An Efficient Heuristics for Minimum Time Control of Continuous Petri nets ⋆

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Abstract: This paper considers the problem of controlling timed continuous Petri nets under infinite server semantics. The proposed control strategy assigns piecewise constant flows to transitions in order to reach the target state. First, by using linear programming, a method driving the system from the initial to the target state through a linear trajectory is developed. Then, in order to improve the time of the trajectory, intermediate states are added by means of bilinear programming.

Keywords: Timed continuous Petri nets, minimum time control, piecewise linear trajectory

1. INTRODUCTION

Petri nets (PNs) are frequently used for modeling, analysis and synthesis of discrete event systems. The distributed state or marking of a PN is given by a vector of natural numbers which represent the number of tokens in each place. Similarly to most formalisms for discrete event systems, PNs suffer from the *state explosion problem*. One possible way to overcome this problem is *fluidification*, a classical relaxation technique.

Continuous Petri nets are a fluid approximation of classical discrete Petri nets (David and Alla (2005); Silva and Recalde (2004)). Unlike conventional PNs, a transition in a continuous Petri net can be fired in a "real" non integer quantity. This leads to continuous markings instead of discrete ones. Similarly to discrete PNs, a time delay can be associated to the firing of transitions, the resulting net is called *timed continuous Petri net* (contPN).

As in discrete PNs, different server semantics can be adopted for the firing of transitions. Among them the most widely used semantics are *finite server* and *infinite server*. In Mahulea et al. (2009), it is shown that infinite server semantics provides a better approximation of the steady state throughput than finite server semantics for a wide class of Petri nets under some general conditions.

A ContPN system with infinite server semantics is a subclass of *Piecewise Linear Systems* (PWL), with input constraints. A PWL system is composed of several linear subsystems together with switching rules. Many works deal with stability analysis or control law design of such systems (Montagner et al. (2006); Yang et al. (2006); Habets et al. (2006)). However, most of these methods cannot be directly applied to contPNs, because of their particular dynamic input constraints.

Control of contPNs with infinite server semantics has already been considered in the literature. In Mahulea et al. (2008b) it is shown that, the optimal steady state control problem of timed contPN system can be solved by means of *Linear Programming Problem* (LPP) when all transitions are assumed to be controllable. The transitory control problem is also solved by means of implicit and explicit MPC control strategy (Mahulea et al. (2008a)). A Lyapunov-function-based dynamic control algorithm was proposed in (Xu et al. (2008)), which can ensure global convergence of both system states and input signals. In contrast to the method in this last reference, the technique proposed here for the computation of control law for direct linear trajectory is based on linear programming instead of nonlinear programming. Although the complexity of this new approach is significantly lower, i.e., it can be solved in polynomial time, the time of the obtained trajectory is very similar in general. Moreover, the method in (Xu et al. (2008)) requires solving a *BiLinear Programming Problem* (BLP) for the computation of intermediate states as our method requires. Refinement of the trajectory by means of intermediate states is carried out by a recursive algorithm which computes new intermediate states until the time can not be improved significantly.

In this work, we design a control strategy which aims at driving the system from an initial state to a target one by minizing the time. The proposed strategy computes first the control actions to drive the system to the target state through a straight line. Such control actions are the result of solving a LPP. Then, as in Xu et al. (2008), in order to obtain faster trajectories, intermediate states, not necessarily on the line connecting the initial and the target marking, are computed by means of a BLP. Moreover,

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an algorithm that recursively refines the initially obtained trajectory is given.

This paper is structured as follows. Section 2 briefly introduces the required concepts of contPN systems and introduces the formulation of applied control. A LPP based method to obtain linear trajectories in order to reach target marking, $\boldsymbol{m}_f$, from initial marking $\boldsymbol{m}_0$ is given in Section 3. In Section 4, the heuristics for minimizing time by means of intermediate states is considered. Finally, some conclusions and future directions are drawn in Section 5.

2. CONTINUOUS PETRI NETS

We assume that the reader is familiar with discrete PNs. In contPN systems, the firing is not restricted to the natural numbers but to the nonnegative real numbers.

2.1 Timed Continuous Petri nets

Definition 1. A continuous PN system is a pair $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ where $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post} \rangle$ is the net structure with the set of places P, the set of transitions T, pre and post matrices $\boldsymbol{Pre}, \boldsymbol{Post} \in \mathbb{R}_{\geq 0}^{|P| \times |T|}$ and $\boldsymbol{m}_0 \in \mathbb{R}_{\geq 0}^{|P|}$ is the initial state.

Let p_i, $i = 1, \ldots, |P|$ and $t_j, j = 1, \ldots, |T|$ denote the places and transitions. For a place $p_i \in P$ and a transition $t_j \in T$, Pre_{ij} and $Post_{ij}$ represent the weights of the arcs from p_i to t_j and from t_j to p_i, respectively. Each place p_i has a marking denoted by $m_i \in \mathbb{R}_{\geq 0}$. The vector of all token loads is called *state* or *marking*, and is denoted by $\boldsymbol{m} \in \mathbb{R}_{\geq 0}^{|P|}$. For every node $v \in P \cup T$, the sets of its input and output nodes are denoted as ${}^\bullet v$ and $v^\bullet$, respectively.

A transition $t_j \in T$ is enabled at $\boldsymbol{m}$ iff $\forall p_i \in {}^\bullet t_j$, $m_i > 0$ and its enabling degree is given by

$$enab(t_j, \boldsymbol{m}) = \min_{p_i \in {}^\bullet t_j} \left\{ \frac{m_i}{Pre_{ij}} \right\}$$

which represents the maximum amount in which t_j can fire. An enabled transition t_j can fire in any real amount α, with $0 < \alpha \leq enab(t_j, \boldsymbol{m})$ leading to a new state $\boldsymbol{m'} = \boldsymbol{m} + \alpha \cdot \boldsymbol{C}_{\cdot j}$ where $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the token flow matrix and $\boldsymbol{C}_{\cdot j}$ is its j^{th} column. If $\boldsymbol{m}$ is reachable from $\boldsymbol{m}_0$ through a finite sequence σ, the state (or fundamental) equation is satisfied: $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma} \in \mathbb{R}_{\geq 0}^{|T|}$ is the firing count vector, i.e., σ_j is the cumulative amount of firings of t_j in the sequence σ.

Definition 2. A contPN system $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m}_0 \rangle$ is a continuous PN system $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$ together with a vector $\boldsymbol{\lambda} \in \mathbb{R}_{>0}^{|T|}$ where λ_j is the firing rate of transition t_j.

The state equation has an explicit dependence on time, denoted by τ: $\boldsymbol{m}(\tau) = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}(\tau)$ which through time differentiation becomes $\dot{\boldsymbol{m}}(\tau) = \boldsymbol{C} \cdot \dot{\boldsymbol{\sigma}}(\tau)$. The derivative of the firing sequence $\boldsymbol{f}(\tau) = \dot{\boldsymbol{\sigma}}(\tau)$ is called the firing flow. Depending on how the flow is defined, many firing semantics appear, being the most used ones *infinite* and *finite* server semantics. This paper deals with infinite server semantics for which the flow of a transition t_j is defined as:

$$f_j(\tau) = \lambda_j \cdot enab(t_j, \boldsymbol{m}(\tau)) = \lambda_j \cdot \min_{p_i \in {}^\bullet t_j} \left\{ \frac{m_i(\tau)}{Pre_{ij}} \right\} \quad (1)$$

Since the flow of a transition depends on its enabling degree which is based on the minimum function, a timed contPN with infinite server semantics is a PWL system with polyhedral regions and everywhere continuous vector field. The state space (or reachability set) $\mathcal{R}$ of a contPN system can be divided into regions [1] as follows: $\mathcal{R} = \mathcal{R}^1 \cup ... \cup \mathcal{R}^\gamma$ where $\gamma \leq \prod_{j=1}^{|T|} |{}^\bullet t_j|$. Intuitively, each $\mathcal{R}^z$ denotes a region where the flow is limited by the same subset of places (one for each transition). More formally, two markings m_a and m_b belong to the same region $\mathcal{R}^z$ iff $[\forall\ t \in T$ and $\forall\ p_u, p_v \in {}^\bullet t$ it holds that $m_a(p_u) \leq m_a(p_v) \leftrightarrow m_b(p_u) \leq m_b(p_v)]$.

For a given $\mathcal{R}^z$, we can define the constraint matrix $\boldsymbol{\Pi}^z \in \mathbb{R}_{\geq 0}^{|P|}$ as follows. Let $\boldsymbol{m}$ be an interior point of $\mathcal{R}^z$, then $\boldsymbol{\Pi}^z$ is defined as:

$$\boldsymbol{\Pi}^z_{ji} = \begin{cases} \dfrac{1}{Pre_{ij}}, \ if \ \dfrac{m_i}{Pre_{ij}} = \min\limits_{p_h \in {}^\bullet t_j} \left\{ \dfrac{m_h}{Pre_{hj}} \right\} \\ 0, \qquad \text{otherwise} \end{cases} \quad (2)$$

If marking $\boldsymbol{m}$ belongs to $\mathcal{R}^z$, we denote $\boldsymbol{\Pi}(\boldsymbol{m}) = \boldsymbol{\Pi}^z$ the corresponding constraint matrix (if $\boldsymbol{m}$ is a border point and belongs to several regions, $\boldsymbol{\Pi}(\boldsymbol{m})$ is the constraint matrix of any of these regions). In the constraint matrix, each row $j = 1, 2..., |T|$ has only one non-null element in the position i that corresponds to the place p_i that restricts the flow of transition t_j. Furthermore, the firing rate of transitions can also be represented by a diagonal matrix $\boldsymbol{\Lambda} = \text{diag}\{\lambda_1, ...\lambda_{|T|}\}$. Using this notation, the nonlinear flow of the transitions at a given state $\boldsymbol{m}$ can be written as:

$$\boldsymbol{f} = \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m}) \cdot \boldsymbol{m} \quad (3)$$

Example 3. Let us consider the PN in Fig. 1. Assume $\boldsymbol{\lambda} = [4\ 1\ 3\ 1]^T$, $\boldsymbol{m}_0 = [7.5\ 4.5\ 4\ 2\ 1.5\ 5\ 4\ 2.5]^T$ and $\boldsymbol{m}_f = [7\ 5\ 5\ 1\ 1\ 4\ 5\ 3]^T$.

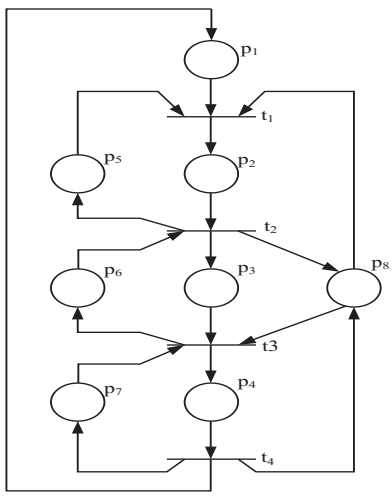


Fig. 1. A PN taken from (Zhou et al. (1990))

[1] These regions are disjoint except possibly on the borders.

Independently of $\boldsymbol{m}_0$, the system dynamics is described as follows:

$$\begin{aligned}
\dot{m}_1 &= f_4 - f_1 = \lambda_4 \cdot m_4 - \lambda_1 \cdot \min\{m_5, m_1, m_8\} \\
\dot{m}_2 &= f_1 - f_2 = \lambda_1 \cdot \min\{m_5, m_1, m_8\} - \lambda_2 \cdot \min\{m_2, m_6\} \\
\dot{m}_3 &= f_2 - f_3 = \lambda_2 \cdot \min\{m_2, m_6\} - \lambda_3 \cdot \min\{m_3, m_7, m_8\} \\
\dot{m}_4 &= f_3 - f_4 = \lambda_3 \cdot \min\{m_3, m_7, m_8\} - \lambda_4 \cdot m_4 \\
\dot{m}_5 &= f_2 - f_1 = \lambda_2 \cdot \min\{m_2, m_6\} - \lambda_1 \cdot \min\{m_5, m_1, m_8\} \\
\dot{m}_6 &= f_3 - f_2 = \lambda_3 \cdot \min\{m_3, m_7, m_8\} - \lambda_2 \cdot \min\{m_2, m_6\} \\
\dot{m}_7 &= f_4 - f_3 = \lambda_4 \cdot m_4 - \lambda_3 \cdot \min\{m_3, m_7, m_8\} \\
\dot{m}_8 &= f_2 + f_4 - f_1 - f_3 = \lambda_2 \cdot \min\{m_2, m_6\} + \lambda_4 \cdot m_4 \\
&\quad -\lambda_1 \cdot \min\{m_5, m_1, m_8\} - \lambda_3 \cdot \min\{m_3, m_7, m_8\}
\end{aligned} \tag{4}$$

Note that $\boldsymbol{m}_0$ and $\boldsymbol{m}_f$ are in different regions: $\boldsymbol{m}_0 \in \mathcal{R}^1 : \{\boldsymbol{m} \mid m_5 \leq m_1,\ m_5 \leq m_8,\ m_8 \leq m_7,\ m_8 \leq m_3,\ m_2 \leq m_6\}$ and $\boldsymbol{m}_f \in \mathcal{R}^2 : \{\boldsymbol{m} \mid m_5 \leq m_1,\ m_5 \leq m_8,\ m_8 \leq m_7,\ m_8 \leq m_3,\ m_6 \leq m_2\}$.

For example, the system dynamics for $\mathcal{R}^1$ are:

$$\begin{cases}
\dot{m}_1 = m_4 - 4 \cdot m_5 \\
\dot{m}_2 = 4 \cdot m_5 - m_2 \\
\dot{m}_3 = m_2 - 3 \cdot m_8 \\
\dot{m}_4 = 3 \cdot m_8 - m_4 \\
\dot{m}_5 = m_2 - 4 \cdot m_5 \\
\dot{m}_6 = 3 \cdot m_8 - m_2 \\
\dot{m}_7 = m_4 - m_8 \\
\dot{m}_8 = m_2 + m_4 - 4 \cdot m_5 + 3 \cdot m_8
\end{cases} \tag{5}$$

2.2 Controlled Timed Continuous Petri Nets

In this section we consider contPN systems subject to external control actions, and assume that the only admissible control law consists in *slowing-down* the firing speed of transitions (Silva and Recalde (2004)). If a transition can be controlled (its flow can be reduced or even stopped), we will say that it is a controllable transition (Mahulea et al. (2008b)).

Definition 4. The controlled flow, $\boldsymbol{w}$, of a timed contPN is defined as $\boldsymbol{w}(\tau) = \boldsymbol{f}(\tau) - \boldsymbol{u}(\tau)$, with $0 \leq \boldsymbol{u}(\tau) \leq \boldsymbol{f}(\tau)$, where $\boldsymbol{f}$ is the flow of the uncontrolled system, i.e., defined as in (1), and $\boldsymbol{u}$ is the control action.

Therefore, the control input $\boldsymbol{u}$ is dynamically upper bounded by the flow $\boldsymbol{f}$ of the corresponding unforced system. Under these conditions, the overall behaviour of the system in which all transitions are controllable is ruled by the following system:

$$\begin{aligned}
&\dot{\boldsymbol{m}} = \boldsymbol{C} \cdot [\boldsymbol{f} \ - \ \boldsymbol{u}] = \boldsymbol{C} \cdot \boldsymbol{w} \\
&0 \leq \boldsymbol{u} \leq \boldsymbol{f}
\end{aligned} \tag{6}$$

The constraint $0 \leq \boldsymbol{u} \leq \boldsymbol{f}$ can be rewritten as $0 \leq \boldsymbol{f} - \boldsymbol{u} \leq \boldsymbol{f}$. Defining $\boldsymbol{w} = \boldsymbol{f} - \boldsymbol{u}$ and using (3), the constraint can be expressed as:

$$0 \ \leq \ \boldsymbol{w} \ \leq \ \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m}) \cdot \boldsymbol{m} \tag{7}$$

The following sections show how to compute a control action $\boldsymbol{u}$ that drives the system from the initial marking $\boldsymbol{m}_0$ to a desired target marking $\boldsymbol{m}_f$. Section 3 describes a method to obtain a linear trajectory. Section 4 makes use of such a method to compute piecewise linear trajectories in order to improve the time to reach $\boldsymbol{m}_f$. Notice that in order to be reachable, $\boldsymbol{m}_f$ necessarily satisfies the state equation, i.e., $\boldsymbol{m}_f = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}$. Our procedure consists of assigning constant controlled flow $\boldsymbol{w}$ that satisfies dynamic upper bounds. We assume that $\boldsymbol{m}_0$ and $\boldsymbol{m}_f$ are interior points of the reachability space, i.e., the markings are strictly positive. The assumption that $\boldsymbol{m}_0$ is positive ensures that the system can move at $\tau = 0$ in the direction of $\boldsymbol{m}_f$; the assumption that $\boldsymbol{m}_f$ is positive ensures that $\boldsymbol{m}_f$ can be reached (Júlvez et al. (2003)) in finite time (Mahulea et al. (2008b)).

3. COMPUTATION OF LINEAR TRAJECTORIES

In this section, we will distinguish two cases: (A) $\boldsymbol{m}_0$ and $\boldsymbol{m}_f$ are in the same region and (B) they are in different regions. In this last case the crossing points of the straight line and borders must be considered.

(A) $\boldsymbol{m}_0$ and $\boldsymbol{m}_f$ are in the same region $\mathcal{R}^z$. Then all the states on the straight line connecting them are also interior points of $\mathcal{R}^z$ since all the regions are convex. The following LPP computes the correspondig control law, where $\boldsymbol{x} = \boldsymbol{w} \cdot \tau_f$,

$$\begin{aligned}
&\min\ \tau_f \\
&\quad \boldsymbol{m}_f = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{x} \qquad (a) \\
&\quad 0 \leq x_j \leq \lambda_j \cdot \Pi^z_{ji} \cdot \min\left\{m_{0i}, m_{f_i}\right\} \cdot \tau_f, \\
&\qquad \forall\ j \in \{1, .., |T|\} \text{ where } i \text{ satisfies } \Pi^z_{ji} \neq 0 \quad (b)
\end{aligned} \tag{8}$$

The equations correspond to: (a) the time dependent equation of the straight line connecting $\boldsymbol{m}_0$ to $\boldsymbol{m}_f$, (b) flow constraints in (7). Notice that (8)(b) is a linear constraint because m_{0_i} and m_{f_i} are known.

(B) $\boldsymbol{m}_0$ and $\boldsymbol{m}_f$ are in different regions. The line connecting $\boldsymbol{m}_0$ and $\boldsymbol{m}_f$ can be divided in several segments, each one starting in a border (or in $\boldsymbol{m}_0$ for the first segment) and ending in a border (or in $\boldsymbol{m}_f$ for the last one). Then LPP (8) is applied on each of these segments.

Algorithm 1 computes the total time τ_f by using LPP in (8) for the linear trajectory ℓ from $\boldsymbol{m}_0$ to $\boldsymbol{m}_f$ which crosses $n \in \{1, 2, .., \gamma\}$ borders. In this algorithm, $\boldsymbol{m}_c^i$ stands for state at the intersection of ℓ and i^{th} crossed border (starting from $\boldsymbol{m}_0$) $i \in \{1, 2, .., n\}$. $\boldsymbol{m}_c^0$ and $\boldsymbol{m}_c^{n+1}$ denote $\boldsymbol{m}_0$ and $\boldsymbol{m}_f$, respectively. $\boldsymbol{w}^i$ is the flow obtained from the state $\boldsymbol{m}_c^i$ to the state $\boldsymbol{m}_c^{i+1}$. Note that, if ℓ does not cross any border, then $n = 0$, that is $\boldsymbol{m}_c^0 = \boldsymbol{m}_0$ and $\boldsymbol{m}_c^1 = \boldsymbol{m}_f$.

Algorithm 1 Linear trajectory

Input: $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$, $\boldsymbol{m}_f$
Compute the line ℓ connecting $\boldsymbol{m}_0$ and $\boldsymbol{m}_f$
Compute the intersection of ℓ and the crossed borders: $\boldsymbol{m}_c^1$, $\boldsymbol{m}_c^2$, ...$\boldsymbol{m}_c^n$
$\boldsymbol{m}_c^0 = \boldsymbol{m}_0$, $\boldsymbol{m}_c^{n+1} = \boldsymbol{m}_f$
for $i = 0$ to n **do**
  Determine τ_i by solving LPP in (8) with $\boldsymbol{m}_0 = \boldsymbol{m}_c^i$ and $\boldsymbol{m}_f = \boldsymbol{m}_c^{i+1}$
end for
Output: $\tau_1, \boldsymbol{w}^1, \tau_2, \boldsymbol{w}^2, ..., \tau_{n+1}, \boldsymbol{w}^{n+1}$, $\tau_f = \sum_{i=1}^{n+1} \tau_i$

Proposition 5. Let $\langle \mathcal{N}, \boldsymbol{\lambda},\ \boldsymbol{m}_0 \rangle$ be a contPN system with $\boldsymbol{m}_0 > 0$. If $\boldsymbol{m}_f$ belongs to $\mathcal{R}$ and $\boldsymbol{m}_f > 0$:

- LPP(8) is feasible
- The control action given by Algorithm 1 ensures that $\boldsymbol{m}_f$ is reached in finite time.

Proof. Since $\boldsymbol{m}_f$ is a reachable marking, then there exists $\boldsymbol{x}$ such that the state equation is satisfied 8(a).

By taking τ_f sufficiently large 8(b) can be satisfied since $\lambda_j \cdot \Pi^z_{ji} \cdot \min\{m_{0i}, m_{fi}\} > 0$. Then, by (Júlvez et al. (2003)) the linear trajectory from $\boldsymbol{m}_0$ to $\boldsymbol{m}_f$ can be followed by the system since $\boldsymbol{m}_0 > 0$ and $\boldsymbol{m}_f > 0$, i.e. , all intermediate states are not spurious. Moreover, since $\boldsymbol{m}_f > 0$, by (Mahulea et al. (2008b)) $\boldsymbol{m}_f$ can be reached in finite time.

Example 6. Let us consider the control law design for the contPN in Ex. 3. Assume the same $\boldsymbol{\lambda}$, $\boldsymbol{m}_0$ and $\boldsymbol{m}_f$. Note that the line ℓ connecting $\boldsymbol{m}_0$ and $\boldsymbol{m}_f$ crosses only one border: $m_2 = m_6$ and the crossing point is calculated as: $\boldsymbol{m}_c^1 = [7.33\ 4.67\ 4.33\ 1.67\ 1.33\ 4.67\ 4.33\ 2.67]^T$, $\boldsymbol{m}_c^0 = \boldsymbol{m}_0$, $\boldsymbol{m}_c^2 = \boldsymbol{m}_f$. The trajectory is illustrated in Fig.2. According

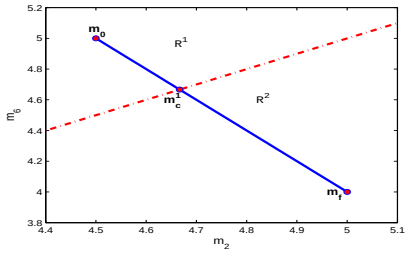


Fig. 2. Trajectory for Ex. 6

to Algorithm 1, first LPP (8) with $\boldsymbol{m}_0 = \boldsymbol{m}_c^0$ and $\boldsymbol{m}_f = \boldsymbol{m}_c^1$ is solved. Its constraints are:

$$\begin{aligned}
7.33 &= x_4 - x_1 + 7.5\\
4.67 &= x_1 - x_2 + 4.5\\
4.33 &= x_2 - x_3 + 4\\
1.67 &= x_3 - x_4 + 2\\
1.33 &= x_2 - x_1 + 1.5\\
4.67 &= x_3 - x_2 + 5\\
4.33 &= x_4 - x_3 + 4\\
2.67 &= x_2 + x_4 - x_1 - x_3 + 2.5
\end{aligned} \qquad (a)$$

$$\begin{aligned}
0 &\le x_1 \le 4 \cdot 1.33 \cdot \tau_f\\
0 &\le x_2 \le 4.5 \cdot \tau_f\\
0 &\le x_3 \le 3 \cdot 2.5\tau_f\\
0 &\le x_4 \le 1.66 \cdot \tau_f
\end{aligned} \qquad (b)$$

The optimal solution is $\tau_f = 0.2$ t.u. and $w_1 = 2.49$, $w_2 = 1.67$, $w_3 = 0$, $w_4 = 1.67$. In the second case, i.e., $\boldsymbol{m}_0 = \boldsymbol{m}_c^1$ and $\boldsymbol{m}_f = \boldsymbol{m}_c^2$:

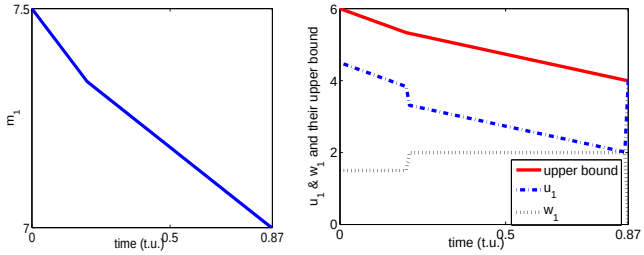
$$\begin{aligned}
7 &= x_4 - x_1 + 7.33\\
5 &= x_1 - x_2 + 4.67\\
5 &= x_2 - x_3 + 4.33\\
1 &= x_3 - x_4 + 1.67\\
1 &= x_2 - x_1 + 1.33\\
4 &= x_3 - x_2 + 4.67\\
5 &= x_4 - x_3 + 4.33\\
3 &= x_2 + x_4 - x_1 - x_3 + 2.67
\end{aligned} \qquad (a)$$

$$\begin{aligned}
0 &\le x_1 \le 4 \cdot \tau_f\\
0 &\le x_2 \le 4 \cdot \tau_f\\
0 &\le x_3 \le 3 \cdot 2.67 \cdot \tau_f\\
0 &\le x_4 \le \tau_f
\end{aligned} \qquad (b)$$

The optimal solution is $\tau_f = 0.67$ t.u. and $w_1 = 1.5$, $w_2 = 1$, $w_3 = 0$, $w_4 = 1$. The total time to go from $\boldsymbol{m}_0$ to $\boldsymbol{m}_f$ through the line ℓ is $\tau_{total} = 0.2 + 0.67 = 0.87$ t.u. And the control is obtained as:

$$\boldsymbol{u}(\tau) = \begin{cases} \begin{bmatrix} 4 \cdot m_5(\tau) - 2.49 \\ m_2(\tau) - 1.67 \\ 3 \cdot m_8(\tau) \\ m_4(\tau) - 1.67 \end{bmatrix}, \text{if } \ 0 \le \tau \le 0.2 \\ \begin{bmatrix} 4 \cdot m_5(\tau) - 1.5 \\ m_2(\tau) - 1 \\ 3 \cdot m_8(\tau) \\ m_4(\tau) - 1 \end{bmatrix}, \text{if } \ 0.2 < \tau \le 0.87 \end{cases} \qquad (9)$$

Fig. 3(a) illustrates the convergence of the marking m_1 under the designed control law, while Fig. 3(b) shows the control signal u_1, w_1 and their upper bound.



(a) (b)

Fig. 3. Evolution of (a) m_1 and (b) u_1, w_1 and and their upper bound for Ex. 6

4. A HEURISTICS FOR MINIMUM TIME CONTROL

This section illustrates how to compose a piecewise linear trajectory by introducing intermediate states in order to improve the time duration to reach $\boldsymbol{m}_f$.

The designed control in the previous subsection takes the system from $\boldsymbol{m}_0$ to $\boldsymbol{m}_f$ through a linear trajectory by minimizing the time duration. In some cases, i.e., when initial or final state have a small value, this may limit the speed of the response. In order to improve the time spent to move from $\boldsymbol{m}_0$ to $\boldsymbol{m}_f$, intermediate states denoted by $\boldsymbol{m}^k$ where $k \in \{1, 2...s\}$, that do not necessarily lie on the line from $\boldsymbol{m}_0$ to $\boldsymbol{m}_f$, are computed. The new trajectory is obtained as the union of $s+1$ segments: $\boldsymbol{m}_0 \to \boldsymbol{m}^1 \to \boldsymbol{m}^2 \to ... \to \boldsymbol{m}^s \to \boldsymbol{m}_f$. In this work we consider the intermediate states in the range:

$$\mathcal{R}^I = \Big\{ \boldsymbol{m} \mid m_i \le \max\{m_{f_i}, m_{0_i}\}, \quad m_i \ge \min\{m_{f_i}, m_{0_i}\}, \\ i \in \{1, 2, .., |P|\} \Big\}, \qquad (10)$$

Because $\boldsymbol{m}_0$, $\boldsymbol{m}_f > 0$, all $\boldsymbol{m} \in \mathcal{R}^I$ are positive and reachable (Júlvez et al. (2003)). We will describe how to compute intermediate states that lie on the borders and the interior of regions, separately.

4.1 Intermediate states on the borders

Let the line from $\boldsymbol{m}_0$ to $\boldsymbol{m}_f$ cross s borders, and pass through $s+1$ different regions: $\mathcal{R}^1$, $\mathcal{R}^2$, ..., $\mathcal{R}^{s+1}$ with corresponding constraint matrices $\boldsymbol{\Pi}^1$, $\boldsymbol{\Pi}^2$...$\boldsymbol{\Pi}^{s+1}$. We assign constant flows for each transitions during each line segments and the intermediate states on each border: $\boldsymbol{m}^1$, $\boldsymbol{m}^2$....$\boldsymbol{m}^s$. Under these assumptions, the following

BLP with $\boldsymbol{m}^0 = \boldsymbol{m}_0$ and $\boldsymbol{m}^{s+1} = \boldsymbol{m}_f$ and with the variables $\boldsymbol{m}^1$, $\boldsymbol{m}^2$,....,$\boldsymbol{m}^{s-1}$, $\boldsymbol{m}^s$, τ_k , $\boldsymbol{x}^k = \boldsymbol{w}^k \cdot \tau_k$, $k \in \{1, ...s\}$ obtains the intermediate states that yield a minimum time trajectory:

$$\min \sum_{k=1}^{s+1} \tau_k$$

$$\boldsymbol{m}^{k+1} = \boldsymbol{m}^k + \boldsymbol{C} \cdot \boldsymbol{x}^{k+1}, \quad k \in \{0, 1, 2, .., s\} \qquad (a)$$

$$\left(\boldsymbol{\Pi}^k - \boldsymbol{\Pi}^{k+1}\right) \cdot \boldsymbol{m}^k = 0, \quad k \in \{1, 2, .., s\} \qquad (b)$$

$$m_i^k \leq m_i^{k+1} \text{ if } m_{0_i} \leq m_{f_i} \quad i \in \{1, 2..|P|\}, \quad k \in \{0, 1, 2, .., s\} \qquad (c)$$

$$m_i^k \geq m_i^{k+1} \text{ if } m_{0_i} \geq m_{f_i} \quad i \in \{1, 2..|P|\}, \quad k \in \{0, 1, 2, .., s\} \qquad (d)$$

$$0 \leq x_j^k \leq \lambda_j \cdot \Pi_{ji}^k \cdot \min\{m_i^{k-1}, m_i^k\} \cdot \tau_k, \text{ with } p_i \text{ st. } \Pi_{ji}^k \neq 0,\ j \in \{1, 2...|T|\}, \quad k \in \{1, 2..s\} \qquad (e)$$

(11)

The constraints correspond to (a) the time dependent equation of the connecting $\boldsymbol{m}^k$ to $\boldsymbol{m}^{k+1}$: $\boldsymbol{m}^{k+1} = \boldsymbol{m}^k + \boldsymbol{C} \cdot \boldsymbol{w}^{k+1} \cdot \tau_f$ and $\boldsymbol{m}^k$ is a reachable marking; (b) $\boldsymbol{m}^k$ is on the crossed border; (c & d) $\boldsymbol{m}^k$ is contained in the hypercube defined by the corners $\boldsymbol{m}^{k-1}$ and $\boldsymbol{m}^{k+1}$; (e) the constraints on the controlled flow.

4.2 Interior intermediate states

In the case that $\boldsymbol{m}_0$ and $\boldsymbol{m}_f$ are in the same region $\mathcal{R}^z$ a recursive method can be used to determine intermediate states. The same recursive method can also be applied to find additional intermediate states between $\boldsymbol{m}^k$ and $\boldsymbol{m}^{k+1}$, since they are on the borders of the same region. In this method we keep computing intermediate states until the time cannot be improved by a considerable amount. In this case, the BLP that will be solved in each step can be simplified to the following problem where $\boldsymbol{m}_0$ and $\boldsymbol{m}_f$ are known; τ_1, τ_2, $\boldsymbol{m}^d$, $\boldsymbol{x}^1 = \boldsymbol{w}^1 \cdot \tau_1, \boldsymbol{x}^2 = \boldsymbol{w}^2 \cdot \tau_2$ are variables:

$$\min\ \tau_1 + \tau_2$$

$$\begin{aligned} &\boldsymbol{m}^d = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{x}^1 \\ &\boldsymbol{m}_f = \boldsymbol{m}^d + \boldsymbol{C} \cdot \boldsymbol{x}^2, \end{aligned} \qquad (a)$$

$$\boldsymbol{m}^d = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}, \quad \boldsymbol{m}^d,\ \boldsymbol{\sigma} \geq 0 \qquad (b)$$

$$\min\{m_{f_i}, m_{0_i}\} \leq m_i^d \leq \max\{m_{f_i}, m_{0_i}\}, \quad \forall i \in \{1, 2...|P|\} \qquad (c)$$

$$\begin{aligned} &0 \leq x_j^1 \leq \lambda_j \cdot \Pi_{ji}^z \cdot \min\{m_{0_i}, m_i^d\} \cdot \tau_1, \text{ with } i \text{ st. } \Pi_{ji}^k \neq 0,\ j \in \{1, 2...|T|\} \\ &0 \leq x_j^2 \leq \lambda_j \cdot \Pi_{ji}^z \cdot \min\{m_i^d, m_{f_i}\} \cdot \tau_2, \text{ with } i \text{ st. } \Pi_{ji}^k \neq 0,\ j \in \{1, 2...|T|\} \end{aligned} \qquad (d)$$

(12)

Algorithm 2 expresses the recursive method we proposed. It computes new intermediate states until the stopping criterion is satisfied. It is satisfied when the time of the trajectory cannot be decreased more than a given value ϵ. In the algorithm, *path* is a global variable storing the ordered set of couples of states in the trajectory and the relative times to reach them. The number of states in *path* is denoted by $size(path)$ and the i^{th} element of the set *path* is denoted by $path(i)$.

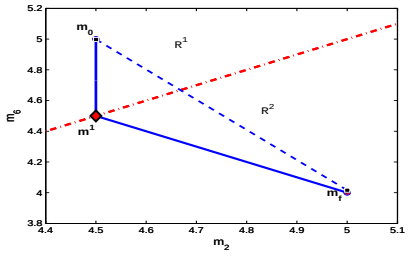
Algorithm 2 Piecewise Linear Trajectory

Input: $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$, $\boldsymbol{m}_f$, ϵ
Global variable: *path*
Compute the line ℓ connecting $\boldsymbol{m}_0$ and $\boldsymbol{m}_f$
Determine the number of crossed borders: s
if $s = 0$ **then**
  Solve (8) to obtain τ_f
  $path_0 = \{(\boldsymbol{m}_0, 0),\ (\boldsymbol{m}_f, \tau_f)\}$
else
  Solve (11) to obtain τ_f
  $path_0 = \{(\boldsymbol{m}_0, 0),\ (\boldsymbol{m}^1, \tau_1), ..., (\boldsymbol{m}_f, \tau_f)\}$
end if
$\tau_x = \tau_f$, $path = path_0$
for $i = 1 : size(path_0) - 1$ **do**
  $(\boldsymbol{m}^x, \tau_x) = path_0(i)$, $(\boldsymbol{m}^y, \tau_y) = path_0(i+1)$
  Call Split function with $((\boldsymbol{m}^x, \tau_x),\ (\boldsymbol{m}^y, \tau_y))$
end for
Calculate the total time for *path*, i.e., τ_y
Output:*path*

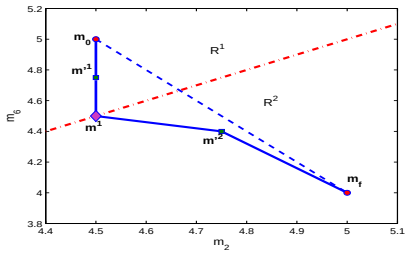
Split function

Input: $(\boldsymbol{m}^x, \tau_x),\ (\boldsymbol{m}^y, \tau_y)$
Solve (12) to obtain $\tau_1,\ \tau_2,\ \boldsymbol{m}^d$
if $\tau_y - (\tau_1 + \tau_2) > \epsilon$ **then**
  Insert $(\boldsymbol{m}^d, \tau_1)$ in *path*
  Change $(\boldsymbol{m}^y, \tau_y)$ in *path* by $(\boldsymbol{m}^y, \tau_2)$
  Call Split function with $((\boldsymbol{m}^x, \tau_x),\ (\boldsymbol{m}^d, \tau_1))$
  Call Split function with $((\boldsymbol{m}^d, \tau_1),\ (\boldsymbol{m}^y, \tau_2))$
end if

Example 7. Let us consider the contPN system considered in Ex. 6, again. Since the linear trajectory between $\boldsymbol{m}_0$ and $\boldsymbol{m}_f$ crosses only one border, $s = 1$ and solving (11) with $s = 1$ yields 0.83 t.u. as a total time. And the new path is $\boldsymbol{m}_0 \rightarrow \boldsymbol{m}^1 \rightarrow \boldsymbol{m}_f$, where $\boldsymbol{m}^1 = [7.5\ 4.5\ 4.5\ 1.5\ 1.5\ 4.5\ 4.5\ 3]^T$. For an additional intermediate state, say $\boldsymbol{m'}^1$, between $\boldsymbol{m}_0$ and $\boldsymbol{m}^1$, we solve (12) for $\boldsymbol{m}_0 = [7.5\ 4.5\ 4\ 2\ 1.5\ 5\ 4\ 2.5]^T$ and $\boldsymbol{m}_f = \boldsymbol{m}^1$. Similarly, for an additional intermediate state between $\boldsymbol{m}^1$ and $\boldsymbol{m}_f$, say $\boldsymbol{m'}^2$, we solve (12) for $\boldsymbol{m}_0 = \boldsymbol{m}^1$ and $\boldsymbol{m}_f = [7\ 5\ 5\ 1\ 1\ 4\ 5\ 3]^T$. And the new path $\boldsymbol{m}_0 \rightarrow \boldsymbol{m'}^1 \rightarrow \boldsymbol{m}^1 \rightarrow \boldsymbol{m'}^2 \rightarrow \boldsymbol{m}_f$ is obtained, where $\boldsymbol{m'}^1 = [7.5\ 4.5\ 4.26\ 1.73\ 1.5\ 4.73\ 4.26\ 2.76]^T$ and $\boldsymbol{m'}^2 = [7.45\ 4.77\ 4.50\ 1.22\ 1.22\ 4.45\ 4.77\ 3]^T$. The total time is reduced to 0.74 t.u. with the control action $u(\tau) = [4 \cdot m_5(\tau) - 1.73\ \ m_2(\tau) - 1.73\ \ 3 \cdot m_8(\tau)\ \ m_4(\tau) - 1.7321]^T$ for $0 \leq \tau \leq 0.15$; $u(\tau) = [4 \cdot m_5(\tau) - 1.5\ \ m_2(\tau) - 1.5\ \ 3 \cdot m_8(\tau)\ \ m_4(\tau) - 1.5]^T$ for $0.15 \leq \tau \leq 0.3$; $u(\tau) = [4 \cdot m_5(\tau) - 1.23\ \ m_6(\tau)\ \ 3 \cdot m_8(\tau)\ \ m_4(\tau) - 1.2247]^T$ for $0.3 \leq \tau \leq 0.51$; $u(\tau) = [4 \cdot m_5(\tau) - 3.21\ \ m_6(\tau) - 2.21\ \ 3 \cdot m_8(\tau)\ \ m_4(\tau) - 1]^T$ for $0.51 \leq \tau \leq 0.74$. The trajectories for one and three intermediate states are illustrated in Fig. 4(a) and Fig. 4(b), respectively. In these figures dotted line shows the border between $\mathcal{R}^1$ and $\mathcal{R}^2$. The total time duration for several intermediate states can be found in Table 1. The experiments were performed by a Matlab program running on a PC with Intel(R) Core(TM)2CPU T5600 @ 1.83GHz, 2.00 GB of RAM.



(a)



(b)

Fig. 4. Trajectory for (a)1 int. state and (b)3 int. state for Ex. 7

Table 1. Intermediate States

Number of int. states	0	1	3	9	13
Total duration (t.u.)	0.87	0.83	0.74	0.73	0.72
CPU time (sec.)	0.03	0.11	0.32	1.65	2.32

Example 8. Let us consider the net in Fig. 5 with $\boldsymbol{\lambda} = [3\ 1\ 1]^T$. By using Algorithm 1, the total time to reach $\boldsymbol{m}_f = [1\ 10\ 6\ 2]^T$ from $\boldsymbol{m}_0 = [13\ 3\ 1\ 10]^T$ through a linear trajectory is obtained as 2.43 t.u., with the control $u(\tau) = [3 \cdot m_4(\tau) - 7 \quad m_2(\tau) \quad m_3(\tau) - 1]^T$ for $0 \leq \tau \leq 0.93$, $u(\tau) = [3 \cdot m_4(\tau) - 12 \quad m_4(\tau) \quad m_3(\tau) - 1.71]^T$ for $0.93 \leq \tau \leq 1.26$, $u(\tau) = [3 \cdot m_1(\tau) - 3 \quad m_2(\tau) \quad m_3(\tau) - 0.42]^T$ for $1.26 \leq \tau \leq 2.43$, while $\boldsymbol{m}_f$ was reached in 4.4 t.u. in (Xu et al. (2008)). The piecewise linear trajectory with 4 intermediate states obtained by Algorithm 2 yields 1.38 t.u.

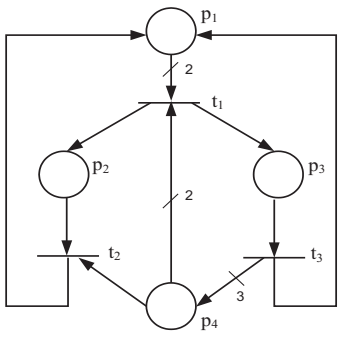


Fig. 5. A PN

5. CONCLUSION

The control problem adressed in the paper consists of reaching a target state in minimum time through a piecewise linear trajectory. Besides the piecewise linear dynamics of continuous Petri nets, the control method handles the fact that the input constraints depend on the current marking, i.e., the inputs are dynamically constrained.

The heuristics proposed in this paper computes first a "rough" piecewise linear trajectory that is refined afterwards in those intervals that allow an improvement. The heuristics makes use of BPPs to obtain intermediate states and LPPs to compute linear trajectories. The refinement of the trajectory is achieved recursively .

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