Steady state control, zero valued poles and token conservation laws in continuous net systems *

Cristian Mahulea[1], Antonio Ramírez Treviño[2], Laura Recalde[1], and Manuel Silva[1]

[1] Dep. de Informática e Ingeniería de Sistemas, Centro Politécnico Superior, Universidad de Zaragoza, María de Luna 1, E-50018 Zaragoza, Spain
{cmahulea,lrecalde,silva}@unizar.es
[2] CINVESTAV-IPN Unidad Guadalajara, 45090 Guadalajara, Mexico
art@gdl.cinvestav.mx

Abstract. The control problem for continuous Petri nets under infinite server semantics is approached in this paper. First, some characterizations of equilibrium points in steady-state are given. Being piecewise linear systems, for every linear system we present results that will characterize the possible steady-states. Second, optimal control for steady-state is studied, a problem that surprisingly can be computed in polynomial time, when all transitions are control-feasible. Third, an interpretation of controllability aspects in the framework of linear dynamic system is presented.

1 Introduction

Continuous Petri Nets (*contPNs*) [1] [2] [3] appear as a promising approach to model, analyze and deal with some synthesis problems in a relaxed setting. The goal of the relaxation is to obtain more efficient algorithms, but the price to be paid is the lost of analyzability of certain properties, or the impossibility to proceed according to some synthesis procedures. On the positive side, for example, reachability is characterized under very general conditions in polynomial time for untimed systems [4]. On the negative part, properties like mutual exclusions or home states are *not* analyzable due to the essential fact of the continuization. Moreover it is well known from some years ago that for basic properties like boundedness, the continuous version only provides sufficient conditions for the original discrete case, while liveness of the continuous case is neither necessary nor sufficient for that of the underlying discrete model [4]. Of course, the above situation can be easily understood if we think on a different area: in continuous models described by ordinary differential equations not all non-linear systems allow a "reasonable" approximation by its linear relaxation. Nevertheless, as it is also well known in the differential equations setting, certain

* This work was partially supported by project CICYT and FEDER DPI2003-06376. Antonio Ramírez Treviño was financed by the Diputación General de Aragón for a stay in the Group of Zaragoza, period in which this work was done.

subclasses of non-linearities accept reasonable linear approximations. Therefore, some net subclasses, like equal conflicts [5], enjoy quite reasonable fluidifications.

This paper is essentially centered on timed continuous models. Assuming that certain systems usually work close to congestion, and having evidence of the gains that in certain cases are obtained by fluidification of discrete models, this paper deals with some control problems under so called *infinite servers semantics* (somehow the continuous and deterministic counter part of markovian flows in discrete Petri Nets(PNs)). Under this firing semantics, due to the *min* operators appearing in the synchronizations, the continuous model is a multilinear switched dynamic system. Starting with the crucial question of how to control?, an approach based in the idea of *slowing down* the firing flow of transitions is considered. One question that immediately appears is: given an initial marking, $\mathbf{m_0}$, and a constant control action, $\mathbf{u_d}$, which steady state is reached? For linear systems the question has a classical and simple answer, but for the class of systems we get many solutions (even infinite) may appear for basic algebraic formulations. This paper explores first the situation for steady state (Section 3). As usual, for particular classes of net systems, unique solutions are obtained. This means that the algebraic characterization is complete. In many cases, even if several steady state markings appear, a single flow is reached. In other words, computationally it may be easy (polynomial time) to deal with the (maximal) flow vector, while several (even infinite) marking conditions may appear. The computation of the steady state of a continuous net system under infinite servers semantics asks for some search procedures like Branch & Bound [6], so even if a relatively efficient algorithm is available, it is not polynomial time. Surprisingly, the computation of an optimal steady state control (maximizing a linear profit function taking into account the throughput, the initial marking and the steady state marking) is polynomial (see Section 4).

In section 5, a bridge between *classical linear theory* based approach to study controllability and some Petri nets concepts is undertaken. The simplifying idea is to keep the fact that the dynamic model is multilinear, but not considering the constraints that must be respected by the action: non-negativity and upper bounded by the enabling degree of the transitions. It is evidenced that net systems generate different token conservation laws some of those leading to uncontrollabilities. Some of those invariances are generated by the P-flows, leading to zero valued *uncontrollable* poles. Others are zero valued *controllable* poles that depend on the net structure, the firing rates and the token load of P-(semi)flows. Finally, some *controllable* non zero poles may generate token conservation laws for particular values of $\mathbf{m_0}$ in $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$.

2 Continuous Petri nets basics

2.1 Untimed continuous Petri nets

We assume that the reader is familiar with discrete PNs. The PNs that will be considered in this paper are *continuous*, relaxation of *discrete* ones. Unlike discrete PN, the amount in which a transition can be fired in *contPN* is not restricted to a natural number. A PN system is a pair $\langle \mathcal{N}, \mathbf{m_0} \rangle$, where $\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post} \rangle$ is a P/T net (P and T are disjoint (finite) sets of places and transitions, and **Pre** and **Post** are $|P| \times |T|$ sized *incidence matrices*) and $\mathbf{m_0}$ is the *initial marking*. In $contPNs$, $\mathbf{m_0}$ is a vector of positive *real* numbers. For every node $v \in P \cup T$, the sets of its input and output nodes are denoted as ${}^\bullet v$ and $v^\bullet$, respectively.

A transition t is *enabled* at $\mathbf{m}$ iff $\forall p \in {}^\bullet t, \mathbf{m}[p] > 0$. The *enabling degree* of t is enab$(t, \mathbf{m})= \min_{p \in {}^\bullet t} \mathbf{m}[p]/\mathbf{Pre}[p, t]$, and t can fire in a certain amount $\alpha \in \mathbb{R}$, $0 \leq \alpha \leq$ enab$(t, \mathbf{m})$ leading to a new marking $\mathbf{m\prime} = \mathbf{m} + \alpha \cdot \mathbf{C}[P, t]$, where $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$ is the *incidence matrix*. If $\mathbf{m}$ is reachable from $\mathbf{m_0}$ through a sequence σ, a fundamental equation can be written: $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma} \in (\mathbb{R}^+ \cup \{0\})^{|T|}$ is the firing count vector.

A $contPN$ is *bounded* when every place is bounded ($\forall p \in P, \exists b_p \in \mathbb{R}$ with $\mathbf{m}[p] \leq b_p$ at every reachable marking $\mathbf{m}$). It is *live* when every transition is *live* (it can ultimately occur from every reachable marking). Liveness may be extended to *lim-live* assuming that infinitely long sequence can be fired. A transition t is *non lim-live* iff a sequence of successively reachable markings exists which converge to a marking such that none of its successors enables a transition t [4].

A net $\mathcal{N}$ is *structurally bounded* when $\langle \mathcal{N}, \mathbf{m_0} \rangle$ is bounded for every initial marking $\mathbf{m_0}$ and is *structurally live* when a $\mathbf{m_0}$ exists such that $\langle \mathcal{N}, \mathbf{m_0} \rangle$ is live. Left and right natural annulers of the incidence matrix $\mathbf{C}$ are called T- and P-semiflows, respectively. The net $\mathcal{N}$ is *conservative* iff $\exists \mathbf{y} > 0, \mathbf{y} \cdot \mathbf{C} = 0$ and it is *consistent* iff $\exists \mathbf{x} > 0, \mathbf{C} \cdot \mathbf{x} = 0$. Left and right real annulers of matrix $\mathbf{C}$ are T- and P-flows, respectively.

If a $contPN$ is consistent and all transitions are fireable, then the (lim)reachable markings are the solutions of the fundamental equation ($\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}$, $\mathbf{m} \geq 0$, $\boldsymbol{\sigma} \geq 0$) [4] [7]. Because of consistency, $\boldsymbol{\sigma} \geq 0$ can be relaxed to $\boldsymbol{\sigma} \in \mathbb{R}^{|T|}$, that is equivalent to $\mathbf{B}^T \cdot \mathbf{m} = \mathbf{B}^T \cdot \mathbf{m_0}$, $\mathbf{m} \geq 0$ with $\mathbf{B}^T$ a basis of P-flows. The set of all reachable markings at the limit is denoted by $lim - RS$. Like in discrete case, nets can be classified according to their structure.

A place p is Choice-Free (CF) iff $|p^\bullet| \leq 1$ (i.e. there is no routing, choice is structurally defined). A transition t is Join-Free (JF) iff $|{}^\bullet t| \leq 1$ (i.e. there is no synchronization on it). Two transitions, t and t', are said to be in Equal Conflict (EQ) relation when $\mathbf{Pre}[P, t] = \mathbf{Pre}[P, t'] \neq 0$. This is an equivalence relation and the set of all the equal conflict sets is denoted by SEQS. In Equal Conflict nets, all outcomes of conflicts have identical precondition, thus choice is local.

Definition 1. *Let $\mathcal{N}$ be a PN.*

1. *$\mathcal{N}$ is a* weighted T-graph *iff* $\forall p \in P : |p^\bullet| = |{}^\bullet p| = 1$.
2. *$\mathcal{N}$ is* Choice-Free *iff* $\forall p \in P : |p^\bullet| \leq 1$.
3. *$\mathcal{N}$ is* Join-Free *iff* $\forall t \in T : |{}^\bullet t| \leq 1$.
4. *$\mathcal{N}$ is* Equal Conflict *iff* ${}^\bullet t \cap {}^\bullet t' \neq 0 \Rightarrow \mathbf{Pre}[P, t] = \mathbf{Pre}[P, t']$.

In this paper we will mainly focus on bounded and lim-live net systems. A bounded and lim-live $contPN$ is consistent, conservative and $rank(\mathbf{C}) \leq$

$|SEQS|-1$ (this is the so called *rank theorem* [8]). For Equal Conflict *contPN*, boundedness and lim-liveness is equivalent to consistency, conservativeness, $rank(\mathbf{C}) = |SEQS|-1$ and the support of every P-semiflow is marked, i.e. $\nexists \mathbf{y} \geq 0$ such that $\mathbf{y} \cdot \mathbf{C} = 0, \mathbf{y} \cdot \mathbf{m_0} = 0$ [4].

2.2 Timed continuous Petri Nets and infinite server semantics

In this section, timing constraints are added to *contPN*. Like in the discrete case, time can be associated to places, to transitions or to arcs. This paper assumes time associated with the transitions. Here we consider first-order approximation (only using the *average* value; i.e. noisy-free) of the fluidified models [9].

Definition 2. *A timed contPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is the untimed contPN $\mathcal{N}$ together with a vector $\boldsymbol{\lambda} \in (\mathbb{R}^+)^{|T|}$, where $\boldsymbol{\lambda}[t_i] = \lambda_i$ is the firing rate of transition t_i.*

Definition 3. *A timed contPN system is a tuple $\Sigma = \langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$, where $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is a timed contPN and $\mathbf{m_0}$ is the initial marking of the net.*

Now, the fundamental equation depends on time τ: $\mathbf{m}(\tau) = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}(\tau)$. Deriving this equation with respect to time we obtain: $\dot{\mathbf{m}}(\tau) = \mathbf{C} \cdot \dot{\boldsymbol{\sigma}}(\tau)$. Using the notation $\mathbf{f}(\tau) = \dot{\boldsymbol{\sigma}}(\tau)$ to represent the flow of the transitions with respect of time, the fundamental equation becomes: $\dot{\mathbf{m}}(\tau) = \mathbf{C} \cdot \mathbf{f}(\tau)$. In this paper we will use the short form: $\dot{\mathbf{m}} = \mathbf{C} \cdot \mathbf{f}$ but the dependence on time is considered.

Depending on the flow definition, there are many firing semantics. *Finite server* (or constant speed) and *infinite server* (or variable speed) [10] [1] are the more frequently used. This paper is focused on *infinite server semantics*, with the flow of each transition defined by:

$$f_i = \mathbf{f}[t_i] = \boldsymbol{\lambda}[t_i] \min_{p_j \in {}^\bullet t_i} \left\{ \frac{\mathbf{m}[p_j]}{\mathbf{Pre}[p_j, t_i]} \right\} \tag{1}$$

Observe that the flow of transition t is proportional to its enabling degree by means of the firing rate $\boldsymbol{\lambda}[t_i] = \lambda_i$.

Remark 1. A timed *contPN* under infinite server semantics is a piecewise linear system due to the minimum operator that appears in the flow definition.

Example 1. Let us consider the net in Figure 1. The flows of the transitions are given by:

$$\begin{cases} f_1 = \boldsymbol{\lambda}[t_1] \cdot \mathbf{m}[p_1] \\ f_2 = \boldsymbol{\lambda}[t_2] \cdot \min(\mathbf{m}[p_2], \mathbf{m}[p_3]) \\ f_3 = \boldsymbol{\lambda}[t_3] \cdot \min(\mathbf{m}[p_4], \mathbf{m}[p_5]) \\ f_4 = \boldsymbol{\lambda}[t_4] \cdot \mathbf{m}[p_6] \end{cases}$$

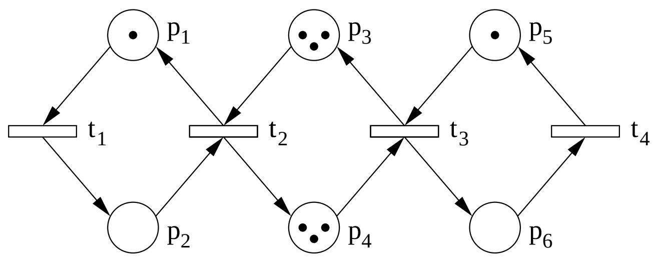

Fig. 1. Timed *contPN* with several equilibria points.

If $\boldsymbol{\lambda} = [1, 1, 1, 1]^T$, for example, we can write:

$$\begin{cases} \dot{\mathbf{m}}[p_1] = f_2 - f_1 = \mathbf{m}[p_2] - \mathbf{m}[p_1] \\ \dot{\mathbf{m}}[p_2] = f_1 - f_2 = \mathbf{m}[p_1] - \min(\mathbf{m}[p_2], \mathbf{m}[p_3]) \\ \dot{\mathbf{m}}[p_3] = f_3 - f_2 = \min(\mathbf{m}[p_4], \mathbf{m}[p_5]) - \min(\mathbf{m}[p_2], \mathbf{m}[p_3]) \\ \dot{\mathbf{m}}[p_4] = f_2 - f_3 = \min(\mathbf{m}[p_2], \mathbf{m}[p_3]) - \min(\mathbf{m}[p_4], \mathbf{m}[p_5]) \\ \dot{\mathbf{m}}[p_5] = f_4 - f_3 = \mathbf{m}[p_6] - \min(\mathbf{m}[p_4], \mathbf{m}[p_5]) \\ \dot{\mathbf{m}}[p_6] = f_3 - f_4 = \min(\mathbf{m}[p_4], \mathbf{m}[p_5]) - \mathbf{m}[p_6] \end{cases} \tag{2}$$

Thus, *nonlinearity* appears due to synchronizations ($|^\bullet t| > 1$). One linear system is defined by the set of arcs in **Pre** limiting the firing of the transitions.

Definition 4. *Let $\Sigma = \langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a timed contPN and $\mathbf{m}$ a reachable marking. It will be said that the arc (p, t) constraints the dynamic of t at $\mathbf{m}$ iff: $\mathbf{f}[t] = \boldsymbol{\lambda}[t] \cdot \frac{\mathbf{m}[p]}{\mathbf{Pre}[p,t]}$.*

Definition 5. *A configuration of Σ at $\mathbf{m}$ is a set of (p, t) arcs describing the effective flow of all the transitions.*

So, a configuration is a *cover* of T by its inputs arcs. One possible representation of a given configuration is a matrix form, $\mathbf{G} \in \{0, 1\}^{|P| \times |T|}$:

$$\mathbf{G}[p_i, t_j] = \begin{cases} 1 \text{ if } p_j \text{ is limiting the flow of } t_i \\ 0 \text{ otherwise} \end{cases} \tag{3}$$

Obviously, $\mathbf{G} \leq \mathbf{Pre}$, even if the net is ordinary (i.e. all arcs have weight one). Each configuration defines an associated linear system.

Example 2. Let us consider the net in Figure 1 with $\boldsymbol{\lambda} = [1, 1, 1, 1]^T$. As we saw in the Example 1, this is a piecewise linear system. For the configuration $\{(p_1, t_1), (p_2, t_2), (p_5, t_3), (p_6, t_6)\}$, $\mathbf{m}[p_2] \leq \mathbf{m}[p_3]$ and $\mathbf{m}[p_5] \leq \mathbf{m}[p_4]$. Then the active linear system is:

$$\begin{cases} \dot{\mathbf{m}}[p_1] = \mathbf{m}[p_2] - \mathbf{m}[p_1] \\ \dot{\mathbf{m}}[p_2] = \mathbf{m}[p_1] - \mathbf{m}[p_2] \\ \dot{\mathbf{m}}[p_3] = \mathbf{m}[p_5] - \mathbf{m}[p_2] \\ \dot{\mathbf{m}}[p_4] = \mathbf{m}[p_2] - \mathbf{m}[p_5] \\ \dot{\mathbf{m}}[p_5] = \mathbf{m}[p_6] - \mathbf{m}[p_5] \\ \dot{\mathbf{m}}[p_6] = \mathbf{m}[p_5] - \mathbf{m}[p_6] \end{cases}$$

or in matrix form:

$$\dot{\mathbf{m}} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \cdot \mathbf{m} \tag{4}$$

Let us now consider the configuration $\{(p_1, t_1), (p_2, t_2), (p_4, t_3), (p_6, t_6)\}$. Then $\mathbf{m}[p_5] \geq \mathbf{m}[p_4]$ and $\mathbf{m}[p_2] \leq \mathbf{m}[p_3]$ and the linear system associated is:

$$\begin{cases} \dot{\mathbf{m}}[p_1] = \mathbf{m}[p_2] - \mathbf{m}[p_1] \\ \dot{\mathbf{m}}[p_2] = \mathbf{m}[p_1] - \mathbf{m}[p_2] \\ \dot{\mathbf{m}}[p_3] = \mathbf{m}[p_4] - \mathbf{m}[p_2] \\ \dot{\mathbf{m}}[p_4] = \mathbf{m}[p_2] - \mathbf{m}[p_4] \\ \dot{\mathbf{m}}[p_5] = \mathbf{m}[p_6] - \mathbf{m}[p_4] \\ \dot{\mathbf{m}}[p_6] = \mathbf{m}[p_4] - \mathbf{m}[p_6] \end{cases}$$

or in matrix form:

$$\dot{\mathbf{m}} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \cdot \mathbf{m} \tag{5}$$

Observe that, depending on the marking of the places, the evolution of the system will be given by one or other linear system. Equations 4 and 5 describe two of these different linear systems.

Any (reachable) marking defines a configuration. When the marking of several places are limiting the firing of the same transition, any of the associated linear systems can be used.

Remark 2. If $\mathcal{N}$ is JF then all arcs in **Pre** are constraining the dynamic of the full system (i.e. all those arcs are essential covers).

Let the set of all configurations be denoted as $\mathcal{G}$, were $\mathcal{G}(\mathbf{m})$ is the matrix representing the configuration associated to $\mathbf{m}$. The number of *minimal* configurations (i.e. only one constraining arc per transition is taken) is bounded by the net structure (i.e. does not depend on the marking) and is equal to $\prod_{t_i \in T} |^\bullet t_i|$.

Timed *contPN* systems evolve from $\mathbf{m_0}$ and may reach a steady state. For unforced *contPN* the computation of bounds for the steady was studied in [6] and it is based on a branch and bound technique, each node corresponding to a linear programming problem (LPP). Usually, the system evolves to a steady state, but for certain cases, oscillations can be maintained forever. For example the evolution of the net system presented in Figure 2 is sketched in Figure 3 and characterizes an oscillatory system.

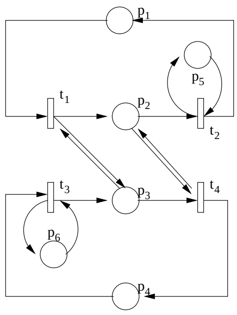

Fig. 2. Timed *contPN* system that has an oscillatory behavior with $\mathbf{m_0} = [100, 0, 100, 0, 1, 1]^T$ and $\boldsymbol{\lambda} = [1, 12, 10, 1]$.

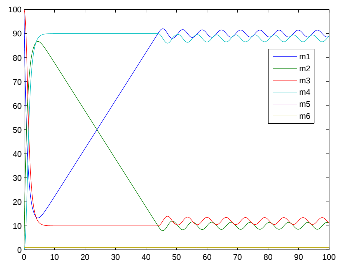

Fig. 3. The evolution of the timed *contPN* system presented in fig. (2)

3 On how to control timed *contPNs*

The parameters $\boldsymbol{\lambda}$ associated with the transitions in timed *contPNs* represent the *maximum firing rate* of each transition. Hence, the only action that can be applied is to *slow down* their firing flow. If a transition can be controlled (its flow reduced or even stopped), we will say that is a *control-feasible* transition. The forced flow of a controlled transition t_i becomes $f_i - u_i$, where f_i is the flow of the *unforced* system (i.e. without control) and u is the control action $0 \le u_i \le f_i$.

In the first part of this section technical notations are presented, while in the second one some considerations regarding the equilibrium points are developed.

3.1 Notation

Definition 6. $\mathbf{H} = [h_{i,j}]$ *is* $|T| \times |P|$ *matrix, where*

$$h_{i,j} = \begin{cases} \frac{1}{\mathbf{Pre}[j,i]}, \; \textit{if}\; \mathbf{Pre}[j,i] > 0 \\ 0, \; \textit{otherwise} \end{cases}$$

Observe that matrix $\mathbf{H}$ is just the transpose of the matrix $\mathbf{Pre}$ where the non null elements are not $\mathbf{Pre}[p,t]$ but their inverses.

Definition 7. *Let R, W and E be three matrices with identical dimensions. The matrix operator $\odot$ is defined as: $R = W \odot E$, where $r_{ij} = w_{ij} \times e_{ij}$*

Definition 8. *The configuration operator is the function $\boldsymbol{\Pi} : RS(\mathcal{N}, \mathbf{m_0}) \longrightarrow \mathbb{R}^{|T|\times|P|}$ such that:*

$$\boldsymbol{\Pi}(\mathbf{m}) = \mathcal{G}(\mathbf{m}) \odot \mathbf{H} \tag{6}$$

where $\mathcal{G}(\mathbf{m})$ is the matrix representing the configuration associated to $\mathbf{m}$.

The configuration operator associates to every marking $\mathbf{m}$ a matrix $|T| \times |P|$, such that each row $i = 1..|T|$ has only one non null element in the position j that corresponds to the place p_j that restricts the flow of transition t_i. The product $\boldsymbol{\Pi}(\mathbf{m}) \cdot \mathbf{m}(\tau)$ is the *enabling* degree of each transition at time τ, $\mathbf{e}(\tau)$.

Through this paper the notations $\boldsymbol{\Pi}(\mathbf{m_d})$ and $\boldsymbol{\Pi}_\mathbf{d}$ will be used indistinctly. The *maximum firing rate matrix* is denoted by: $\boldsymbol{\Lambda} = diag(\lambda_1, .., \lambda_m)$.

3.2 The state equation of controlled timed *contPN*

According to the above notation, the controlled flow vector is $\mathbf{f} = \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\mathbf{m}) \cdot \mathbf{m} - \mathbf{u} \geq 0$, with $u_i = 0$ if t_i is not a control-feasible transition. Thus the state equation of controlled timed *contPNs* (i.e. net systems in which all the transitions are control-feasible: $\forall t \in T$, $\mathbf{u}[t] > 0$ is possible at certain instant) becomes:

$$\begin{cases} \dot{\mathbf{m}} = \mathbf{C} \cdot (\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\mathbf{m}) \cdot \mathbf{m} - \mathbf{u}) \\ 0 \leq \mathbf{u} \leq \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\mathbf{m}) \cdot \mathbf{m} \end{cases} \tag{7}$$

Unless otherwise stated, *in the following we will assume that all transitions are control-feasible*. Controlling the set of transitions, almost all reachable markings of an untimed system can be reached in the timed system. The only problem is at the borders when the marking of one place is zero. In this case, the marking is reached at the limit (this is like the discharging of a capacitor in an electrical RC-circuit: theoretical total discharging takes an infinite amount of time). For example, in the net system in Figure 4 the marking $[0, 1, 1]^T$ is reached in the untimed model. Considering now the timed model, stoping transitions t_2 and t_3 ($\mathbf{u}[t_2] = \mathbf{f}[t_2]$ and $\mathbf{u}[t_3] = \mathbf{f}[t_3]$) and setting $\mathbf{u}[t_1] = 0$, the marking $[0, 1, 1]^T$ is reached at the limit because $\dot{\mathbf{m}}[t_1](\tau) = -\boldsymbol{\lambda}[t_1] \cdot \mathbf{m}[p_1](\tau) \rightarrow \mathbf{m}[t_1](\tau) = e^{-\boldsymbol{\lambda}[t_1] \cdot \tau}$. Note that it takes an infinite amount of time to empty p_1. The following can be stated:

Proposition 1. *If all transitions are control-feasible, then all the reachable markings of the untimed contPN can be reached in the timed model, maybe at the limit. (lim-*RS_{timed} *= lim-*$RS_{untimed}$*).*

The steady-state markings we are interested *to obtain* (reference markings for the control loop) are positive (if the marking of a place is zero then the flows of the output transitions are zero, meaning total inactivity of the machine or processor being controlled). Defining $RS^+(\mathcal{N}, \mathbf{m_0}) = \{\mathbf{m} \in RS | \mathbf{m} > 0\}$ these markings can be reached in finite time in the timed model. If the *contPN* is consistent and all transitions can be fired at least once, then $\mathbf{m} \in RS^+$ iff $\mathbf{B}^T \cdot \mathbf{m} = \mathbf{B}^T \cdot \mathbf{m_0}, \mathbf{m} > 0$ [4].

Definition 9. *Let* $\mathbf{m_d} \in RS^+$ *and* $\mathbf{0} \leq \mathbf{u_d} \leq \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\mathbf{m_d}) \cdot \mathbf{m_d}$*. Then* $\mathbf{m_d}$ *is an equilibrium point for* $\mathbf{u_d}$ *if* $\dot{\mathbf{m}}_\mathbf{d} = \mathbf{0}$.

An equilibrium point represents a state in which the system can be maintained using the defined action. Given an initial marking and $\mathbf{m_d}$ a desired marking, one control problem is to reach $\mathbf{m_d}$ and then keep it. In this section this transient control will not be considered.

Obviously, taking into account (7), $\mathbf{m_d}$ is a equilibrium marking if $\mathbf{C} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\mathbf{m_d}) \cdot \mathbf{m_d} - \mathbf{u_d} = \mathbf{0}$. Therefore, the flow of a controlled timed *contPN* ($\mathbf{f} = \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\mathbf{m_d}) \cdot \mathbf{m_d} - \mathbf{u_d}$) is a T-semiflow.

Given a marking $\mathbf{m_d} \in RS^+$, the control input $\mathbf{u_d}$ ensuring that $\mathbf{m_d}$ is an equilibrium point can be computed solving the following system:

$$\begin{array}{l} \mathbf{C} \cdot (\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\mathbf{m_d}) \cdot \mathbf{m_d} - \mathbf{u_d}) = \mathbf{0} \\ \mathbf{0} \leq \mathbf{u_d} \leq \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\mathbf{m_d}) \cdot \mathbf{m_d} \end{array} \tag{8}$$

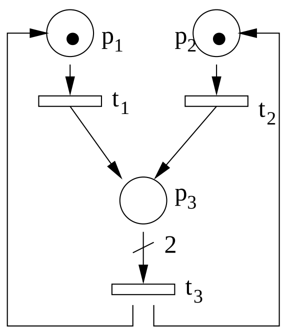

Fig. 4. Timed continuous Join-Free system with $\boldsymbol{\lambda} = [1, 1, 1]^T$ and unique equilibrium point for a given $\mathbf{u_d}$ (for example $\mathbf{m_d} = [0.66, 0.66, 0.66]^T$ for $\mathbf{u_d} = [0, 0, 0]^T$).

Given a $\mathbf{u_d}$, let us denote as $\mathbf{M_{u_d}}$ all the equilibrium states it could maintain. That is, $\mathbf{M_{u_d}} = \{\mathbf{m} \in RS^+ | \mathbf{C} \cdot (\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\mathbf{m}) \cdot \mathbf{m} - \mathbf{u_d}) = 0, 0 \leq \mathbf{u_d} \leq \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\mathbf{m}) \cdot \mathbf{m}\}$. The set $\mathbf{M_{u_d}}$ can have one single element (Figure 4) or an infinite number of equilibrium points in a single configuration (configuration $\{(p_1, t_1), (p_2, t_2), (p_3, t_3), (p_4, t_4), (p_5, t_5), (p_7, t_6)\}$ in Figure 6), or in several configurations (configurations $\{(p_1, t_1), (p_4, t_2), (p_7, t_3), (p_5, t_4), (p_6, t_5), (p_8, t_6)\}$ and $\{(p_1, t_1), (p_4, t_2), (p_7, t_3), (p_5, t_4), (p_6, t_5), (p_9, t_6)\}$ in Figure 8).

Next proposition characterizes all the equilibrium points of a net system with the same control action in steady state, $\mathbf{u_d}$.

Proposition 2. *Let $\Sigma = \langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a consistent timed contPN system with all transitions fireable at least once and $\mathbf{m_d}$ an equilibrium point for $\mathbf{u_d}$. Then $\mathbf{m_i}$ is also an equilibrium point for $\mathbf{u_d}$ iff:*

$$\begin{cases} \mathbf{B}^T \cdot (\mathbf{m_d} - \mathbf{m_i}) = 0 & (1) \\ \mathbf{C} \cdot \boldsymbol{\Lambda} \cdot (\boldsymbol{\Pi}_\mathbf{d} \cdot \mathbf{m_d} - \boldsymbol{\Pi}_\mathbf{i} \cdot \mathbf{m_i}) = 0 & (2) \\ \mathbf{m_i} \geq 0 & (3) \\ \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}_\mathbf{i} \cdot \mathbf{m_i} \geq \mathbf{u_d} \geq 0 & (4) \end{cases} \tag{9}$$

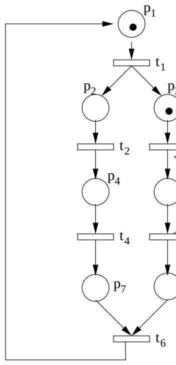

Fig. 5. Timed continuous Marked Graph system with $\boldsymbol{\lambda} = [1, 1, 1, 1, 1, 1]^T$ and many equilibrium points in the same configuration for a given $\mathbf{u_d}$.

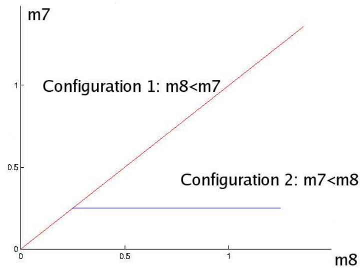

Fig. 6. Equilibrium points of the timed continuous Marked Graph system in Figure 5 for $\mathbf{u_d} = [0, 0, 0, 0, 0, 0]^T$.

Proof. $\Longrightarrow$ If $\mathbf{m_i}$ is an equilibrium point it is a reachable marking. The system is consistent so: $\mathbf{B}^T \cdot \mathbf{m_i} = \mathbf{B}^T \cdot \mathbf{m_d}$, i.e. (9.1) is necessary.

Both markings are equilibrium points: $\mathbf{C} \cdot (\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}_\mathbf{d} \cdot \mathbf{m_d} - \mathbf{u_d}) = 0$ and $\mathbf{C} \cdot (\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}_\mathbf{i} \cdot \mathbf{m_i} - \mathbf{u_d}) = 0$. Eliminating $\mathbf{u_d}$ with both equations, (9.2) is obtained.

$\Longleftarrow$ Equations (9.1) and (9.3) ensure the reachability of $\mathbf{m_i}$ because the net is consistent and every transition is fireable. The control input $\mathbf{u_d}$ can be applied (9.4) and using (9.2) $\mathbf{m_i}$ is an equilibrium marking. □

Lemma 1. *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a timed contPN system and $\mathbf{m_d}$, $\mathbf{m_i}$ two equilibrium points for $\mathbf{u_d}$. The flows at these markings are equal iff $\boldsymbol{\Pi}_\mathbf{d} \cdot \mathbf{m_d} = \boldsymbol{\Pi}_\mathbf{i} \cdot \mathbf{m_i}$.*

Proof. Flows are equal: iff $\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}_\mathbf{d} \cdot \mathbf{m_d} - \mathbf{u_d} = \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}_\mathbf{i} \cdot \mathbf{m_i} - \mathbf{u_d}$, that is iff $\boldsymbol{\Lambda} \cdot (\boldsymbol{\Pi}_\mathbf{d} \cdot \mathbf{m_d} - \boldsymbol{\Pi}_\mathbf{i} \cdot \mathbf{m_i}) = 0$. Since $\boldsymbol{\Lambda}$ is a full rank matrix (by definition is a diagonal matrix with diagonal elements greater than zero), this can happen iff $\boldsymbol{\Pi}_\mathbf{d} \cdot \mathbf{m_d} = \boldsymbol{\Pi}_\mathbf{i} \cdot \mathbf{m_i}$. □

Example 3. For the timed *contPN* system depicted in Figure 9 the optimal flow $\mathbf{f}_{max} = [0.2, 0.2, 0.6, 0.2]^T$ is obtained with $\mathbf{u_d} = [0, 0, 0, 0]^T$ and marking $\mathbf{m_d} = [0.2, 0.6, 0.6, 0.6]^T$. Marking $\mathbf{m'} = [0.1, 0.3, 1.8, 0.3]^T$, is also an equilibrium point, and the flow is different $\mathbf{f'} = [0.1, 0.1, 0.3, 0.1]^T$. Obviously, the conditions of the lemma do not hold, $\boldsymbol{\Pi}_\mathbf{d} \cdot \mathbf{m_d} \neq \boldsymbol{\Pi}' \cdot \mathbf{m}'$.

Theorem 1. *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a consistent timed contPN system with all transitions fireable at least once. In one configuration $\boldsymbol{\Pi}$ all the equilibrium points for a given $\mathbf{u}$ have the same flow if*

$$rank \begin{bmatrix} \boldsymbol{\Pi} & | & -\boldsymbol{\Lambda}^{-1} \cdot X_1 \; ... \; -\boldsymbol{\Lambda}^{-1} \cdot X_k \\ \mathbf{B^T} & | & 0 \; ... \qquad\qquad 0 \end{bmatrix} = rank \begin{bmatrix} \boldsymbol{\Pi} \\ \mathbf{B}^T \end{bmatrix} + k$$

where $k = |T| - rank(\mathbf{C})$ and $X_1, ..., X_k$ is a T-flow basis.

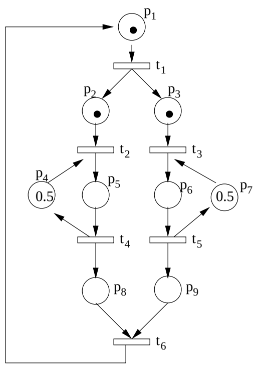

Fig. 7. Timed *contPN* system with $\boldsymbol{\lambda} = [1, 1, 1, 1, 1, 1]^T$ and many equilibrium points in the same configuration for a fixed $\mathbf{u_d}$.

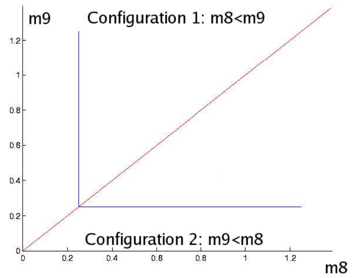

Fig. 8. Equilibrium points of the timed *contPN* system in Figure 7 for $\mathbf{u_d} = [0, 0, 0, 0, 0, 0]^T$.

Proof. Let us assume that $\mathbf{m_a}$ and $\mathbf{m_b}$ are two equilibrium points under $\boldsymbol{\Pi}$ for the same control $\mathbf{u}$. Obviously, the flow in steady state will be a T-semiflow: $\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi} \cdot \mathbf{m_a} - \mathbf{u} = \alpha_1 \cdot X_1 + ... + \alpha_k \cdot X_k$, $\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi} \cdot \mathbf{m_b} - \mathbf{u} = \beta_1 \cdot X_1 + ... + \beta_k \cdot X_k$. Now, we can write: $\boldsymbol{\Pi} \cdot \Delta\mathbf{m} - \boldsymbol{\Lambda}^{-1} \cdot (\zeta_1 \cdot X_1 - ... - \zeta_k \cdot X_k) = 0$ ($\Delta\mathbf{m} = \mathbf{m_a} - \mathbf{m_b}, \zeta_i = \alpha_i - \beta_i, \forall i$). Moreover, since these markings are reachable, $\mathbf{B}^T \cdot \Delta\mathbf{m} = 0$.

$$\begin{bmatrix} \boldsymbol{\Pi} \mid -\boldsymbol{\Lambda}^{-1} \cdot X_1 \; ... \; -\boldsymbol{\Lambda}^{-1} \cdot X_k \\ \mathbf{B}^T \mid \qquad 0 \; ... \qquad 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta\mathbf{m} \\ \zeta_1 \\ ... \\ \zeta_k \end{bmatrix} = 0 \tag{10}$$

Under the rank condition for every solution of this system $\zeta_i = 0$ $\forall i$. Therefore $\mathbf{m_a}$ and $\mathbf{m_b}$ have the same flow. □

Example 4. Let us consider the *contPN* system in Figure 10 with $\boldsymbol{\lambda} = [2, 1, 1]^T$. The configuration $\{(p_4, t_1), (p_4, t_2), (p_3, t_3)\}$ with associated matrix $\boldsymbol{\Pi}$ can have several equilibrium points with different flows because the conditions of Theorem 1 are not satisfied. For this system, $\boldsymbol{\Pi} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $\mathbf{B}^T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$, $\boldsymbol{\Lambda}^{-1} \cdot \mathbf{X} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$ and $rank \begin{bmatrix} \boldsymbol{\Pi} & -\boldsymbol{\Lambda} \cdot \mathbf{X} \\ \mathbf{B}^T & 0 \end{bmatrix} = rank \begin{bmatrix} \boldsymbol{\Pi} \\ \mathbf{B}^T \end{bmatrix} = 4$. If $\mathbf{u} = [0, 0, 0]^T$, the equilibrium markings $\mathbf{m}_1 = [15.25, 1, 0.75, 0.75]^T$ and $\mathbf{m}_2 = [15.5, 0.8, 0.7, 0.7]^T$ belonging to this configuration have the flows $\mathbf{f}_1 = [0.75, 0.75, 0.75]^T$ and $\mathbf{f}_2 = [0.7, 0.7, 0.7]^T$ respectively. Thus, any intermediate value is also possible.

For the class of Equal Conflict *contPN* and non-controlled conflict transitions, we can prove that all equilibrium points have the same flow under the same configuration.

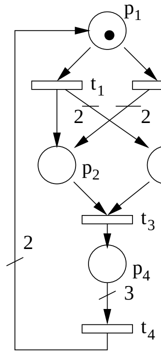

Fig. 9. Conservative but not lim-live continuous EQ system with several non optimal equilibrium points for $\boldsymbol{\lambda} = [1,1,1,1]^T$.

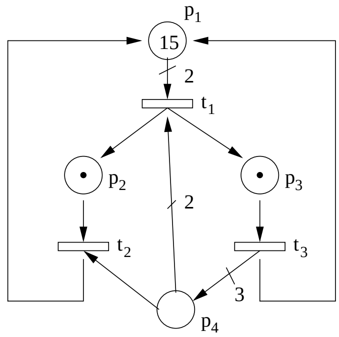

Fig. 10. Bounded and lim-live *contPN* that has several equilibrium points with distinct flow.

Theorem 2. *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a bounded and lim-live EQ timed contPN system. Given* $\mathbf{u}$ *in which transitions in conflict are not controlled, there exists at least one equilibrium point, and even if there are more, all of them have the same flow.*

Proof. The throughput in steady state for unforced ($\mathbf{u} = 0$) continuous EQ nets can be computed using a linear programming problem ([9]). More precisely, the throughput is obtained looking for the slowest P-semiflow. The solution is unique with respect to the flow, but there can exist more than one marking that respect the P-semiflows and have the same associated flow.

Assume ${}^\bullet t = p$ (i.e. non synchronizing transition) and $\mathbf{u}[t] \neq 0$. If the steady-state marking of p is $\mathbf{m}[p]$, we can reduce the value of $\mathbf{m}[p]$ and transform the system into an equivalent one with the same steady-state flow with the marking $\mathbf{m}'[p] = \mathbf{m}[p] - \frac{\mathbf{Pre}[p,t]}{\boldsymbol{\lambda}[t]} \cdot \mathbf{u}[t]$. The flow will be: $\boldsymbol{\lambda}[t] \cdot \frac{\mathbf{m}'}{\mathbf{Pre}[p,t]} = \boldsymbol{\lambda}[t] \cdot \frac{\mathbf{m}}{\mathbf{Pre}[p,t]} - \mathbf{u}[t]$, the same as in the original system (with $\mathbf{u}[t] \neq 0$). For every controlled transition we can apply the same technique (in the case of synchronizations we remove tokens from all input places) obtaining an equivalent system with $\mathbf{u} = 0$. For this system all the equilibrium points have the same flow. □

This theorem ensures that all equilibrium points of the CF *contPNs* systems in Figure 4, Figure 5 and Figure 7 have the same flow for any constant control input $\mathbf{u}$. The number of equilibrium points in one configuration are characterized in the following theorem.

Theorem 3. *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a bounded and lim-live EQ timed contPN system. If rank $\begin{bmatrix} \boldsymbol{\Pi}_\mathbf{i} \\ \mathbf{B}^T \end{bmatrix} = |P|$ and conflict transitions are not controlled, then at most one equilibrium point exists under $\boldsymbol{\Pi}_\mathbf{i}$.*

Proof. Let $rank \begin{bmatrix} \boldsymbol{\Pi}_\mathbf{i} \\ \mathbf{B}^T \end{bmatrix} = |P|$ and $\mathbf{m_d}$, $\mathbf{m_i}$ ($\mathbf{m_i} = \mathbf{m_d} + \Delta\mathbf{m}$) two equilibrium points under $\boldsymbol{\Pi}_\mathbf{i}$ for $\mathbf{u_d}$. Using Theorem (2) all equilibrium points with the same action $\mathbf{u_d}$ have the same flow, i.e. $\boldsymbol{\Pi}_\mathbf{i} \cdot \mathbf{m_d} = \boldsymbol{\Pi}_\mathbf{i} \cdot \mathbf{m_i}$ or $\boldsymbol{\Pi}_\mathbf{i} \cdot \Delta\mathbf{m} = 0$. Moreover, $\mathbf{B}^T \cdot \mathbf{m_i} = \mathbf{B}^T \cdot \mathbf{m_d}$, or $\mathbf{B}^T \cdot \Delta\mathbf{m} = 0$.

Under the rank assumption, the previous system has only one solution, $\Delta\mathbf{m} = 0$. So $\mathbf{m_d} = \mathbf{m_i}$. Hence, $\boldsymbol{\Pi}_\mathbf{i}$ has at most one equilibrium point. □

Example 5. Let us consider the net in Figure 7 and $\boldsymbol{\Pi} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1\,0 \end{bmatrix}$ one configuration. One P-flow basis is : $\mathbf{B}^T = \begin{bmatrix} 0\,0\,0\,1\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,1\,0\,0 \\ 1\,1\,0\,0\,1\,0\,0\,1\,0 \\ 1\,0\,1\,0\,0\,1\,0\,0\,1 \end{bmatrix}$. Because $rank \begin{bmatrix} \boldsymbol{\Pi}_\mathbf{i} \\ \mathbf{B}^T \end{bmatrix} = 8 < 9$ (the number of places) this configuration has an infinite number of equilibrium points (configuration 1= $\{(p_1,t_1),(p_4,t_2),(p_7,t_3),(p_5,t_4),(p_6,t_5),(p_8,t_6)\}$ in Figure 8).

Corollary 1. *Let $\mathcal{N}$ be a conservative and consistent JF contPN. Given $\mathbf{u_d}$, only one equilibrium point exists in $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$.*

4 Optimal control for steady state

In production control is frequent the case that the profit function depends on production (benefits in selling), working process and amortization of investments. Under linear hypothesis for fixed machines (i.e. $\boldsymbol{\lambda}$ defined), the profit function may have the following form: $\mathbf{w}^T \cdot \mathbf{f} - \mathbf{z}^T \cdot \mathbf{m} - \mathbf{q}^T \cdot \mathbf{m}_0$, where $\mathbf{f}$ is the throughput vector, $\mathbf{m}$ the average marking, $\mathbf{w}^T$ a gain vector w.r.t. flows, $\mathbf{z}^T$ is the cost vector due to immobilization to maintain the production flow and $\mathbf{q}^T$ represents depreciations or amortization of the initial investments.

Let us consider the following linear programming problem:

$$\begin{cases} \max \{\mathbf{w^T} \cdot \mathbf{f} - \mathbf{z^T} \cdot \mathbf{m} - \mathbf{q^T} \cdot \mathbf{m_0}\} & \\ \mathbf{C} \cdot \mathbf{f} = 0,\ \mathbf{f} \geq 0 & \text{(a)} \\ \mathbf{m} = \mathbf{m}_0 + \mathbf{C} \cdot \boldsymbol{\sigma},\ \mathbf{m}, \boldsymbol{\sigma} \geq 0 & \text{(b)} \\ \mathbf{f}[t_i] = \lambda_i \cdot \left(\frac{\mathbf{m}[p_j]}{\mathbf{Pre}[p_j,t_i]} \right) - \mathbf{v}[p_j,t_i], \forall p_j \in {}^\bullet t_i,\ \mathbf{v}[p_j,t_i] \geq 0 & \text{(c)} \end{cases} \tag{11}$$

where $\mathbf{v}[p_j, t_i]$ are *slack* variables.

The equations correspond to: (a) $\mathbf{f}$ is a T-semiflow; (b) fundamental equation ($\mathbf{m}$ is a reachable marking); (c) firing law for infinite servers semantics. In the case of $|{}^\bullet t_i| = 1$ the corresponding slack variable is the same as the control input.

Theorem 4. *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a timed contPN system and $\langle \mathbf{f}, \mathbf{m}, \mathbf{v} \rangle$ be a solution of LPP (11), then*

1. *For every transition t_i, let $u_i = \min\limits_{p_j \in {}^\bullet t_i} \mathbf{v}[p_j, t_i]$ be its control input. Then $\mathbf{u}$ is the control in steady-state for $\mathbf{m}$.*
2. *If for every $u_i > 0$ transition t_i is control-feasible, then $\mathbf{u}$ is an optimal control.*

Proof. In steady state, $\mathbf{f}[t_i] = \boldsymbol{\lambda}[t_i] \cdot \min\limits_{p_j \in {}^\bullet t_i} \left(\frac{\mathbf{m}[p_j]}{\mathbf{Pre}[p_j, t_i]} \right) - u_i$. Choosing $u_i = \min\limits_{p_j \in {}^\bullet t_i} \mathbf{v}[p_j, t_i]$ for all transitions, the equation (11.c) is verified. If all t_i with $u_i \neq 0$ can be controlled, the control can be applied in steady state; then the command is optimal. □

For mono T-semiflow nets (nets that have an unique minimal T-semiflow) (or reducible to [6]), the equation (11a) can be replaced with the equivalent one: $\mathbf{f} = \alpha \cdot \mathbf{X}$ with $\mathbf{X}$ the minimal T-semiflow.

If the net is consistent and every transition can be fired at least once, the equation (11b) is equivalent to: $\mathbf{B}^T \cdot \mathbf{m} = \mathbf{B}^T \cdot \mathbf{m_0}$, $\mathbf{m} \geq 0$.

Example 6. The solution of LPP 11 is not necessarily unique (as we saw in the previous section). In order to obtain the maximum throughput in steady-state for the *contPN* in Figure 1 with $\boldsymbol{\lambda} = [1, 1, 1, 1]^T$ and $\mathbf{m_0} = [1, 0, 3, 3, 1, 0]^T$. LPP (11) leads to:

$$\begin{cases} \max f_1 \\ f_1 = f_2 = f_3 = f_4 \\ m_1 + m_2 = 1 \\ m_3 + m_4 = 6 \\ m_5 + m_6 = 1 \\ f_1 = m_1 - u_1 \\ f_2 = m_2 - v_{22} \\ f_2 = m_3 - v_{23} \\ f_3 = m_4 - v_{34} \\ f_3 = m_5 - v_{35} \\ f_4 = m_6 - u_4 \\ \mathbf{f}, \mathbf{m}, \mathbf{v} \geq 0 \end{cases} \tag{12}$$

One optimal solution of this LPP is: $f_1 = 0.5$, $\mathbf{m_d} = [0.5\ 0.5\ 3.5\ 2.5\ 0.5\ 0.5]^T$ and $\mathbf{v} = [0, 0, 3, 2, 0, 0]^T$. Therefore $u_2 = \min(v_{22}, v_{23}) = 0$, $u_3 = \min(v_{34}, v_{35}) = 0$ and $\mathbf{u_d} = [0\ 0\ 0\ 0]^T$ is an optimal control in steady state.

For sure the solution is not unique, all the markings that satisfy (13) are also solution of (12).

$$\begin{cases} m_1 = m_2 = m_5 = m_6 = 0.5 \\ m_3 + m_4 = 6 \\ m_3, m_4 \geq 0.5 \end{cases} \tag{13}$$

Corollary 2. *For JF contPN, the solution of the system in (11) is unique and* $\mathbf{u} = \mathbf{0}$ *(monotonicity).*

Proof. For JF *contPN* we have only persistent transitions, so the solution is unique. Maximizing the flow, $\mathbf{u} = \mathbf{0}$ is solution of (11) and is the same as the steady state of the unforced net. □

5 Approaching dynamic control: on controllability and marking invariance laws

The dynamic systems under study are described by the equations in (7). The classical control theory for linear systems cannot be applied for our system because we are working inside a polytope (not in a *vectorial space* as in classical linear system) and our control input is bounded. In this section, a relaxation of the equations modelling the system is proposed, eliminating the restrictions regarding the bounds for the control input. The goal is to try to interpret "classical results" in the *contPN* case. Therefore, the system under study is reduced to the non-linear equations:

$$\dot{\mathbf{m}} = \mathbf{C} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\mathbf{m}) \cdot \mathbf{m} - \mathbf{C} \cdot \mathbf{u} \tag{14}$$

For classical linear systems *controllability* has been thoroughly studied (see, for example [11], [12]). A dynamic system is said to be *completely state controllable* if for any time τ_0, it is possible to construct an *unconstrained* control vector $\mathbf{u}(\tau)$ that will transfer a given initial state $\mathbf{x}(\tau_0)$ to a final state $\mathbf{x}(\tau)$ in a finite time interval $\tau_0 < \tau$.

In system theory, a very well-known controllability criterion exists which allows to decide whether a continuous linear system is controllable or not. Given a linear system $\dot{\mathbf{x}}(\tau) = \mathbf{A} \cdot \mathbf{x}(\tau) + \mathbf{S} \cdot \mathbf{u}(\tau)$, the *controllability matrix is*:

$$\mathbb{C} = [\mathbf{S} \cdots \mathbf{A}^k \mathbf{S} \cdots \mathbf{A}^{(n-1)} \mathbf{S}] \tag{15}$$

Proposition 3. *[11] A linear continuous-time system* $\dot{\mathbf{x}}(\tau) = \mathbf{A} \cdot \mathbf{x}(\tau) + \mathbf{S} \cdot \mathbf{u}(\tau)$ *is completely controllable iff the controllability matrix* $\mathbb{C}$ *has a full rank. If* $\mathbb{C}$ *is not a full rank matrix then the system has only* $rank(\mathbb{C})$ *controllable state variables.*

For *contPN* systems, every $\boldsymbol{\Pi}(\mathbf{m})$ leads to a linear and time-invariant dynamic system with controllability matrix $\mathbb{C}(\mathbf{m})$:

$$\mathbb{C}(\mathbf{m}) = -\left[\mathbf{C} \cdots\ (\mathbf{C} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\mathbf{m}))^{n-1} \cdot \mathbf{C}\right] \tag{16}$$

Proposition 4. *If all transitions are control-feasible,* $\forall \mathbf{m}$, *the space generated by the columns of* $\mathbb{C}(\mathbf{m})$ *and* $\mathbf{C}$ *are equal. Thus* $rank(\mathbb{C}(\mathbf{m})) = rank(\mathbf{C}) = |P| - \dim(\mathbf{B})$.

Proof. Observe that $(\mathbf{C}\cdot\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m}))^{n-1}\cdot\mathbf{C} = \mathbf{C}\cdot(\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\cdot\mathbf{C})^{n-1}$. Thus, $rank(\mathbf{C}) = rank([\mathbf{C}\cdots\mathbf{C}\cdot(\boldsymbol{\Lambda}\cdot\boldsymbol{\Pi}(\mathbf{m})\cdot\mathbf{C})^{n-1}])$. Given that $\mathbf{C}$ is included in $\mathbb{C}(\mathbf{m})$ the first part is true. □

Notice that $\mathbb{C}(\mathbf{m})$ depends on $\boldsymbol{\Pi}(\mathbf{m})$, but the space generated by its columns is always the same, just that one defined by that of matrix $\mathbf{C}$. This is something that can be easily expected because all transitions are have assumed to be control-feasible.

Nets with at least one P-flow are non controllable in the classical sense of dynamic system for any firing rate $\boldsymbol{\lambda}$ and any initial marking $\mathbf{m_0}$.P-flows based token conservation laws make $|P| - rank(\mathbf{C})$ places linearly-redundant (even if they constraint the behaviour of the net system model).

Let us consider first a motivating example.

Example 7. Let us consider the *contPN* system in Figure 1 with $\boldsymbol{\lambda} = [\alpha, \beta, \gamma, \delta]^T$. This net has three independent token conservation laws derived from P-(semi)flows: $\mathbf{m}[p_1] + \mathbf{m}[p_2] = 1, \mathbf{m}[p_3] + \mathbf{m}[p_4] = 6$ and $\mathbf{m}[p_5] + \mathbf{m}[p_6] = 1$. Thus $\dot{\mathbf{m}}[p_1] + \dot{\mathbf{m}}[p_2] = \dot{\mathbf{m}}[p_3] + \dot{\mathbf{m}}[p_4] = \dot{\mathbf{m}}[p_5] + \dot{\mathbf{m}}[p_6] = 0$, that means that three uncontrollable zero valued poles will appear. If we fix $\mathbf{m}[p_2], \mathbf{m}[p_3]$ and $\mathbf{m}[p_5]$ as state variables then $\mathbf{m}[p_1], \mathbf{m}[p_4]$ and $\mathbf{m}[p_6]$ are redundant. The linear dynamic system corresponding to the configuration $\{(p_1, t_1), (p_2, t_2), (p_5, t_3), (p_6, t_6)\}$ is:

$$\begin{cases} \dot{m}_2 = -\beta\cdot m_2 + \alpha\cdot(1 - m_2) = -(\alpha+\beta)\cdot m_2 + \alpha \\ \dot{m}_3 = -\beta\cdot m_2 + \gamma\cdot m_5 \\ \dot{m}_5 = -\gamma\cdot m_5 + \delta\cdot(1 - m_5) = -(\gamma+\delta)\cdot m_5 + \delta \end{cases}$$

Eliminating all variables in the right hand side:

$$-\frac{\beta}{\alpha+\beta}\cdot\dot{m}_2 + \frac{\gamma}{\gamma+\delta}\cdot\dot{m}_5 + \dot{m}_3 = \frac{\gamma\cdot\delta}{\gamma+\delta} - \frac{\alpha\cdot\beta}{\alpha+\beta} = q$$

Therefore, if $q = 0$ an additional zero valued pole is obtained. It is not rooted in a P-flow, but depends also on $\boldsymbol{\lambda}$ and $\mathbf{B}^T\mathbf{m_0}$ (i.e. not on a particular $\mathbf{m_0}$ but on the token load of P-flows).

If $q \neq 0$, sooner or later the above configuration will be left. This clear since at least one of the variables ($\mathbf{m}[p_2], \mathbf{m}[p_3]$ or $\mathbf{m}[p_5]$) will grow (or decrease) indefinitely. This can also be deduced using that the steady state flow has to be a T-semiflow of the net. Since it has only one minimal T-semiflow $[1, 1, 1, 1]^T$, in steady state: $f_1 = f_2 = f_3 = f_4$.

$$f_1 = f_2 \Longrightarrow \alpha\cdot m_1 = \beta\cdot(1 - m_1) \Longrightarrow f_1 = \frac{\alpha\cdot\beta}{\alpha+\beta}$$

$$f_3 = f_4 \Longrightarrow \gamma\cdot m_5 = \delta\cdot(1 - m_5) \Longrightarrow f_3 = \frac{\gamma\cdot\delta}{\gamma+\delta}$$

$$f_1 = f_3 \Longleftrightarrow \frac{\alpha\cdot\beta}{\alpha+\beta} = \frac{\gamma\cdot\delta}{\gamma+\delta} \Longleftrightarrow q = 0$$

Thus, if $q \neq 0$ this will not be the equilibrium configuration.

The following transformation matrix is used to change the reference in which the marking vector is expressed. This will be useful to approach the controllability of the system. The kind of transformation matrix to be considered will have in this context a particular structure.

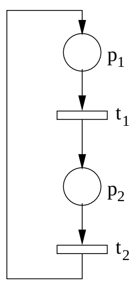

Fig. 11. Join-Free timed *contPN*.

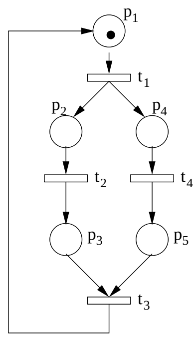

Fig. 12. Choice-Free timed *contPN* system for example 11.

Definition 10. *Let $\mathcal{N}$ be a contPN. A transformation matrix $\mathbb{Q}_{\mathcal{N}}$, is a full rank matrix, where the first rows form a basis of $P-flows$ and the remaining rows are completed with elementary vectors in order to build a full rank matrix $\mathbb{Q}_{\mathcal{N}}$.*

Example 8. For the timed models in Figure 11 and Figure 1, $P-Flow$ basis are:

$$\mathbf{B}_1^T = \begin{bmatrix} 1\,1 \end{bmatrix} \text{ and } \mathbf{B}_2^T = \begin{bmatrix} 1\,1\,0\,0\,0\,0 \\ 0\,0\,1\,1\,0\,0 \\ 0\,0\,0\,0\,1\,1 \end{bmatrix} \tag{17}$$

Adding elementary vectors, $\mathbb{Q}$ matrices can be, for example:

$$\mathbb{Q}_1 = \begin{bmatrix} 1\,1 \\ 0\,1 \end{bmatrix} \text{ and } \mathbb{Q}_2 = \begin{bmatrix} 1\,1\,0\,0\,0\,0 \\ 0\,0\,1\,1\,0\,0 \\ 0\,0\,0\,0\,1\,1 \\ 0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,1 \end{bmatrix} \tag{18}$$

The system described by equation (14) can be rewritten in new coordinates $\bar{\mathbf{m}}$, when matrix $\mathbb{Q}_{\mathcal{N}}$ is used as a state vector transformation matrix. Let $\bar{\mathbf{m}} = \mathbb{Q}_{\mathcal{N}} \cdot \mathbf{m}$.

Definition 11. *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a timed contPN described by equation (14), where $\mathbb{Q}_{\mathcal{N}}$ is a transformation matrix of $\mathcal{N}$. Then*

$$\overset{\bullet}{\bar{\mathbf{m}}}=\mathbb{Q}_{\mathcal{N}}\mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbb{Q}_{\mathcal{N}}^{-1}\bar{\mathbf{m}}-\mathbb{Q}_{\mathcal{N}}\mathbf{C}\mathbf{u} \tag{19}$$

will be named a $\mathbb{Q}$-canonical representation of equation (14).

Theorem 5. *Let $\Sigma = \langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ be a contPN system, then:*

1. *In $\boldsymbol{\Pi}(\mathbf{m})$ the number of zero valued poles is given by the dimension of the right annulers of $\mathbf{C} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\mathbf{m}) \cdot \mathbf{v}$, $\mathbf{v} \in \mathbb{R}^{|P|}$.*
2. *The number of non controllable poles is $|P| - rank(\mathbf{C})$ and are zero valued.*

Proof. 1. The zero eigenvalues of the matrix $\mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})$ are:

$$\mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m}) \cdot \mathbf{v} = 0 \cdot \mathbf{v} = 0$$

Therefore, the space generated by the column of $\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})$ is contained in the kernel of $\mathbf{C}$. For sure, the number of zero valued poles is given by the dimension of the right annulers of the matrix $\mathbf{C} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\mathbf{m}) \cdot \mathbf{v}$.

2. Making the change of variables:

$$\bar{\mathbf{m}} = \mathbb{Q}_{\mathcal{N}} \cdot \mathbf{m} \tag{20}$$

the $\mathbb{Q}$-canonical representation of equation (14) is obtained:

$$\overset{\bullet}{\bar{\mathbf{m}}} = \mathbb{Q}_{\mathcal{N}} \cdot \mathbf{C} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\mathbf{m}) \cdot \mathbb{Q}_{\mathcal{N}}^{-1} \cdot \bar{\mathbf{m}} - \mathbb{Q}_{\mathcal{N}} \cdot \mathbf{C} \cdot \mathbf{u} \tag{21}$$

Since $\mathbb{Q}_{\mathcal{N}}$ contains a basis for the P-flows of $\mathbf{C}$, then one zero row of the matrix

$$\mathbb{Q}_{\mathcal{N}} \cdot \mathbf{C}$$

implies one zero row (in the same position) in the matrix:

$$\mathbb{Q}_{\mathcal{N}}\mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbb{Q}_{\mathcal{N}}^{-1}$$

Without loss of generality, assume that the row i of $\mathbb{Q}_{\mathcal{N}} \cdot \mathbf{C}$ is zero, then the row i of $\mathbb{Q}_{\mathcal{N}}\mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbb{Q}_{\mathcal{N}}^{-1}$ is zero. Therefore the value of the state variable $\bar{\mathbf{m}}_i$ is never affected by other state variables, or by the input, thus $\bar{\mathbf{m}}_i$ is uncontrollable. Intuitively, each one of these $\bar{\mathbf{m}}_i$ comes from a P-flow equation, a linear constraints among variables (i.e. token conservation law: $\mathbf{b_i} \cdot \bar{\mathbf{m}} = \mathbf{b_i} \cdot \bar{\mathbf{m}}_\mathbf{0}$). Thus the pole value associated to $\bar{\mathbf{m}}_i$ is zero and there exists $\dim(\mathbf{B})$ uncontrollable zero valued poles. Using Proposition 4, $rank(\mathbb{C}(\mathbf{m})) = |P| - \dim(\mathbf{B})$, then there exist no more uncontrollable poles. If there are more zero valued poles, they are controllable.

□

Example 9. Let us consider the *contPN* system in Figure 11, where $\boldsymbol{\lambda} = [1, 1]^T$. It has the following equation:

$$\dot{\mathbf{m}} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \mathbf{m} - \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \mathbf{u} \tag{22}$$

The controllability matrix of this net is the following:

$$\mathbb{C} = \begin{bmatrix} -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix}$$

The rank of this matrix is one, then it has only one controllable pole (equal to -2) and one non controllable pole (equal to 0). A transformation matrix is:

$$\mathbb{Q} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

then, the corresponding $\mathbb{Q}$-canonical representation of equation (22) is:

$$\dot{\bar{\mathbf{m}}} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \bar{\mathbf{m}} - \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \mathbf{u}$$

Example 10. The timed *contPN* system shown in Figure 1 with $\boldsymbol{\lambda} = [1,1,1,1]^T$ and the configuration defined by the marking in the figure, has the following equation:

$$\dot{\mathbf{m}} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \mathbf{m} - \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \mathbf{u} \tag{23}$$

One possible transformation matrix is:

$$\mathbb{Q} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

then, the corresponding $\mathbb{Q}$-canonical representation of equation (23) is:

$$\dot{\bar{\mathbf{m}}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix} \bar{\mathbf{m}} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \mathbf{u}$$

Clearly, $\dot{\bar{m}}_1 = \dot{\bar{m}}_2 = \dot{\bar{m}}_3 = 0$, or in the original settings $\dot{m}_1 + \dot{m}_2 = \dot{m}_3 + \dot{m}_4 = \dot{m}_5 + \dot{m}_6 = 0$, as already mentioned. Thus, they are zero valued uncontrollable poles (they only depend on the net structure).

Globally speaking, this new system has the following poles: (0,0,0,-2,0,-2) and three linearly independent P-flows. The fourth zero value pole depends on $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ and $\mathbf{B}^T \mathbf{m_0}$, not only on $\mathcal{N}$ as the uncontrollable ones.

Obviously, the non controllable poles appear in all the configurations. The controllable poles can have different values. For example, if we consider the same system and the configuration: $\{(p_1,t_1),(p_3,t_2),(p_5,t_3),(p_6,t_4)\}$, the canonical representation is given by:

$$\dot{\bar{\mathbf{m}}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix} \bar{\mathbf{m}} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \mathbf{u}$$

with the poles: (0,0,0,-1,-1,-2).

But new, token-invariant laws may appear depending on $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m_0} \rangle$ (i.e. not depending on the net structure as these derived from P-semiflows, but from the precise marking $\mathbf{m_0}$. Let us present a simple case.

Example 11. Consider now the *contPN* in Figure 12 with $\boldsymbol{\lambda} = [\alpha, \beta, \delta, \gamma]^T$. There exist two P-semiflows: $m_1 + m_2 + m_3 = 1$ and $m_1 + m_4 + m_5 = 1$. Then there are only three state variables, for example m_1, m_3 and m_5. The dynamic linear system associated with the configuration $\{(p_1,t_1),(p_2,t_2),(p_3,t_3),(p_4,t_4)\}$ is:

$$\begin{cases} \dot{m}_1 = \delta \cdot m_3 - \alpha \cdot m_1 \\ \dot{m}_3 = -\delta \cdot m_3 + \beta \cdot (1 - m_1 - m_3) \\ \dot{m}_5 = -\delta \cdot m_3 + \gamma \cdot (1 - m_1 - m_5) \end{cases}$$

Nevertheless, if $\beta = \gamma$, $\dot{\mathbf{m}}_3 - \dot{\mathbf{m}}_5 = -\beta \cdot (\mathbf{m}_3 - \mathbf{m}_5)$. Making a linear transformation in order to compute: $\bar{\mathbf{m}}_{35} = \mathbf{m}_3 - \mathbf{m}_5$, then $\dot{\bar{\mathbf{m}}}_{35} = -\beta \cdot \mathbf{m}_{35}$. If $\mathbf{m_0}[p_3] = \mathbf{m_0}[p_5] \Longrightarrow \dot{\bar{\mathbf{m}}}_{35} = 0$. In this case, the pole is different from 0, and depend on $\mathbf{m_0}$, thus $\mathbf{m}_3 = \mathbf{m}_5$ is a token conservation that is not rooted in a zero valued pole.

6 Conclusions

This work dealt with some control problems of continuous Petri nets. Necessary conditions for the equilibrium points in steady-state are some easy algebraic equations. For continuous EQ nets the steady-state flow is unique, even if several steady-state markings are possible. For general *contPNs*, a necessary condition of steady-state markings with different flows is presented. Optimal steady-state flow and input control is addressed by means of a LPP, that can be solved in polynomial time. In the last part of the paper, classical controllability theory of linear dynamic system is used to provide a first kind of interpretation to the class of systems that appear in our field, for the particular case in which all transitions have been assumed to be control-feasible. Controllability in more general framework and control schemes are currently topics under consideration.

References

1. Silva, M., Recalde, L.: Petri nets and integrality relaxations: A view of continuous Petri nets. IEEE Trans. on Systems, Man, and Cybernetics **32** (2002) 314–327
2. David, R., Alla, H.: Discrete, Continuous and Hybrid Petri Nets. Springer-Verlag (2004)
3. Silva, M., Recalde, L.: On fluidification of Petri net models: from discrete to hybrid and continuous models. Annual Reviews in Control **28** (2004) 253–266
4. Recalde, L., Teruel, E., Silva, M.: Autonomous continuous P/T systems. In S. Donatelli, J.K., ed.: Application and Theory of Petri Nets 1999. Volume 1639 of Lecture Notes in Computer Science., Springer (1999) 107–126
5. Teruel, E., Silva, M.: Structure theory of equal conflict systems. Theoretical Computer Science **153** (1996) 271–300
6. Júlvez, J., Recalde, L., Silva, M.: Steady-state performance evaluation of continuous mono-t-semiflow Petri nets. Automatica **41** (2005) 605–616
7. Júlvez, J., Recalde, L., Silva, M.: On reachability in autonomous continuous Petri net systems. In van der Aalst, W., Best, E., eds.: 24^{th} International Conference on Application and Theory of Petri Nets (ICATPN 2003). Volume 2679 of Lecture Notes in Computer Science. Springer, Eindhoven, The Netherlands (2003) 221–240
8. Silva, M., Teruel, E., Colom, J.M.: Linear algebraic and linear programming techniques for the analysis of net systems. In Rozenberg, G., Reisig, W., eds.: Lectures in Petri Nets. I: Basic Models. Volume 1491 of Lecture Notes in Computer Science. Springer (1998) 309–373
9. Recalde, L., Silva, M.: Petri Nets fluidification revisited: Semantics and steady state. APII-JESA **35** (2001) 435–449
10. Alla, H., David, R.: Continuous and hybrid Petri nets. Journal of Circuits, Systems, and Computers **8** (1998) 159–188
11. Luenberger, D.G.: Introduction to Dynamic Systems. Theory, Models, and Applications. John Wiley & Sons (1979)
12. Ogata, K.: Discrete-Time Control Systems, 2nd. ed. Prentice Hall (1995)