Hybrid Petri nets as an approximation to Markovian Petri nets *

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Abstract: A Markovian Petri net is a stochastic discrete event system frequently used for analysis and performance evaluation purposes. In the past, the fluidification has been proposed as a relaxation technique for avoiding the "state explosion problem". Following the same approach, in this paper a hybrid Petri net model is defined as a partial relaxation of an original Markovian Petri net. It is shown through a simple example that such partial relaxation can be worse than a full relaxation (given by a fully continuous Petri net). Therefore, the rest of the paper is devoted to deal with the approximation of a Markovian Petri net system by means of a hybrid model.

Keywords: Petri-Nets, Hybrid, Markov models, Approximate analysis.

1. INTRODUCTION

Different authors have proposed hybrid models based on Petri nets (PN). David and Alla introduced fluid and hybrid Petri nets with constant and variable speeds, for which they have explored their modeling capabilities (Alla & David (1998)). Following a different approach, Trivedi and his group introduced the so called Fluid Stochastic Petri Nets (Trivedi & Kulkarni (1993)): stochastically timed hybrid models defined for performance evaluation purposes. Alternative approaches are provided by Valette (Valette et al. (1998)) and Demongodin (Demongodin & Koussoulas (1998)), by adding continuous modeling capabilities to PN’s. For instance, in the Valette’s approach, differential equations associated to places are introduced (as a generalization to hybrid automata).

In this paper, an approach similar to that introduced by Alla and David is considered. In (Silva & Recalde (2004); Jiménez et al. (2004)) fluid and hybrid Petri nets are introduced as a relaxation of an original discrete PN, rather than considering them as models per se. In this way, the analysis of the relaxed version of the system can provide information about the original one, but avoiding the so called “state explosion problem”, which frequently appears in discrete event systems. Moreover, in the relaxed model it appears an interesting advantage: techniques from both PN’s and Control theories can be applied for analysis and design purposes (Vázquez, Ramírez, Recalde & Silva (2008); Mahulea et al. (2008)).

Under such approach, the approximation of Markovian Petri nets MPN (i.e., stochastic Petri nets under exponential services assumption) by the corresponding fully continuous PN was studied in (Vázquez, Recalde & Silva (2008)). In the present paper, those results are extended to partially relaxed models. The goal is to provide sufficiency rules for guaranteeing the approximation of a MPN system by the corresponding hybrid relaxation.

This paper is structured as follows: In Section 2 some basic concepts on continuous and Markovian Petri nets are introduced. After that, results related to the approximation of timed continuous Petri nets to Markovian Petri nets are recalled in Section 3. The hybrid Petri net model under study is introduced in Section 4. The approximation to MPN by the hybrid relaxation is analyzed in Section 5. Finally, sufficient conditions for the approximation are provided as conclusions in Section 6.

2. BASIC CONCEPTS ON CONTINUOUS AND MARKOVIAN PETRI NETS

We assume that the reader is familiar with PN’s (see for instance Silva (1993)). The structure \( \langle N, T, P, \text{Pre}, \text{Post} \rangle \) of continuous Petri nets (CPN) is the same as the structure of discrete PNs. That is, \( P \) is a finite set of places, \( T \) is a finite set of transitions with \( P \cap T = \emptyset \), \( \text{Pre} \) and \( \text{Post} \) are \( |P| \times |T| \) sized, natural valued, pre- and post- incidence matrices. We assume that \( N \) is connected and that every place has a successor, i.e., \( |\text{Pre}| \geq 1 \). The usual PN system, \( \langle N, M_0 \rangle \) with \( M_0 \in \mathbb{N}^{|P|} \), will be said to be discrete so as to distinguish it from a continuous PN system \( \langle N, m_0 \rangle \), in which \( m_0 \in \mathbb{R}^{|P|}_{\geq 0} \).

In the following, the marking of a CPN will be denoted in lower case \( \text{m} \), while the marking of the corresponding discrete one will be denoted in upper case \( \text{M} \). The main difference between both formalisms is in the evolution rule, since in continuous PNs firing is not restricted to be done in integer amounts. As a consequence the marking is not forced to be integer. More precisely, a transition \( t \) is enabled
at \( m \) iff for every \( p \in T \), \( m(p) > 0 \), and its enabling degree is \( \text{enab}(t, m) = \min_{p \in \text{enab}(t)} \{m(p)/\text{Pre}(p, t)\} \). The firing of \( t \) in a certain amount \( \alpha \leq \text{enab}(t, m) \) leads to a new marking \( m' = m + \alpha \cdot C \), where \( C = \text{Post} - \text{Pre} \) is the token-flow matrix. As in discrete systems, right and left integer annulators of the token flow matrix are called \( T \)- and \( P \)-flows, respectively. When they are non-negative, they are called \( T \)- and \( P \)-semiflows. If there exists \( y > 0 \) such that \( y \cdot C = 0 \), the net is said to be conservative, and if there exists \( x > 0 \) such that \( C \cdot x = 0 \) the net is said to be consistent. Here, we consider net systems whose initial marking marks all \( P \)-semiflows.

A Markovian Stochastic Petri Net system (MPN) is a discrete system in which the transitions fire at independently exponentially distributed random time delays. Then, the firing time of each transition is characterized by its firing rate. In this way, a MPN is a tuple \( \langle N, M_0, \lambda, \sigma \rangle \), where \( \lambda \in \mathbb{R}^{|T|}_+ \) represents the transition rates. Transitions (like stations in queueing networks) are the meeting points of clients and servers. In this paper, we will assume infinite server semantics for all transitions. Then, the time to fire a transition \( t_i \), at a given marking \( M \), is an exponentially distributed random variable with parameter \( \lambda_i \cdot \text{Enab}(t_i, M) \), where the integer enabling degree is \( \text{Enab}(t_i, M) = \min_{t_j \in \text{enab}(t_i)} \{ |M(p)/\text{Pre}(p, t_j)| \} \). \( \text{Enab}(t_i, M) \) also represents the number of active servers of \( t_i \) at marking \( M \). We suppose that a unique steady-state behavior exists, and we restrict our study to bounded in average and reversible (therefore ergodic) MPN systems.

3. TIMED CONTINUOUS PETRI NETS AS AN APPROXIMATION TO MPN

The approximation of a MPN by means of the corresponding CPN was studied in Vázquez, Recalde & Silva (2008). There, the MPN was analyzed by means of the fundamental equation. The main ideas are recalled next.

3.1 Fundamental equation for Markovian Petri nets

Consider a MPN system with structure \( N \), timing rates \( \lambda \), and initial marking \( M_0 \). Denote the initial time as \( \tau_0 \) and consider a particular transition \( t \). By definition, at any marking the time to fire each active server of \( t \) is characterized by a random variable having an exponential p.d.f. with parameter \( \lambda \). Now, consider a fixed time interval \( \Delta \tau \). If a server remains active during \( \Delta \tau \) then the number of its firings (the number of jobs done) during \( \Delta \tau \) is characterized by a r.v. having a Poisson p.d.f. with parameter \( \lambda \cdot \Delta \tau \). Furthermore, since we are considering infinite server semantics, the number of firings of \( t \) during \( \Delta \tau \) is the sum of the number of firings of each of its servers during this time interval. If \( \Delta \tau \) is small enough then the number of active servers of \( t \) during this time interval remains almost constant. Therefore, the number of firings of \( t \), during the time interval \( (\tau_0, \tau_0 + \Delta \tau) \), can be approximated by a r.v. \( \Delta \sigma(f(\tau_0)) \) having a Poisson p.d.f. with parameter \( \Delta \tau \cdot \lambda \cdot \text{Enab}(t_i, M_0) \), where \( \text{Enab}(t_i, M_0) \) is the number of active servers of \( t_i \) at \( M_0 \) (the sum of independent Poisson distributed r.v.’s is also a Poisson distributed r.v., whose parameter is the sum of the parameters of the summands).

Now, considering the firing count vector \( \Delta \sigma(\Delta F(\tau_0)) \), whose elements are the corresponding r.v.’s \( \Delta \sigma(f(\tau_0)) \) of each transition, the marking at time \( \tau_0 + \Delta \tau \) can be approximated by using the fundamental equation, i.e.

\[
M(\tau_0 + \Delta \tau) \approx M_0 + C \cdot \Delta \sigma(\Delta F(\tau_0))
\]

which can be generalized as:

\[
M(k + 1) \approx M(k) + C \cdot \Delta \sigma(\Delta F(k))
\]

(1)

where \( k \) denotes \( \tau_0 + k \Delta \tau \) and the parameters are given by \( \Delta F(k) = \Delta \tau \cdot \lambda_i \cdot \text{Enab}(t_i, M(k)) \). This equation constitutes a useful representation of the MPN.

3.2 Timed Continuous Petri nets

A Timed Continuous Petri Net (TCPN) is a continuous PN together with a vector \( \lambda \in \mathbb{R}^{|T|}_+ \). Different semantics have been defined for timed continuous transitions, the two most important being infinite server or variable speed, and finite server or constant speed. Here infinite server semantics will be considered. Like in purely Markovian discrete net models, under infinite server semantics the flow through a timed transition \( t_i \) is the product of the rate, \( \lambda_i \), and \( \text{enab}(t_i, M) \), the instantaneous enabling of the transition, i.e., \( f_i(m) = \lambda_i \cdot \text{enab}(t_i, m) = \lambda_i \cdot \min_{t_j \in \text{enab}(t_i)} \{m(p)/\text{Pre}(p, t_j)\} \). Observe that \( \text{Enab}(t_i, M) \in \mathbb{N} \) while \( \text{enab}(t_i, m) \in \mathbb{R}^{|T|}_+ \). For the flow to be well defined, every transition must have at least one input place, hence in the following we will assume \( \forall t \in T, |t| \geq 1 \). The “min” in the definition leads to the concept of configurations: a configuration assigns to each transition one place that, for some markings, will control its firing speed. An upper bound for the number of configurations is \( |t| \). The reachability space is divided into regions according to the configurations. These regions are polyhedrons (in bounded systems), and are disjoint, except on the borders.

The flow through the transitions can be written in a vectorial form as \( f(m) = \Pi \Lambda(m)m \), where \( \Lambda \) is a diagonal matrix whose elements are those of \( \lambda \), and \( \Pi(m) \) is the configuration operator matrix at \( m \), which is defined such that the \( i \)-th entry of the vector \( \Pi(m)m \) is equal to the enabling degree of transition \( t_i \) (more details can be found, for instance, in Mahulea et al. (2008)). Therefore, the state equation of a TCPN system, which is linear inside each region, is given by:

\[
m = \Pi \Lambda (m)m
\]

(2)

3.3 Approximation of MPN by TCPN

In order to study the approximation of the MPN by means of the TCPN, in (Vázquez, Recalde & Silva (2008)) the continuous system was analyzed in discrete-time, obtaining the following difference equation:

\[
m(k+1) \approx m(k) + \Pi \Lambda (m)\Delta m(k)\Delta \tau
\]

(3)

In that paper, it was proved that given \( m_0 = M_0 \), the marking of a TCPN system \( \langle N, \lambda, m_0 \rangle \) (3) approximates the expected value of the marking of the MPN \( \langle N, \lambda, M_0 \rangle \) (1), during the time interval \( (\tau_0, \tau_0 + n \Delta \tau) \), if the following conditions are fulfilled at \( M(\tau_0 + k \Delta \tau) \) for any time step \( k \) in the interval \( (\tau_0, \tau_0 + n \Delta \tau) 

\text{Condition 1) The probability that each transition of the MPN is enabled is near to one.}

Condition 2) The probability that the marking is outside the region of $M_0$ is near to zero.

Even if the quality of the approximation decreases when a change of regions occurs (i.e., Condition 2 does not hold during a certain time interval) and/or the transitions are not enabled during a certain time period (Condition 1), the approximation could be good enough for analysis and control purposes. Then, both Conditions should be considered just as sufficient for the mean value approximation.

In order to improve the approximation when Condition 2 does not hold, a noise column vector $v_k$ is added to the flow of the TCPN model, leading to a Markovian continuous Petri net (MCPN). The noise vector has as entries independent normally distributed random variables with mean and covariance matrix:

$E[v_k] = 0, \quad \sum v_k = \text{diag}[\Lambda \Pi(M_k) m_k \Delta \tau]$  \hspace{1cm} (4)

Then the MCPN model is defined as:

$m_{k+1} = m_k + \Lambda \Pi(m_k) m_k \Delta \tau + C v_k$  \hspace{1cm} (5)

By analysing the moments of this system and applying the Central Limit Theorem, it was shown that the first two moments (mean value and covariance) of the marking of the MCPN system approximate those of the marking of the corresponding MPN during a time interval $(n_0, n_0 + n \Delta t)$, if $m_0 = M_0$ and Condition 1 is fulfilled.

4. MARKOVIAN HYBRID PETRI NET MODEL

According to the results of the previous section, if some transitions are not enabled during all the time with a probability near to 1 (Condition 1) then significant errors may appear in the continuous approximation. In such cases, it makes sense to fluidly only those transitions for which Condition 1 holds, obtaining a hybrid Petri net model.

Hybrid Petri nets were introduced by Alla & David (1998). There, the discrete part of the hybrid PN model is defined as a timed PN (i.e., with constant delays at the transitions), while the continuous part is a continuous PN with constant speed (finite server semantics). In order to be consistent with the MPN model, the hybrid PN system considered in this paper must include the marking behavior of the MPN at the discrete transitions, and the infinite server semantics in the continuous part. Therefore, the following hybrid model is proposed as a Markovian timing for the autonomous hybrid PN already introduced in (Silva & Recalde (2004)).

Definition 1. A Markovian hybrid Petri net (MHPN), under infinite server semantics, is a tuple $(N, M_0, \lambda, \Lambda)$. $N$ is the structure of the PN, in which the set of places $P$ (transitions $T$) is partitioned into the set of continuous $P^c$ ($T^c$) and discrete $P^d$ ($T^d$) ones (i.e., $P = P^c \cup P^d$, $P^c \cap P^d = \emptyset$ and $T = T^c \cup T^d$, $T^c \cap T^d = \emptyset$). Since the fluidification is introduced through transitions, it is imposed in the model that fluid transitions only can have input or output fluid places, and each fluid place must have at least one input or output fluid transition, i.e., $P^c = T^c \cup T^d$ (it is possible to make all the places fluid by fluidifying only some transitions). $M_0 \in \mathbb{N}^{|P|}$ represents the initial marking, and $\lambda \in \mathbb{R}^{|T|}$ represents the transition rates. Each discrete transition $t_i \in T^d$ fires in discrete amounts with exponentially distributed random time delays, with parameter $\lambda_i \cdot \text{Enab}(t_i, M)$, as in the MPN model. Each continuous transition $t_i \in T^c$ fires with the flow $f_i(M) = \lambda_i \cdot \text{enab}(t_i, M)$, as in the TCPN model.

Under this definition, the fundamental equation introduced in subsection 3.1 can be used for representing the behavior of the discrete part of the system (the firing of discrete transitions), while (3) can be used for describing the continuous behavior. Without loss of generality, let us suppose that the first columns of matrix $C$ are related to the discrete transitions, while the last columns correspond to fluid ones. Then the MHPN can be represented as:

$M(k + 1) \simeq M(k) + \left[ C^d \ C^c \right] \cdot \left[ \Delta \sigma(\Delta F(k)) \right] \Lambda^c \Pi(M(k)) M(k) \Delta \tau$

where $M(k)$ represents the whole marking and $C^d$ ($C^c$) represents the restriction of $C$ to the discrete (continuous) transitions (i.e., $C = [C^d \ C^c]$). In the same way, the firing rate matrix $\Lambda$ is divided into a matrix for the discrete transitions ($\Lambda^d$) and other one for the continuous transitions ($\Lambda^c$). The firing count vector $\sigma(\Delta F(k))$, having as elements random variables with Poisson p.d.f. with parameters $\Delta F(k) = A^d \cdot \text{Enab}(M(k)) \cdot \Delta \tau$, is defined just for the discrete transitions, while the configuration matrix $\Pi(\cdot)$ is defined just for fluid transitions.

Now, let us suppose that the first rows of the incidence matrix corresponds to the discrete places, while the last rows to fluid ones. Then, the marking can be represented as $M(k) = [\mu_k^d \ \mu_k^c]^T$, where $\mu_k^d$ ($\mu_k^c$) corresponds to the marking of the discrete (fluid) places. In the same way, the incidence matrices can be written as $C^d = [\dot{C}^d \ C^c]^T$ and $C^c = [\dot{C}^c \ C^c]^T$, where $\dot{C}^d$ ($\dot{C}^c$) represents the restriction of $C^d$ to the discrete (continuous) places, and both $\dot{C}^d$ and $\dot{C}^c$ are defined in a similar way. However, since $T^d \cap (T^c \cup T^* \cup T^c) = \emptyset$ then $\dot{C}^c = 0$. Therefore, the MHPN can be rewritten as two different systems but connected:

$\mu_{k+1} \simeq \mu_k + C^d \cdot \sigma(\Delta F(k))$

$m_{k+1} \simeq m_k + C^c \cdot \Lambda \Pi(m_k) m_k \Delta \tau + C^c \cdot \nu_k$

(6)

Notice that the flow of the fluid transitions only depends on the marking at the fluid places. On the contrary, the firing of discrete transitions depends on the marking of both discrete and fluid places, because the parameters of the Poisson random variables are $\Delta F(k) = A^d \cdot \text{Enab}(M(k)) \cdot \Delta \tau$ (i.e., is a function of the whole marking $M(k)$).

In the system given by (6) discrete transitions fire with random delays, while the continuous ones are deterministic w.r.t. the fluid marking. However, it is possible to add uncorrelated gaussian noise to the continuous transitions in order to improve the approximation of the flow at these (as done in TCPN model), obtaining the following system:

$\mu_{k+1} \simeq \mu_k + C^d \cdot \sigma(\Delta F(k))$

$m_{k+1} \simeq m_k + C^c \cdot \Lambda \Pi(m_k) m_k \Delta \tau + C^c \cdot \nu_k$

(7)

where $\nu_k$ is defined for the fluid transitions as in (4). In the sequel, model (7) will be denoted as $MHPN + \nu_k$. 
5. APPROXIMATION OF THE MPN MODEL BY THE CORRESPONDING MHPN

The MHPN model is defined as a partial relaxation of the MPN, so, one could think that the approximation provided by the hybrid system to the original discrete one should be better than that provided by the totally relaxed continuous model. However, that is not always the case.

Consider for instance the PN system of fig. 1(a) with rates \( \lambda = [1 \ 3 \ 1 \ 2]^T \). This PN was simulated 1000 times as a discrete, fluid and hybrid system, in order to obtain mean trajectories of the marking at \( p_1 \). As a hybrid model, nodes \( t_1, t_2, p_4, p_5 \) are discrete, while others are continuous. Fig. 1(b) shows the resulting mean trajectories. It can be seen that fluid models TCPN (3) and MCPN (5) provide a better approximation to the Markovian PN (denoted as \( E\{M\} \)) than hybrid models MHPN (6) and MHPN + \( v_k \) (7), i.e., a partial relaxation can be worse than a full relaxation! Let us analyze this in the following subsection.

5.1 Approximation analysis

The dynamical behavior of the MPN is achieved by the firing of its transitions, which is characterized by the firing count vector \( \Delta \sigma (\Delta F(k)) \) in (1). In this way, if at some time step \( k \), the average marking of the MPN is well approximated by the average marking of a given relaxed model (either fluid or hybrid) and their transitions fire in the same amount in both (the MPN and the relaxed model), then the marking approximation will hold for the next time step \( k + 1 \). Therefore, following an inductive reasoning, if the initial condition of both systems coincide and the firing count vector of the relaxed model approximates that of the MPN system through the time, then the marking approximation is achieved (errors are not accumulated because, roughly speaking, the ergodicity of the MPN implies asymptotic stability in the relaxed model, i.e., early errors will not affect the long term behavior). However, it is important to remember that the firing count vector of the MPN is a random variable, then the corresponding firing count vector of the relaxed model should approximate the moments of the original one, i.e., mean value and covariance. Let us focus first in the mean value approximation though this subsection.

The mean value of the firing count vector of the MPN is approximated by the flow (but multiplied by \( \Delta \tau \)) at the continuous transitions in the relaxed fluid model. In (Vázquez, Recalde & Silva (2008)) it was found that such approximation is effective if Conditions 1 and 2 hold. A similar reasoning holds for the fluid transitions in the hybrid models, so no more analysis in these is required.

On the other hand, in the hybrid models, discrete transitions can have as input places either discrete or continuous ones. If discrete transitions have only discrete input places no problem occurs (there is no relaxation, then the approximation is perfect at these transitions). However, if a discrete transition has input continuous places then it can lead to a bad approximation, as in the case of the system of fig. 1(a). Now, consider the synchronization of fig. 2(a).

Transition \( t_1 \) is a discrete transition having as input places \( p_1 \in P^c \) and \( p_2 \in P^f \). The expected number of firings of \( t_1 \) during a time interval \( \Delta \tau \) is proportional to the expected value of its enabling degree, i.e., \( E[\Delta \sigma (\Delta F_1)] \propto \lambda_1 \cdot \Delta \tau \cdot E\{Enab(t_1)\} \), which can be computed by using the total probability theorem for the MPN as:

\[
E\{Enab(t_1)\} = E\{min(|M(p_1)|/Pre(p_1, t_1), |M(p_2)|/Pre(p_2, t_1))\}
= \sum \sum \min(M_1,M_2) \cdot \text{Prob}(M_1|M_2)\text{Prob}(M_2)
\]

and for the hybrid Petri net as:

\[
E\{Enab(t_1)\} = \sum \int \min(|x|,M_2)f_{j_1}(x)dx \cdot \text{Prob}(M_2)
\]

where \( S_{M_1}, (S_{M_2}) \) denotes all the possible values for the marking at \( p_1 \) (\( p_2 \)), and \( f_{j_1}(\cdot) \) is the probability density function of the marking at fluid place \( p_1 \) given \( M(p_2) = M_2 \) (\( M_1, M_2 \) denote fixed values for \( M(p_1), M(p_2) \)). Then, the approximation of the firing count vector is achieved if the fluid marking at \( p_1 \) is representative of the marking (the value of the marking, not the mean value of this) of the original MPN w.r.t. the enabling degree function, i.e., if the value of \( \sum S_{M_1} \min(M_1,M_2)\text{Prob}(M_1|M_2) \) in (8) is close to the value of \( \int \min(|x|,M_2)f_{j_1}(x)dx \) in (9). For instance, in the synchronization of fig. 2(a), markings at places \( p_1 \) and \( p_2 \) are random variables, but given the current marking, for the most probable values of \( p_1 \) and \( p_2 \), \( p_2 \) will constraint \( t_1 \), i.e., in (9) \( \int \min(|x|,M_2)f_{j_1}(x)dx \simeq M_2 \). Following a similar reasoning, in the original MPN the probability that \( p_1 \) constraints \( t_1 \) is negligible, i.e., in (8) \( \sum S_{M_1} \min(M_1,M_2) \cdot \text{Prob}(M_1|M_2) \simeq M_2 \), so the enabling degree of \( t_1 \), and thus the firing count, is well approximated in the relaxed model. This case can be generalized as a sufficient condition for obtaining a good approximation, during a time interval \( (\tau_0, \tau_0 + n\Delta \tau) \):

Condition 3) The probability that the discrete transitions are constrained by discrete places at \( M(\tau_0 + k\Delta \tau) \), for any time step \( k \) in the interval \( (\tau_0, \tau_0 + n\Delta \tau) \), is near to 1.

Condition 3 implies that the arcs between fluid places and discrete transitions are temporarily implicit (i.e., the discrete subnet evolves independently of the marking at fluid places). Furthermore, this condition can be generalized. Consider again the discrete transition of fig. 2(a). If at the current time step \( k \) the distribution of the marking at \( p_1 \) in the MPN were well approximated by the distribution of the fluid marking for the same place but in the MHPN,
then the expected enabling degree of $t_1$ in both models would be similar. Such condition can be stated as:

$$Prob(M_t) \simeq \int_{M_1}^{M_1+1} f_1(x)dx \quad \forall M_1 \in S_{M_1}$$

(10)

In order to obtain such approximation, it is required that the marking at $p_1$ be large enough, i.e., that $S_{M_1}$ consists in several probable values for the marking at $p_1$, so the marking approximation errors will be small w.r.t. their mean values. Furthermore, it is very important that $p_1$ at least enables $t_1$ with probability near to 1, otherwise a minimum error in the marking can lead to a big one in the firing count. For instance, consider the PN of fig. 2(b). Notice that as a hybrid model $t_1$ is enabled only at infinite time (only in infinite time the marking at $p_1$ is 1), but in the discrete $MPN$ $t_1$ is enabled with a significant probability (since the mean value of the marking in the $MPN$ is close to 0,9, the probability that $M(p_1) = 1$ is significant). Then, Condition 3 is generalized as:

**Condition 4** Discrete transitions can be constrained by either discrete or fluid places, but fluid places constraining discrete transitions enable them with probability near to 1 at $M(\tau + k\Delta \tau)$, for any time step $k$ in the interval $(\tau_0, \tau_0 + n\Delta \tau)$, i.e., such output discrete transitions are always enabled. The larger the marking at such fluid places, the better the approximation.

For instance, consider again the system of figure 1(a) with the same firing rates. The $MPN$ and $MHPN$ systems have been simulated 1000 times for different initial markings at $p_1$, while the initial markings for the other places remain as in fig. 1(a). Table 1 resumes the results thus obtained. The first column represents the initial marking. Columns 2 and 3 are the expected values at the steady state of $p_1$ for the $MPN$ and $MHPN$, respectively. Next column deals with to the approximation error, while the last column is the probability that $p_5$ constraints $t_1$ (i.e., Condition 3 for $t_1$). As expected, the error is lower when $p_1$ does not constraint $t_1$ (i.e., when $P.C3 \to 1$). On the other hand, in the last experiment an initial marking of $M_0 = [10,0,0,10,0]^T$ was used. It can be seen that the approximation by the hybrid model is good, even if the probability that $p_5$, the discrete place, constraints $t_1$ is almost 0, i.e., $p_1$ constraints $t_1$ so Condition 3 is not fulfilled. However, in this last case the value of $p_1$ is large enough, which means that Condition 4 holds.

### 5.2 Improvement of MHPN by adding noise

As it was pointed out in subsection 3.3, the addition of uncorrelated gaussian noise to the $TCPN$ model improves the approximation to the $MPN$. Such improvement becomes more significative when the marking is around to the border of two or more regions. Let us detail this.

Consider a synchronization as in fig. 2(a). Suppose that $t_1$, $p_1$ and $p_2$ are continuous, and that at some time instant $\tau$, the marking is 1 at $p_1$ and $p_2$. In such case, the enabling degree of $t_1$ would be $\min(m(p_1), m(p_2)) = 1$, so $t_1$ would fire with a speed of $\lambda_1 = 1$. Now, let us analyze the behavior of $t_1$ in the $MPN$. Suppose that at the same time instant $\tau$ the relaxed model approximates the $MPN$ one, then a value of 1 at $p_1$ and $p_2$ in the relaxed model means that the average marking of the $MPN$ is $[E(M(p_1)), E(M(p_2))]. \text{Now, the expected enabling degree is } E(\text{Enab}(t_1)) = E(\min(m(p_1), m(p_2)))$, but notice that $\min(\cdot, \cdot)$ is not a linear operator, so, in general $E(\text{Enab}(t_1)) \neq \min(E(\{M(p_1), E(\{M(p_2)\})) = 1$. In the $MPN$, the expected value of the enabling degree depends on the particular distribution of the marking, which is in general unknown. Then, even if the relaxed model approximates well the $MPN$ at $\tau$, the approximation of the enabling degree, and thus the firing count, can be bad.

As recalled in subsection 3.3, the addition of noise $v_k$ (4) to the fluid transitions in the $TCPN$ model improves the approximation of the firing count of the discrete transitions in the $MPN$. Moreover, the approximation is achieved not only at the mean value but also at the covariance. Following a similar reasoning, the addition of noise to the continuous transitions in the $MHPN$ model may improve the approximation to the original $MPN$, obtaining thus the $MHPN + v_k$ model (7). Nevertheless, the difference between the approximation provided by the $MHPN$ model (6) and the hybrid model with noise (7) is not so important as in the case of fully continuous systems.

For instance, consider the system of fig. 3. As hybrid, transitions $t_6$, $t_7$ and places $p_4$, $p_5$, $p_6$, $p_7$ and $p_8$ are continuous, while other transitions and places are discrete. Consider rates as $\lambda = [30,30,30,30,30,30,30,30,30]^T$. The system was simulated 400 times as discrete ($MPN$), continuous ($TCPN$ and $MCPN$) and hybrid ($MHPN$ and $MHPN + v_k$). The mean trajectories for the marking at place $p_6$ are shown in fig. 4(a). As it can be seen, the addition of the noise in the $MCPN$ model represents an important improvement to the approximation of the $MPN$ (denoted as $E(M)$) with respect to the $TCPN$ system. However, the improvement in the hybrid models (compare $MHPN$ against $MHPN + v_k$) is not so important. The reason for that is the stochastic behavior of the discrete transitions in the hybrid model, i.e., the stochastic behavior of the firing of discrete transitions in the $MHPN$ makes the marking at continuous places be also stochastic, so it approximates not only the mean value but also the covariance (in certain degree) of the marking in the $MPN$.

### Table 1. Initial and steady state markings at $p_1$ for the $MPN$ and $MHPN$ of fig. 1(a)

<table>
<thead>
<tr>
<th>$M_0(p_1)$</th>
<th>$MPN$</th>
<th>$MHPN$</th>
<th>error</th>
<th>P.C3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.95</td>
<td>2.40</td>
<td>23.1%</td>
<td>0.42</td>
</tr>
<tr>
<td>3</td>
<td>2.54</td>
<td>2.74</td>
<td>7.87%</td>
<td>0.52</td>
</tr>
<tr>
<td>4</td>
<td>3.27</td>
<td>3.36</td>
<td>2.8%</td>
<td>0.65</td>
</tr>
<tr>
<td>5</td>
<td>4.04</td>
<td>4.02</td>
<td>0.5%</td>
<td>0.78</td>
</tr>
<tr>
<td>10</td>
<td>8.52</td>
<td>8.58</td>
<td>0.7%</td>
<td>0.98</td>
</tr>
</tbody>
</table>

| $M_0(p_1) = M_0(p_4) = 10$ | 9.50 | 9.8 | 3.2% | 0.01 |
In this paper, a hybrid Petri net model $MHPN$ is introduced as a partial relaxation of a $MPN$. Such hybrid model is enriched by adding gaussian noise to the continuous transitions, in order to improve the approximation, obtaining thus another hybrid system $MHPN + v_k$. It was found that in order to approximate a $MPN$ by a hybrid relaxation, next conditions should be taken into account:

1. All the fluid transitions should be enabled with probability near to 1 (Condition 1).
2. Discrete transitions should be constrained by discrete places with probability near to 1, i.e., Condition 3.
3. Otherwise, continuous places that constraint discrete transitions should enable such transitions with probability near to 1; the larger the marking at those fluid places, the better the approximation (Condition 4).
4. If Condition 3 does not hold for some discrete transitions, and Condition 4 is barely fulfilled, then the addition of the gaussian noise (4) becomes important for the approximation, i.e., the $MHPN + v_k$ model should be considered.

6. CONCLUSIONS

In this paper, a hybrid Petri net model $MHPN$ is introduced as a partial relaxation of a $MPN$. Such hybrid

REFERENCES