mathbf{M}(p)/\mathbf{Pre}(p,t_i)\rfloor\}$. $Enab(t_i, \mathbf{M})$ also represents the number of active servers of $t_i$ at marking $\mathbf{M}$. We suppose that a unique steady-state behavior exists, and we restrict our study to bounded in average and reversible (therefore ergodic) $MPN$ systems.

# 3. TIMED CONTINUOUS PETRI NETS AS AN APPROXIMATION TO MPN

The approximation of a $MPN$ by means of the corresponding $CPN$ was studied in Vázquez, Recalde & Silva (2008). There, the $MPN$ was analyzed by means of the fundamental equation. The main ideas are recalled next.

## *3.1 Fundamental equation for Markovian Petri nets*

Consider a $MPN$ system with structure $\mathcal{N}$, timing rates $\boldsymbol{\lambda}$, and initial marking $\mathbf{M}_0$. Denote the initial time as $\tau_0$ and consider a particular transition $t_i$. By definition, at any marking the time to fire each active server of $t_i$ is characterized by a random variable having an exponential p.d.f. with parameter $\lambda_i$. Now, consider a fixed time interval $\Delta\tau$. If a server remains active during $\Delta\tau$ then the number of its firings (the number of jobs done) during $\Delta\tau$ is characterized by a r.v. having a Poisson p.d.f. with parameter $\lambda_i \cdot \Delta\tau$. Furthermore, since we are considering *infinite server semantics*, the number of firings of $t_i$ during $\Delta\tau$ is the sum of the number of firings of each of its servers during this time interval. If $\Delta\tau$ is small enough then the number of active servers of $t_i$ during this time interval remains almost constant. Therefore, the number of firings of $t_i$, during the time interval $(\tau_0, \tau_0 + \Delta\tau)$, can be approximated by a r.v. $\Delta\boldsymbol{\sigma}_i(\Delta\mathbf{F}_i(\tau_0))$ having a Poisson p.d.f. with parameter $\Delta\mathbf{F}_i(\tau_0) = \Delta\tau \cdot \lambda_i \cdot Enab(t_i, \mathbf{M}_0)$, where $Enab(t_i, \mathbf{M}_0)$ is the number of active servers of $t_i$ at $\mathbf{M}_0$ (the sum of independent Poisson distributed r.v.'s is also a Poisson distributed r.v., whose parameter is the sum of the parameters of the summands).

Now, considering the firing count vector $\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(\tau_0))$, whose elements are the corresponding r.v.'s $\Delta\boldsymbol{\sigma}_i(\Delta\mathbf{F}_i(\tau_0))$ of each transition, the marking at time $\tau_0 + \Delta\tau$ can be approximated by using the fundamental equation, i.e.

$$\mathbf{M}(\tau_0 + \Delta\tau) \simeq \mathbf{M}_0 + \mathbf{C}\cdot\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(\tau_0))$$

which can be generalized as:

$$\mathbf{M}(k+1) \simeq \mathbf{M}(k) + \mathbf{C}\cdot\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(k)) \tag{1}$$

where $k$ denotes $\tau_0 + k\Delta\tau$ and the parameters are given by $\Delta\mathbf{F}_i(k) = \Delta\tau\cdot\lambda_i\cdot Enab(t_i, \mathbf{M}(k))$. This equation constitutes a useful representation of the $MPN$.

## *3.2 Timed Continuous Petri nets*

A *Timed Continuous Petri Net (TCPN)* is a *continuous* PN together with a vector $\boldsymbol{\lambda} \in \mathbb{R}^{|T|}_{>0}$. Different semantics have been defined for timed *continuous* transitions, the two most important being *infinite server* or *variable speed*, and *finite server* or *constant speed*. Here *infinite server semantics* will be considered. Like in purely Markovian *discrete* net models, under *infinite server semantics* the flow through a timed transition $t_i$ is the product of the rate, $\lambda_i$, and $enab(t_i, \mathbf{m})$, the instantaneous enabling of the transition, i.e., $\mathbf{f}_i(\mathbf{m}) = \lambda_i \cdot enab(t_i, \mathbf{m}) = \lambda_i \cdot \min_{p\in^\bullet t_i}\{\mathbf{m}_p/\mathbf{Pre}(p,t_i)\}$. Observe that $Enab(t_i, \mathbf{M}) \in \mathbb{N}$ while $enab(t_i, \mathbf{m}) \in \mathbb{R}_{\geq 0}$. For the flow to be well defined, every transition must have at least one input place, hence in the following we will assume $\forall t \in T, |^\bullet t| \geq 1$. The "min" in the definition leads to the concept of *configurations*: a configuration assigns to each transition one place that, for some markings, will control its firing speed. An upper bound for the number of configurations is $\prod_{t\in T}|^\bullet t|$. The reachability space is divided into *regions* according to the *configurations*. These *regions* are polyhedrons (in bounded systems), and are disjoint, except on the borders.

The flow through the transitions can be written in a vectorial form as $\mathbf{f}(\mathbf{m}) = \boldsymbol{\Lambda\Pi}(\mathbf{m})\mathbf{m}$, where $\boldsymbol{\Lambda}$ is a diagonal matrix whose elements are those of $\boldsymbol{\lambda}$, and $\boldsymbol{\Pi}(\mathbf{m})$ is the configuration operator matrix at $\mathbf{m}$, which is defined such that the i-th entry of the vector $\boldsymbol{\Pi}(\mathbf{m})\mathbf{m}$ is equal to the enabling degree of transition $t_i$ (more details can be found, for instance, in Mahulea et al. (2008)). Therefore, the state equation of a $TCPN$ system, which is linear inside each *region*, is given by:

$$\overset{\bullet}{\mathbf{m}} = \mathbf{C}\boldsymbol{\Lambda\Pi}(\mathbf{m})\mathbf{m} \tag{2}$$

## *3.3 Approximation of MPN by TCPN*

In order to study the approximation of the $MPN$ by means of the $TCPN$, in (Vázquez, Recalde & Silva (2008)) the continuous system was analyzed in discrete-time, obtaining the following difference equation:

$$\mathbf{m}_{k+1} \simeq \mathbf{m}_k + \mathbf{C}\boldsymbol{\Lambda\Pi}(\mathbf{m}_k)\mathbf{m}_k\Delta\tau \tag{3}$$

In that paper, it was proved that given $\mathbf{m}_0 = \mathbf{M}_0$, the marking of a $TCPN$ system $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0\rangle$ (3) approximates the expected value of the marking of the $MPN$ $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{M}_0\rangle$ (1), during the time interval $(\tau_0, \tau_0 + n\Delta\tau)$, if the following conditions are fulfilled at $\mathbf{M}(\tau_0 + k\Delta\tau)$ for any time step $k$ in the interval $(\tau_0, \tau_0 + n\Delta\tau)$:
*Condition 1)* The probability that each transition of the $MPN$ is enabled is near to one.

*Condition 2)* The probability that the marking is outside the *region* of $\mathbf{M}_0$ is near to zero.

Even if the quality of the approximation decreases when a change of *regions* occurs (i.e., Condition 2 does not hold during certain time) and/or the transitions are not enabled during certain time period (Condition 1), the approximation could be good enough for analysis and control purposes. Then, both Conditions should be consider just as sufficient for the mean value approximation.

In order to improve the approximation when Condition 2 does not hold, a noise column vector $\mathbf{v}_k$ is added to the flow of the $TCPN$ model, leading to a *Markovian continous* Petri net ($MCPN$). The noise vector has as entries independent normally distributed random variables with mean and covariance matrix:

$$E\{\mathbf{v}_k\} = \mathbf{0}, \qquad \mathbf{\Sigma}_{\mathbf{v}_k} = diag[\mathbf{\Lambda\Pi}(\mathbf{m}_k)\mathbf{m}_k\Delta\tau] \tag{4}$$

Then the $MCPN$ model is defined as:

$$\mathbf{m}_{k+1} = \mathbf{m}_k + \mathbf{C\Lambda\Pi}(\mathbf{m}_k)\mathbf{m}_k\Delta\tau + \mathbf{C}\mathbf{v}_k \tag{5}$$

By analyzing the moments of this system and applying the Central Limit Theorem, it was shown that the first two moments (mean value and covariance) of the marking of the $MCPN$ system (5) *approximate* those of the marking of the corresponding $MPN$ (1) during a time interval $(\tau_0, \tau_0 + n\Delta\tau)$, if $\mathbf{m}_0 = \mathbf{M}_0$ and Condition 1 is fulfilled.

## 4. MARKOVIAN HYBRID PETRI NET MODEL

According to the results of the previous section, if some transitions are not enabled during all the time with a probability near to 1 (Condition 1) then significant errors may appear in the continuous approximation. In such case, it makes sense to fluidify only those transitions for which Condition 1 holds, obtaining thus a *hybrid* Petri net model.

*Hybrid* Petri nets were introduced by Alla & David (1998). There, the *discrete* part of the *hybrid PN* model is defined as a timed $PN$ (i.e., with constant delays at the transitions), while the *continuous* part is a *continuous PN* with constant speed (finite server semantics). In order to be consistent with the $MPN$ model, the *hybrid PN* system considered in this paper must include the random behavior of the $MPN$ at the *discrete* transitions, and the infinite server semantics in the *continuous* part. Therefore, the following hybrid model is proposed as a $Markovian$ timing for the autonomous hybrid $PN$ already introduced in (Silva & Recalde (2004)).

*Definition 1.* A *Markovian hybrid* Petri net ($MHPN$), under infinite server semantics, is a tuple $\langle \mathcal{N}, \mathbf{M}_0, \boldsymbol{\lambda}\rangle$. $\mathcal{N}$ is the structure of the $PN$, in which the set of places $P$ (transitions $T$) is partitioned into the set of continuous $P^c$ ($T^c$) and discrete $P^d$ ($T^d$) ones (i.e., $P = P^c \cup P^d$, $P^c \cap P^d = \emptyset$ and $T = T^c \cup T^d$, $T^c \cap T^d = \emptyset$). Since the fluidification is introduced through transitions, it is imposed in the model that fluid transitions only can have input or output fluid places, and each fluid place must have at least one input or output fluid transition, i.e., $P^c = ^\bullet T^c \cup T^{c\bullet}$ (it is possible to make all the places fluid by fluidifying only some transitions). $\mathbf{M}_0 \in \mathbb{N}^{|P|}$ represents the initial marking, and $\boldsymbol{\lambda} \in \mathbb{R}^{|T|}_{>0}$ represents the transition rates. Each discrete transition $t_i \in T^d$ fires in discrete amounts with exponentially distributed random time delays, with parameter $\lambda_i \cdot Enab(t_i, \mathbf{M})$, as in the $MPN$ model. Each continuous transition $t_i \in T^c$ fires with the flow $\mathbf{f}_i(\mathbf{m}) = \lambda_i \cdot enab(t_i, \mathbf{m})$, as in the $TCPN$ model.

Under this definition, the fundamental equation introduced in subsection 3.1 can be used for representing the behavior of the discrete part of the system (the firing of discrete transitions), while (3) can be used for describing the continuous behavior. Without loss of generality, let us suppose that the first columns of matrix $\mathbf{C}$ are related to the discrete transitions, while the last columns correspond to fluid ones. Then the $MHPN$ can be represented as:

$$\mathbf{M}(k+1) \simeq \mathbf{M}(k) + \begin{bmatrix}\mathbf{C}^d & \mathbf{C}^c\end{bmatrix} \cdot \begin{bmatrix} \Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(k)) \\ \mathbf{\Lambda}^c\mathbf{\Pi}(\mathbf{M}(k))\mathbf{M}(k)\Delta\tau \end{bmatrix}$$

where $\mathbf{M}(k)$ represents the whole marking and $\mathbf{C}^d$ ($\mathbf{C}^c$) represents the restriction of $\mathbf{C}$ to the discrete (continuous) transitions (i.e., $\mathbf{C} = [\mathbf{C}^d \quad \mathbf{C}^c]$). In the same way, the firing rate matrix $\mathbf{\Lambda}$ is divided into a matrix for the discrete transitions ($\mathbf{\Lambda}^d$) and other one for the continuous transitions ($\mathbf{\Lambda}^c$). The firing count vector $\boldsymbol{\sigma}(\Delta\mathbf{F}(k))$, having as elements random variables with Poisson p.d.f. with parameters $\Delta\mathbf{F}(k) = \mathbf{\Lambda}^d \cdot Enab(\mathbf{M}(k)) \cdot \Delta\tau$, is defined just for the discrete transitions, while the configuration matrix $\mathbf{\Pi}(\cdot)$ is defined just for fluid transitions.

Now, let us suppose that the first rows of the incidence matrix corresponds to the discrete places, while the last rows to fluid ones. Then, the marking can be represented as $\mathbf{M}(k) = [\mu_k^T \quad \mathbf{m}_k^T]^T$, where $\mu_k$ ($\mathbf{m}_k$) corresponds to the marking of the discrete (fluid) places. In the same way, the incidence matrices can be written as $\mathbf{C}^d = [({}^d\mathbf{C}^d)^T \quad ({}^c\mathbf{C}^d)^T]^T$ and $\mathbf{C}^c = [({}^d\mathbf{C}^c)^T \quad ({}^c\mathbf{C}^c)^T]^T$, where ${}^d\mathbf{C}^d$ (${}^c\mathbf{C}^d$) represents the restriction of $\mathbf{C}^d$ to the discrete (continuous) places, and both ${}^d\mathbf{C}^c$ and ${}^c\mathbf{C}^c$ are defined in a similar way. However, since $P^d \cap (^\bullet T^c \cup T^{c\bullet}) = \emptyset$ then ${}^d\mathbf{C}^c = \mathbf{0}$. Therefore, the $MHPN$ can be rewritten as two different systems but connected:

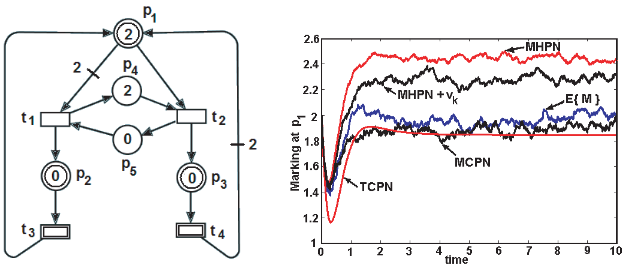
$$\begin{aligned} \mu_{\mathbf{k+1}} &\simeq \mu_{\mathbf{k}} +^d \mathbf{C}^d \cdot \Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(k)) \\ \mathbf{m_{k+1}} &\simeq \mathbf{m_k} +^c \mathbf{C}^c \cdot \mathbf{\Lambda\Pi}(\mathbf{m}_k)\mathbf{m}_k\Delta\tau +^c \mathbf{C}^d \cdot \Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(k)) \end{aligned} \tag{6}$$

Notice that the flow of the fluid transitions only depends on the marking at the fluid places. On the contrary, the firing of discrete transitions depends on the marking of both discrete and fluid places, because the parameters of the Poisson random variables are $\Delta\mathbf{F}(k) = \mathbf{\Lambda}^d \cdot Enab(\mathbf{M}(k)) \cdot \Delta\tau$ (i.e., is a function of the whole marking $\mathbf{M}(k)$).

In the system given by (6) discrete transitions fire with random delays, while the continuous ones are deterministic w.r.t. the fluid marking. However, it is possible to add uncorrelated gaussian noise to the continuous transitions in order to improve the approximation of the flow at these (as done in $TCPN$ model), obtaining the following system:

$$\begin{aligned} \mu_{\mathbf{k+1}} &\simeq \mu_{\mathbf{k}} +^d \mathbf{C}^d \cdot \Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(k)) \\ \mathbf{m_{k+1}} &\simeq \mathbf{m_k} +^c \mathbf{C}^c \cdot \mathbf{\Lambda\Pi}(\mathbf{m}_k)\mathbf{m}_k\Delta\tau +^c \mathbf{C}^c \cdot \mathbf{v}_k \\ &\quad +^c\mathbf{C}^d \cdot \Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(k)) \end{aligned} \tag{7}$$

where $\mathbf{v}_k$ is defined for the fluid transitions as in (4). In the sequel, model (7) will be denoted as $MHPN + \mathbf{v}_k$.




(a) (b)

Fig. 1. a) A $PN$ system with $\lambda = [1 \quad 3 \quad 1 \quad 2]^T$. b) As a hybrid model, nodes $t_1, t_2, p_4$ and $p_5$ are discrete. $E\{\mathbf{M}\}$ corresponds to the original system $MPN$, while others represents the corresponding relaxations.

## 5. APPROXIMATION OF THE $MPN$ MODEL BY THE CORRESPONDING $MHPN$

The $MHPN$ model is defined as a partial relaxation of the $MPN$, so, one could think that the approximation provided by the *hybrid* system to the original *discrete* one should be better than that provided by the totally relaxed *continuous* model. However, that is not always the case.

Consider for instance the $PN$ system of fig. 1(a) with rates $\lambda = [1 \quad 3 \quad 1 \quad 2]^T$. This $PN$ was simulated 1000 times as a discrete, fluid and hybrid system, in order to obtain mean trajectories of the marking at $p_1$. As a hybrid model, nodes $t_1, t_2, p_4, p_5$ are discrete, while others are continuous. Fig. 1(b) shows the resulting mean trajectories. It can be seen that fluid models $TCPN$ (3) and $MCPN$ (5) provide a better approximation to the Markovian $PN$ (denoted as $E\{\mathbf{M}\}$) than hybrid models $MHPN$ (6) and $MHPN + \mathbf{v}_k$ (7), i.e., *a partial relaxation can be worse than a full relaxation!* Let us analyze this in the following subsection.

### *5.1 Approximation analysis*

The dynamical behavior of the $MPN$ is achieved by the firing of its transitions, which is characterized by the firing count vector $\Delta\sigma(\mathbf{\Delta F}(k))$ in (1). In this way, if at some time step $k$, the average marking of the $MPN$ is well approximated by the average marking of a given relaxed model (either fluid or hybrid) and their transitions fire in the same amount in both (the $MPN$ and the relaxed model), then the marking approximation will hold for the next time step $k+1$. Therefore, following an inductive reasoning, if the initial condition of both systems coincide and the firing count vector of the relaxed model approximates that of the $MPN$ system through the time, then the marking approximation is achieved (errors are not accumulated because, roughly speaking, the ergodicity of the $MPN$ implies asymptotic stability in the relaxed model, i.e., early errors will not affect the long term behavior). However, it is important to remember that the firing count vector of the $MPN$ is a random variable, then the corresponding firing count vector of the relaxed model should approximate the moments of the original one, i.e, mean value and covariance. Let us focus first in the mean value approximation though this subsection.

The mean value of the firing count vector of the $MPN$ is approximated by the flow (but multiplied by $\Delta\tau$) at the continuous transitions in the relaxed fluid model. In (Vázquez, Recalde & Silva (2008)) it was found that such approximation is effective if Conditions 1 and 2 hold. A similar reasoning holds for the fluid transitions in the hybrid models, so no more analysis in these is required.

On the other hand, in the hybrid models, discrete transitions can have as input places either discrete or continuous ones. If discrete transitions have only discrete input places no problem occurs (there is no relaxation, then the approximation is perfect at these transitions). However, if a discrete transition has input continuous places then it can lead to a bad approximation, as in the case of the system of fig. 1(a). Now, consider the synchronization of fig. 2(a). Transition $t_1$ is a discrete transition having as input places $p_1 \in P^c$ and $p_2 \in P^d$. The expected number of firings of $t_1$ during a time interval $\Delta\tau$ is proportional to the expected value of its enabling degree, i.e., $E\{\Delta\sigma(\mathbf{\Delta F}_1)\} = \lambda_1 \cdot \Delta\tau \cdot E\{Enab(t_1)\}$, which can be computed by using the total probability theorem for the $MPN$ as:

$$
\begin{aligned}
E\ \{Enab(t_1)\} =& \\
&= E\{min(\lfloor \mathbf{M}(p_1)/Pre(p_1,t_1)\rfloor, \lfloor \mathbf{M}(p_2)/Pre(p_2,t_1)\rfloor)\} \\
&= \sum_{S_{\mathbf{M}_2}} \sum_{S_{\mathbf{M}_1}} min(M_1,M_2) \cdot Prob(M_1|M_2) Prob(M_2)
\end{aligned} \tag{8}
$$

and for the hybrid Petri net as:

$$
E\{Enab(t_1)\} = \sum_{S_{\mathbf{M}_2}} \int min(\lfloor x \rfloor, M_2) f_{1|2}(x)dx \cdot Prob(M_2) \tag{9}
$$

where $S_{\mathbf{M}_1}$ ($S_{\mathbf{M}_2}$) denotes all the possible values for the marking at $p_1$ ($p_2$), and $f_{1|2}(\cdot)$ is the probability density function of the marking at fluid place $p_1$ given $\mathbf{M}(p_2) = M_2$ ($M_1, M_2$ denote fixed values for $\mathbf{M}(p_1), \mathbf{M}(p_2)$). Then, the approximation of the firing count vector is achieved if the fluid marking at $p_1$ is representative of the marking (the value of the marking, not the mean value of this) of the *original* $MPN$ w.r.t. the enabling degree function, i.e., if the value of $\sum_{S_{\mathbf{M}_1}} min(M_1,M_2) Prob(M_1|M_2)$ in (8) is close to the value of $\int min(\lfloor x \rfloor, M_2) f_{1|2}(x)dx$ in (9). For instance, in the synchronization of fig. 2(a), markings at places $p_1$ and $p_2$ are random variables, but given the current marking, for the most probable values of $p_1$ and $p_2$, $p_2$ will constraint $t_1$, i.e., in (9) $\int min(\lfloor x \rfloor, M_2) f_{1|2}(x)dx \simeq M_2$. Following a similar reasoning, in the original $MPN$ the probability that $p_1$ constraints $t_1$ is negligible, i.e., in (8) $\sum_{S_{\mathbf{M}_1}} min(M_1,M_2) \cdot Prob(M_1|M_2) \simeq M_2$, so the enabling degree of $t_1$, and thus the firing count, is well approximated in the relaxed model. This case can be generalized as a sufficient condition for obtaining a good approximation, during a time interval $(\tau_0, \tau_0 + n\Delta\tau)$:
*Condition 3)* The probability that the discrete transitions be constrained by discrete places at $\mathbf{M}(\tau_0 + k\Delta\tau)$, for any time step $k$ in the interval $(\tau_0, \tau_0 + n\Delta\tau)$, is near to 1.

Condition 3 implies that the arcs between fluid places and discrete transitions are temporarily implicit (i.e., the discrete subnet evolves independently of the marking at fluid places). Furthermore, this condition can be generalized. Consider again the discrete transition of fig. 2(a). If at the current time step $k$ the distribution of the marking at $p_1$ in the $MPN$ were well approximated by the distribution of the fluid marking for the same place but in the $MHPN$,

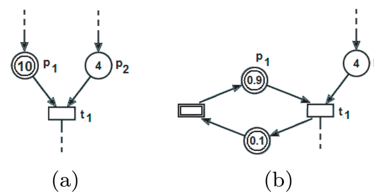



Fig. 2. a) Discrete transition in a $MHPN$ with a continuous and a discrete input places. b) $t_1$ only is enabled only at infinite time as a hybrid model.

then the expected enabling degree of $t_1$ in both models would be similar. Such condition can be stated as:

$$Prob(M_1) \simeq \int_{M_1}^{M_1+1} f_1(x)dx \qquad \forall M_1 \in S_{\mathbf{M}_1} \quad (10)$$

In order to obtain such approximation, it is required that the marking at $p_1$ be large enough, i.e., that $S_{\mathbf{M}_1}$ consists in several probable values for the marking at $p_1$, so the marking approximation errors will be small w.r.t. their mean values. Furthermore, it is very important that $p_1$ at least enables $t_1$ with probability near to 1, otherwise a minimum error in the marking can lead to a big one in the firing count. For instance, consider the $PN$ of fig. 2(b). Notice that as a hybrid model $t_1$ is enabled only at infinite time (only in infinite time the marking at $p_1$ is 1), but in the discrete $MPN$ $t_1$ is enabled with a significant probability (since the mean value of the marking in the $MPN$ is close to 0.9, the probability that $\mathbf{M}(p_1) = 1$ is significant). Then, Condition 3 is generalized as:

*Condition 4)* Discrete transitions can be constrained by either discrete or fluid places, but fluid places constraining discrete transitions enable them with probability near to 1 at $\mathbf{M}(\tau_0 + k\Delta\tau)$, for any time step $k$ in the interval $(\tau_0, \tau_0 + n\Delta\tau)$, i.e., such output discrete transitions are always enabled. The larger the marking at such fluid places, the better the approximation.

For instance, consider again the system of figure 1(a) with the same firing rates. The $MPN$ and $MHPN$ systems have been simulated 1000 times for different initial markings at $p_1$, while the initial markings for the other places remain as in fig. 1(a). Table 1 resumes the results thus obtained. The first column represents the initial marking. Columns 2 and 3 are the expected values at the steady state of $p_1$ for the $MPN$ and $MHPN$, respectively. Next column deals with to the approximation error, while the last column is the probability that $p_5$ constraints $t_1$ (i.e., Condition 3 for $t_1$). As expected, the error is lower when $p_1$ does not constraint $t_1$ (i.e., when $P.C3 \to 1$). On the other hand, in the last experiment an initial marking of $\mathbf{M}_0 = [10, 0, 0, 10, 0]^T$ was used. It can be seen that the approximation by the *hybrid* model is good, even if the probability that $p_5$, the discrete place, constraints $t_1$ is almost 0, i.e., $p_1$ constraints $t_1$ so Condition 3 is not fulfilled. However, in this last case the value of $p_1$ is large enough, which means that Condition 4 holds.

### *5.2 Improvement of MHPN by adding noise*

As it was pointed out in subsection 3.3, the addition of uncorrelated gaussian noise to the $TCPN$ model improves the approximation to the $MPN$. Such improvement becomes more significative when the marking is around to the border of two or more *regions*. Let us detail this.

Table 1. Initial and steady state markings at $p_1$ for the $MPN$ and $MHPN$ of fig. 1(a)

| $\mathbf{M}_0(p_1)$ | MPN | MHPN | error | P. C3 |
|---|---|---|---|---|
| 2 | 1.95 | 2.40 | 23.1% | 0.42 |
| 3 | 2.54 | 2.74 | 7.87% | 0.52 |
| 4 | 3.27 | 3.36 | 2.8% | 0.65 |
| 5 | 4.04 | 4.02 | 0.5% | 0.78 |
| 10 | 8.52 | 8.58 | 0.7% | 0.98 |
| $\mathbf{M}_0(p_1) = \mathbf{M}_0(p_4) = 10$ | 9.50 | 9.8 | 3.2% | 0.01 |

Consider a synchronization as in fig. 2(a). Suppose that $t_1, p_1$ and $p_2$ are continuous, and that at some time instant $\tau$, the marking is 1 at $p_1$ and $p_2$. In such case, the enabling degree of $t_1$ would be $min(\mathbf{m}(p_1), \mathbf{m}(p_2)) = 1$, so $t_1$ would fire with a speed of $\lambda_1 \cdot 1$. Now, let us analyze the behavior of $t_1$ in the $MPN$. Suppose that at the same time instant $\tau$ the relaxed model approximates the $MPN$ one, then a value of 1 at $p_1$ and $p_2$ in the relaxed model means that the average marking of the $MPN$ is $[E\{\mathbf{M}(p_1)\}, E\{\mathbf{M}(p_2)\}]^T = [1, 1]^T$. Now, the expected enabling degree is $E\{Enab(t_1)\} = E\{min(\mathbf{M}(p_1), \mathbf{M}(p_2))\}$, but notice that $min(\lfloor\cdot\rfloor, \lfloor\cdot\rfloor)$ is not a linear operator, so, in general $E\{Enab(t_1)\} \neq min(E\{\mathbf{M}(p_1)\}, E\{\mathbf{M}(p_2)\}) = 1$. In the $MPN$, the expected value of the enabling degree depends on the particular distribution of the marking, which is in general unknown. Then, even if the relaxed model approximates well the $MPN$ at $\tau$, the approximation of the enabling degree, and thus the firing count, can be bad.

As recalled in subsection 3.3, the addition of noise $\mathbf{v}_k$ (4) to the fluid transitions in the $TCPN$ model improves the approximation of the firing count of the discrete transitions in the $MPN$. Moreover, the approximation is achieved not only at the mean value but also at the covariance. Following a similar reasoning, the addition of noise to the continuous transitions in the $MHPN$ model may improve the approximation to the original $MPN$, obtaining thus the $MHPN + \mathbf{v}_k$ model (7). Nevertheless, the difference between the approximation provided by the $MHPN$ model (6) and the hybrid model with noise (7) is not so important as in the case of fully continuous systems.

For instance, consider the system of fig. 3. As hybrid, transitions $t_6$, $t_7$ and places $p_4$, $p_5$, $p_6$, $p_7$, $p_8$ and $p_9$ are continuous, while other transitions and places are discrete. Consider rates as $\lambda = [30, 2, 30, 30, 30, 3.5, 10]^T$. The system was simulated 400 times as discrete ($MPN$), continuous ($TCPN$ and $MCPN$) and hybrid ($MHPN$ and $MHPN + \mathbf{v}_k$). The mean trajectories for the marking at place $p_6$ are shown in fig. 4(a). As it can be seen, the addition of the noise in the $MCPN$ model represents an important improvement to the approximation of the $MPN$ (denoted as $E\{\mathbf{M}\}$) with respect to the $TCPN$ system. However, the improvement in the hybrid models (compare $MHPN$ against $MHPN + \mathbf{v}_k$) is not so important. The reason for that is the stochastic behavior of the discrete transitions in the hybrid model, i.e., the stochastic behavior of the firing of discrete transitions in the $MHPN$ makes the marking at continuous places be also stochastic, so it approximates not only the mean value but also the covariance (in certain degree) of the marking in the $MPN$.

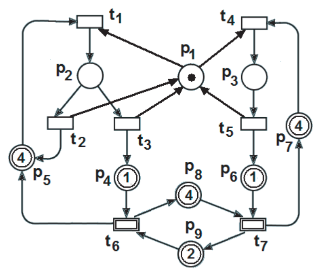
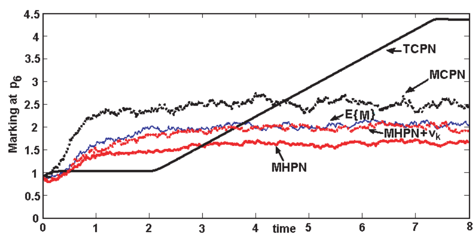
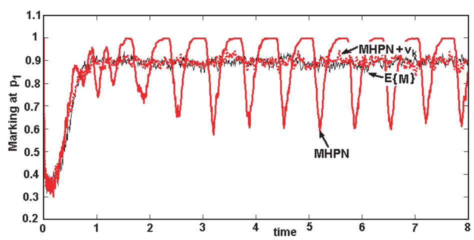



Fig. 3. As a hybrid $PN$, nodes $t_6, t_7, p_4, p_5, p_6, p_7, p_8, p_9$ are continuous while others are discrete.




(a)




(b)

Fig. 4. a) Expected marking at $p_6$ of the $PN$ of fig. 3, with $\lambda = [30, 2, 30, 30, 30, 3.5, 10]^T$. b) Expected marking at $p_1$ with $\lambda = [30, 2, 30, 30, 30, 0.5, 0.5]^T$. In both figures, $E\{\mathbf{M}\}$ corresponds to the $MPN$, while other curves represent the corresponding relaxations.

However, the addition of noise in the hybrid model becomes important when discrete transitions are constrained by continuous places (i.e., Condition 3 does not hold) and the marking at those places is low (i.e., Condition 4 is barely fulfilled). For example, consider again the system of fig. 3 with $\lambda = [30, 2, 30, 30, 30, 0.5, 0.5]^T$. Fig. 4(b) shows the mean trajectory of 400 simulations, of the marking at $p_1$ for the discrete ($MPN$) and hybrid ($MHPN$ and $MHPN + \mathbf{v}_k$) systems (the periodic behavior of the $MHPN$ is explained by the low rates of $t_6$ and $t_7$ that increase the marking at $p_5$ and $p_7$ during a long deterministic period until enabling $t_1$ and $t_4$, which fire almost instantaneously w.r.t. the fluid dynamics). It can be seen that $MHPN + \mathbf{v}_k$ provides a better approximation to the $MPN$ than $MHPN$. In this case, the noise added to the continuous transitions makes the p.d.f. of the continuous marking at $p_5$ and $p_7$ approximate better the p.d.f. of the corresponding markings in the $MPN$, because in this not only the mean value of the marking is being approximated but also the covariance, so (10) is closer to be fulfilled.

## 6. CONCLUSIONS

In this paper, a hybrid Petri net model $MHPN$ is introduced as a partial relaxation of a $MPN$. Such hybrid model is enriched by adding gaussian noise to the continuous transitions, in order to improve the approximation, obtaining thus another hybrid system $MHPN + \mathbf{v}_k$. It was found that in order to approximate a $MPN$ by a hybrid relaxation, next conditions should be taken into account:

(1) All the fluid transitions should be enabled with probability near to 1 (Condition 1).
(2) Discrete transitions should be constrained by discrete places with probability near to 1, i.e., Condition 3. Otherwise, continuous places that constraint discrete transitions should enable such transitions with probability near to 1; the larger the marking at those fluid places, the better the approximation (Condition 4).
(3) For all fluid transition representing a synchronization, if more than one place constraint such transition (i.e., Condition 2 does not hold for the regions defined by the fluid subnet), then gaussian noise (4) should be added to the fluid transitions for improving the approximation, obtaining thus a $MHPN + \mathbf{v}_k$ model.
(4) If Condition 3 does not hold for some discrete transitions, and Condition 4 is barely fulfilled, then the addition of the gaussian noise (4) becomes important for the approximation, i.e., the $MHPN + \mathbf{v}_k$ model should be considered.

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