Steady state control reference and token conservation laws in continuous Petri net systems

Cristian Mahulea, Antonio Ramírez-Treviño, Laura Recalde, and Manuel Silva *†

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Abstract

This paper addresses several questions related to the control of timed continuous Petri Nets under infinite server semantics. First, some results regarding equilibrium states and control actions are given. In particular, it is shown that the considered systems are piecewise linear, and for every linear subsystem the possible steady-states are characterized. Second, optimal steady-state control is studied, a problem that surprisingly can be computed in polynomial time, when all transitions are controllable and the objective function is linear. Third, an interpretation of some controllability aspects in the framework of linear dynamic systems is presented. An interesting finding is that non controllable poles are zero valued.

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*C. Mahulea, L. Recalde and M. Silva are with Instituto de Investigación en Ingeniería de Aragón (I3A) and Departamento de Informática e Ingeniería de Sistemas, Universidad de Zaragoza, María de Luna 1, Zaragoza 50018, Spain, email: {cmahulea,lrecalde,silva}@unizar.es.

†A. Ramírez-Treviño is with CINVESTAV Unidad Guadalajara, Guadalajara 45010, México, email: art@gdl.cinvestav.mx.

1 Introduction

Continuous Petri Nets (contPNs) [**?**] [**?**] [**?**] appear as a promising approach to model, analyze and deal with some synthesis problems of discrete event systems in a relaxed setting. The goal of the relaxation is to obtain more efficient algorithms. For instance, reachability can be characterized in polynomial time under quite general conditions in the contPN model [**?**]. However, one must be careful since there are properties that cannot be captured by the contPN model. Properties like mutual exclusions or existence of home states are essentially *not* analyzable under fluidification. Moreover it is well known that for basic properties like boundedness, the continuous version only provides sufficient conditions for the original discrete case, while liveness of the continuous case is neither necessary nor sufficient for that of the underlying discrete model [?]. ContPNs have been mainly used in the domain of manufacturing [**?**, **?**, **?**], although others applications have been proposed, like traffic systems [**?**].

Time can be introduced in the model, leading to the so called timed contPN. In timed contPN two server semantics are mostly used. These are inspired from the discrete stochastic Petri nets, namely *finite server semantics* and *infinite server semantics* (see [?] for a more detailed discussion).

Under finite server semantics [?], also called *constant speed contPNs* [?], the flow (or throughput) of the transitions is piecewise-constant. A very similar formalism is considered in [**?**] where the optimal instantaneous flow is computed using a linear programming problem (LPP). Infinite server semantics [?], also called *variable speed continuous Petri nets* [**?**], considers that the flow of a transition is piecewise-linear.

In the literature, other semantics have been considered for timed continuous Petri nets. For example in [**?**, **?**] time is associated with places and (max,+) algebra can be used in some cases to describe the system evolution. The optimal steady-state is computed using dynamical programming [?] or linear programming [?]. In [**?**], an approximation of variable speed contPNs, called *asymptotic continuous Petri nets* is introduced where the instantaneous firing speed vector is piecewise-constant. Unfortunately, when the net has choices, the conflict resolution policy must be defined "a priori". Moreover, for an unsaturated system an oscillatory behavior may be obtained in steady-state.

Assuming that certain discrete event systems usually work close to congestion, and having evidence of the gains that in certain cases are obtained by fluidification, this paper deals with some control problems of timed continuous PNs under so called *infinite server semantics* (the continuous and first order deterministic approximation of markovian flows in discrete Petri Nets (PNs)). Under this firing semantics, due to the *min* operator occurring in the synchronizations, the continuous model is a multilinear switched dynamic system.

Starting with the crucial question of how to control, an approach based in the idea of *slowing down* the firing flow of transitions is considered [?]. A first question is which markings can be equilibrium points, assuming a constant control action $\boldsymbol{u_d}$ in steady-state. This paper explores this problem in Section 3. For some particular net subclasses, unique solutions are algebraically obtained (thus their characterization is complete). If several steady state markings appear, in many cases they produce the same flow in steady state. In particular, it may be computationally easy (polynomial time) to compute the (maximal) flow vector, even if several (even infinite) steady-state markings appear.

The second main contribution is the computation of an optimal steady state control reference, maximizing a linear profit function that takes into account the throughput, the initial marking and the steady state marking (see Section 4). If all transitions are controllable, this problem is solved in polynomial time, using a LPP. If some transitions are not controllable the problem becomes more complicated and a Branch & Bound algorithm can be used, like in [**?**].

In Section 5, a bridge between controllability in *classical linear theory* and Petri nets is established. The simplifying idea is to keep the fact that the dynamic model is multilinear, but ignore the constraints that must be respected by the action: non-negative and upper bounded by a function of the marking (state). It is shown that net systems generate different token conservation laws, some of them leading to uncontrollability. Some conservation laws are generated by the P-flows (which depend only on the net structure) and zero valued poles appear in the *uncontrollable* part of the system. Other zero valued *controllable* poles are related to conservation laws that depend on the net structure, the firing rates and the token load of P-(semi)flows. Finally, some *controllable* non zero poles may generate token conservation laws for particular values of $\boldsymbol{m_0}$.

2 Continuous Petri nets basics

2.1 Untimed continuous Petri nets

We assume that the reader is familiar with discrete PNs. Nevertheless, the PNs that will be considered in this paper are *continuous*, a straightforward relaxation of *discrete* ones. Unlike discrete PN, the amount in which a transition can be fired in contPN is not restricted to a natural number. A PN system is a pair $\langle\mathcal{N}, \boldsymbol{m_0}\rangle$, where $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post}\rangle$ is a P/T net (P and T are disjoint finite sets of places and transitions, and $\boldsymbol{Pre}$ and $\boldsymbol{Post}$ are $|P| \times |T|$ sized *incidence matrices*) and $\boldsymbol{m_0}$ is the *initial marking.* In contPNs, $\boldsymbol{m_0}$ is a vector of non negative *real* numbers. For every node $v \in P \cup T$, the sets of its input and output nodes are denoted as ${}^{\bullet}v$ and $v^{\bullet}$, respectively.

A transition t is *enabled* at $\boldsymbol{m}$ iff $\forall p \in {}^{\bullet}t, m[p] > 0$. The *enabling degree* of t is enab$(t, \boldsymbol{m})$= $\min_{p\in{}^{\bullet}t}\{m[p]/\mathrm{Pre}[p, t]\} = \max\{k|k \cdot \mathrm{Pre}[\cdot, t] \leq \boldsymbol{m}\}$, and t can fire in a certain amount $\alpha \in \mathrm{R}$, $0 \leq \alpha \leq$ enab$(t, \boldsymbol{m})$ leading to a new marking $\boldsymbol{m'} = \boldsymbol{m} + \alpha \cdot \boldsymbol{C}[P, t]$, where $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the *token flow matrix.* If $\boldsymbol{m}$ is reachable from $\boldsymbol{m_0}$ through a sequence σ, a fundamental equation can be written: $\boldsymbol{m} = \boldsymbol{m_0} + \boldsymbol{C} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma} \in (\mathrm{R}^+ \cup \{0\})^{|T|}$ is the firing count vector. The set of all reachable markings is denoted by $RS_{untimed}(\mathcal{N}, \boldsymbol{m_0})$.

Proposition 1 *[?] A marking $\boldsymbol{m} \in RS_{untimed}(\mathcal{N}, \boldsymbol{m_0})$ iff*

1. $\boldsymbol{m} = \boldsymbol{m_0} + \boldsymbol{C} \cdot \boldsymbol{\sigma}$*,* $\boldsymbol{\sigma} \geq \boldsymbol{0}$
2. *a sequence can be fired from $\boldsymbol{m_0}$ that contains all the transitions in the support of $\boldsymbol{\sigma}$*
3. *there is no empty trap (a set of places Θ such that $\Theta^{\bullet} \subseteq {}^{\bullet}\Theta$) in $\mathcal{N}_{\boldsymbol{\sigma}}$ at $\boldsymbol{m}$*

where $\mathcal{N}_{\boldsymbol{\sigma}}$ denotes the subnet obtained from $\mathcal{N}$ removing the transitions not in the support of $\boldsymbol{\sigma}$ and the resulting isolated places.

A contPN is *bounded* when every place is bounded ($\forall p \in P, \exists b_p \in \mathrm{R}$ with $m[p] \leq b_p$ at every reachable marking $\boldsymbol{m}$). It is *live* when every transition is *live* (it can ultimately occur from every reachable marking). Reachability may be extended to *lim-reachability* assuming that infinitely long sequences can be fired. The set of all reachable markings at the limit is denoted by $lim - RS$. A transition t is *lim-live* iff no sequence of successively reachable markings exists which converges to a marking such that none of its successors enables t [?].

A net $\mathcal{N}$ is *structurally bounded* when $\langle\mathcal{N}, \boldsymbol{m_0}\rangle$ is bounded for every initial marking $\boldsymbol{m_0}$ and is *structurally live* when a $\boldsymbol{m_0}$ exists such that $\langle\mathcal{N}, \boldsymbol{m_0}\rangle$ is live. Left and right natural annullers of the token flow matrix $\boldsymbol{C}$ are called P- and T-semiflows, respectively. The net $\mathcal{N}$ is *conservative* iff

$\exists \boldsymbol{y} > 0,\ \boldsymbol{y} \cdot \boldsymbol{C} = 0$ and it is *consistent* iff $\exists \boldsymbol{x} > 0,\ \boldsymbol{C} \cdot \boldsymbol{x} = 0$. Left and right real annullers of matrix $\boldsymbol{C}$ are P- and T-flows, respectively. The support of a P- (T-) (semi)flow $\boldsymbol{s}$ is denoted by $||\boldsymbol{s}||$.

Proposition 2 *[?] [?] Let $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ be a contPN system. If it is consistent and all transitions are fireable the following statements are equivalent:*

1. *$\boldsymbol{m}$ is lim-reachable*
2. *$\exists \boldsymbol{\sigma} \geq 0$ s.t. $\boldsymbol{m} = \boldsymbol{m_0} + \boldsymbol{C} \cdot \boldsymbol{\sigma} \geq \boldsymbol{0}$*
3. *${\boldsymbol{B_y}}^T \cdot \boldsymbol{m} = {\boldsymbol{B_y}}^T \cdot \boldsymbol{m_0}$, $\boldsymbol{m} \geq 0$ where $\boldsymbol{B_y}$ is a basis of P-flows.*

Like in discrete case, nets can be classified according to their structure. A place p is Choice-Free (CF) iff $|p^\bullet| \leq 1$ (i.e. there is no routing, choice is structurally defined). A transition t is Join-Free (JF) iff $|{}^\bullet t| \leq 1$ (i.e. there is no synchronization on it). Two transitions, t and t', are said to be in Equal Conflict (EQ) relation when $\boldsymbol{Pre}[P,t] = \boldsymbol{Pre}[P,t'] \neq 0$. This is an equivalence relation and the set of all the equal conflict sets is denoted by SEQS.

Definition 3 *Let $\mathcal{N}$ be a PN.*

1. *$\mathcal{N}$ is* ordinary *iff all arcs have weight one.*
2. *$\mathcal{N}$ is* pure *iff $\forall t \in T, {}^\bullet t \cap t^\bullet = \emptyset$.*
3. *$\mathcal{N}$ is a* weighted T-graph *iff $\forall p \in P : |p^\bullet| = |{}^\bullet p| = 1$.*
4. *$\mathcal{N}$ is* Choice-Free *iff $\forall p \in P : |p^\bullet| \leq 1$.*
5. *$\mathcal{N}$ is* Join-Free *iff $\forall t \in T : |{}^\bullet t| \leq 1$.*
6. *$\mathcal{N}$ is* Equal Conflict *iff ${}^\bullet t \cap {}^\bullet t' \neq 0 \Rightarrow \boldsymbol{Pre}[P,t] = \boldsymbol{Pre}[P,t']$.*

In this paper we will mainly focus on bounded and lim-live net systems.

Proposition 4 *[?] A bounded and lim-live contPN is consistent, conservative and $rank(\boldsymbol{C}) \leq |SEQS| - 1$. (This is one of the so called* rank theorems*.)*

2.2 Timed continuous Petri Nets and infinite server semantics

In this section, timing constraints are added to contPN. Like in the discrete case, time can be associated with places, with transitions or with arcs. This paper assumes time associated with the transitions. Here we consider first-order approximations (only using the *average* value; i.e. noise-free) of the fluidified models.

Definition 5 *A timed contPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is the untimed contPN $\mathcal{N}$ together with a vector $\boldsymbol{\lambda} \in (\mathrm{R}^+)^{|T|}$, where $\lambda[t_i] = \lambda_i$ is the firing rate of transition t_i.*

Definition 6 *A timed contPN system is a tuple $\Sigma = \langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$, where $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is a timed contPN and $\boldsymbol{m_0}$ is the initial marking of the net.*

Now, the fundamental equation explicitly depends on time τ: $\boldsymbol{m}(\tau) = \boldsymbol{m_0} + \boldsymbol{C} \cdot \boldsymbol{\sigma}(\tau)$. Deriving it with respect to time we obtain: $\dot{\boldsymbol{m}}(\tau) = \boldsymbol{C} \cdot \dot{\boldsymbol{\sigma}}(\tau)$. Using the notation $\boldsymbol{f}(\tau) = \dot{\boldsymbol{\sigma}}(\tau)$ to represent the flow of the transitions with respect of time, the state equation becomes: $\dot{\boldsymbol{m}}(\tau) = \boldsymbol{C} \cdot \boldsymbol{f}(\tau)$ (to simplify notation, in the sequel time τ will not be written if no ambiguity exists).

Depending on the flow definition, there are many firing semantics. *Finite server* (or constant speed) and *infinite server* (or variable speed) [**?**] [?] are the more frequently used. In discrete Petri nets, infinite server semantics and finite server semantics are equivalent if the servers are made explicit. That is, finite server can be simulated with infinite server, hence the later is more general. This is not true in the continuous case. Therefore, if the servers are made explicit, the two semantics correspond to two different approximations of the discrete net system.

In [?], the authors have observed that infinite server semantics frequently provides a very good approximation of a discrete model. But net systems exist for which finite server semantics provides a better approximation of the discrete model. Nevertheless, for a broad class of nets it is formally proved in [**?**] that infinite server semantics always provides a better approximation than finite server semantics. In particular, for all net systems in figures 1, 4, 5, 7, 11, 12 and 13 of this paper, infinite server semantics is better approximation of the underlying discrete model than finite server semantics.

Thus, this paper is focused on *infinite server semantics.* The flow of each transition being defined by:

$$f_i = f[t_i] = \lambda_i \cdot enab(t_i, \boldsymbol{m}) = \lambda_i \min_{p_j \in {}^{\bullet}t_i} \left\{ \frac{m_j}{\mathrm{Pre}[p_j, t_i]} \right\} \tag{1}$$

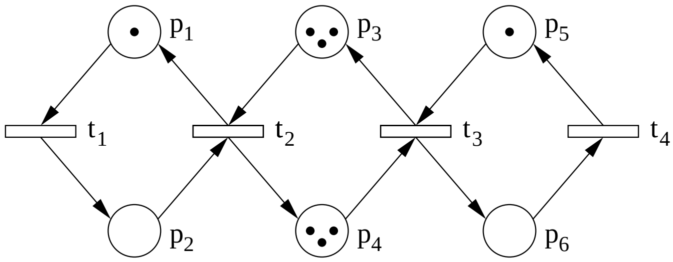

Figure 1: Timed contPN (marked graph) with several equilibrium points.

where, $m_j = m[p_j]$ is the marking of place p_j.

Remark 7 *Due to the minimum operator that appears in the flow definition, a timed contPN under infinite server semantics is a piecewise linear system.*

Example 8 *Let us consider the unforced net system in Figure 1. The flows of the transitions are given by:*

$$\begin{cases} f_1 = \lambda_1 \cdot m_1 \\ f_2 = \lambda_2 \cdot \min(m_2, m_3) \\ f_3 = \lambda_3 \cdot \min(m_4, m_5) \\ f_4 = \lambda_4 \cdot m_6 \end{cases}$$

If $\boldsymbol{\lambda} = [1, 1, 1, 1]^T$*, for example, we can write:*

$$\begin{cases} \dot{m}[p_1] = f_2 - f_1 = \min(m_2, m_3) - m_1 \\ \dot{m}[p_2] = f_1 - f_2 = m_1 - \min(m_2, m_3) \\ \dot{m}[p_3] = f_3 - f_2 = \min(m_4, m_5) - \min(m_2, m_3) \\ \dot{m}[p_4] = f_2 - f_3 = \min(m_2, m_3) - \min(m_4, m_5) \\ \dot{m}[p_5] = f_4 - f_3 = m_6 - \min(m_4, m_5) \\ \dot{m}[p_6] = f_3 - f_4 = \min(m_4, m_5) - m_6 \end{cases} \tag{2}$$

Thus, *nonlinearity* appears due to synchronizations ($|{}^{\bullet}t| > 1$). One linear system is defined by the set of arcs in $\boldsymbol{Pre}$ constraining the firing of the transitions.

Definition 9 *Let* $\Sigma = \langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$ *be a timed contPN system and* $\boldsymbol{m}$ *a reachable marking. It will be said that the arc* (p, t) constrains *the dynamic of* t *at* $\boldsymbol{m}$ *iff:* $f[t] = \lambda[t] \cdot \frac{m[p]}{Pre[p,t]}$.

Definition 10 *A* configuration *of Σ at $\boldsymbol{m}$ is a set of (p,t) arcs, one per transition, constraining the dynamic of the system.*

So, a configuration is a *cover* of T by its inputs arcs. Each configuration defines an embedded linear dynamic system. One possible representation of a given configuration is a $|T| \times |P|$ matrix:

$$\mathbf{\Pi}[t_j, p_i] = \begin{cases} \frac{1}{\mathrm{Pre}[p_i,t_j]} & \text{if } p_i \text{ is constraining the flow of } t_j \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The configuration operator associates to every marking $\boldsymbol{m}$ a matrix $|T| \times |P|$, such that each row $i = 1..|T|$ has only one non null element in the position j that corresponds to the place p_j that restricts the flow of transition t_i (if more than one place is restricting the flow, any of them can be used, but only one is taken). The matrix that represents the configuration of a marking $\boldsymbol{m}$ will be denoted as $\mathbf{\Pi}(\boldsymbol{m})$. The product $\mathbf{\Pi}(\boldsymbol{m}) \cdot \boldsymbol{m}(\tau)$ is the *enabling* degree of each transition at time τ, $\boldsymbol{e}(\tau)$.

Through this paper the notations $\mathbf{\Pi}(\boldsymbol{m_d})$ and $\mathbf{\Pi_d}$ will be used indistinctly. The *firing rate matrix* is denoted by: $\mathbf{\Lambda} = diag(\lambda_1, .., \lambda_{|T|})$.

Example 11 *Let us consider the net system in Figure 1 with $\boldsymbol{\lambda} = [1,1,1,1]^T$. As we saw in Example 8, this is a piecewise linear system. For $m_2 \leq m_3$ and $m_5 \leq m_4$ the active configuration is $\{(p_1,t_1),(p_2,t_2),(p_5,t_3),(p_6,t_4)\}$ and the corresponding linear system is:*

$$\begin{cases} \dot{m}_1 = m_2 - m_1 \\ \dot{m}_2 = m_1 - m_2 \\ \dot{m}_3 = m_5 - m_2 \\ \dot{m}_4 = m_2 - m_5 \\ \dot{m}_5 = m_6 - m_5 \\ \dot{m}_6 = m_5 - m_6 \end{cases}$$

or in matrix form:

$$\dot{\boldsymbol{m}} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \cdot \boldsymbol{m} \tag{4}$$

If $m_2 \leq m_3$ but $m_4 \leq m_5$ the active configuration is $\{(p_1, t_1), (p_2, t_2), (p_4, t_3), (p_6, t_4)\}$, *and the associated linear system is:*

$$\begin{cases} \dot{m}_1 = m_2 - m_1 \\ \dot{m}_2 = m_1 - m_2 \\ \dot{m}_3 = m_4 - m_2 \\ \dot{m}_4 = m_2 - m_4 \\ \dot{m}_5 = m_6 - m_4 \\ \dot{m}_6 = m_4 - m_6 \end{cases}$$

or in matrix form:

$$\dot{\boldsymbol{m}} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \cdot \boldsymbol{m} \tag{5}$$

Observe that, depending on the marking of the places, the evolution of the system will be given by one or other linear system. Equations 4 and 5 describe two of these different linear systems.

Any (reachable) marking $\boldsymbol{m}$ defines a configuration, $\boldsymbol{\Pi}(\boldsymbol{m})$. When the markings of several places are simultaneously limiting the firing of the same transition, any of the associated linear systems can be used.

Remark 12 *If $\mathcal{N}$ is JF then all arcs in $\boldsymbol{Pre}$ are constraining the dynamic of the full system (i.e. all those arcs are essential covers). Hence, there exist only one configuration.*

Independently of markings, an upper bound of the number of configurations can be computed taking one input arc per transition, thus $\gamma = \prod_{t_i \in T} |^{\bullet}t_i|$ different configurations can be structurally defined. Nevertheless, implicit places [?, **?**] can reduce the number of effective configurations. For nets in figures 1, 2, 4, 5, 7, 9 and 10: $\gamma_1 = \gamma_5 = \gamma_{10} = 4$, $\gamma_2 = 16$, $\gamma_4 = 1$, $\gamma_7 = 8$, $\gamma_9 = 2$.

Timed contPN systems evolve from $\boldsymbol{m_0}$ and may reach a steady state. For unforced contPN the computation of bounds for the steady state was studied in [?] and it is based on a branch and bound technique, each node corresponding to a LPP. Usually, the system evolves to a steady state, but for certain cases, oscillations can be maintained forever. For example the oscillatory evolution of the net system presented in Figure 2 is sketched in Figure 3.

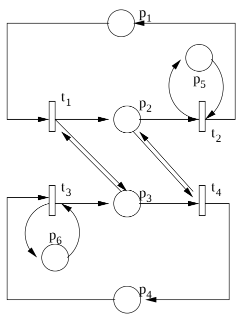

Figure 2: Timed contPN system that has an oscillatory behavior with $\boldsymbol{m_0} = [100, 0, 100, 0, 1, 1]^T$ and $\boldsymbol{\lambda} = [1, 12, 10, 1]^T$.

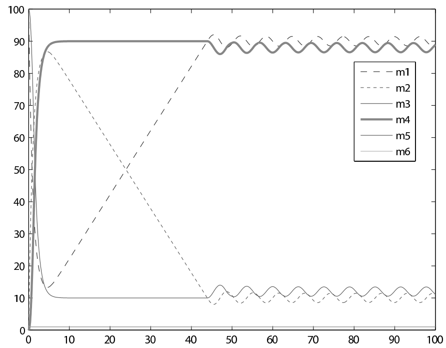

Figure 3: The evolution of the timed contPN system presented in Fig. 2

3 Control of timed contPNs and characterization of steady-states

The parameters $\boldsymbol{\lambda}$ associated with the transitions in timed contPNs represent their *firing rate*. We assume that the only action that can be applied is to *reduce* their firing flow i.e. throughput. If a transition can be controlled (its flow reduced or even stopped), we will say that it is a *controllable* transition. The flow of a controlled transition t_i becomes $f_i - u_i$, where f_i is the flow of the *unforced* system (i.e. defined as in Eq. (1)) and u is the control action $0 \leq u_i \leq f_i$.

According to the above notation, the controlled flow vector is $\boldsymbol{\varphi} = \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m}) \cdot \boldsymbol{m} - \boldsymbol{u} \geq 0$, with $u_i = 0$ if t_i is not controllable. Thus the state equation of controlled timed contPNs (i.e. net systems in which all the transitions are controllable: $\forall t \in T$, $\boldsymbol{u}[t] > 0$ is possible at certain instant) becomes:

$$\begin{cases} \dot{\boldsymbol{m}} = \boldsymbol{C} \cdot (\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m}) \cdot \boldsymbol{m} - \boldsymbol{u}) \\ 0 \leq \boldsymbol{u} \leq \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m}) \cdot \boldsymbol{m} \end{cases} \tag{6}$$

Unless otherwise stated, *in the following we will assume that all transitions are controllable.* Controlling all transitions, almost all reachable markings of an untimed system can be reached in the timed one. The only problem is at the borders when the marking of one place is zero. In this case, the marking is reached at the limit (this is like the discharging of a capacitor in an electrical RC-circuit: theoretical total discharging takes an infinite amount of time). For example, in the net system in Figure 4 the marking $[0, 1, 1]^T$ is reachable in the untimed model. Considering now the timed model, stopping transitions t_2 and t_3 ($u_2 = f_2$ and $u_3 = f_3$) and setting $u_1 = 0$, the marking $[0, 1, 1]^T$ is reached at the limit because $\dot{m}_1(\tau) = -\lambda_1 \cdot m_1(\tau) \Rightarrow m_1(\tau) = e^{-\lambda_1 \cdot \tau} \cdot m_1(0)$. Note that it takes an infinite amount of time to empty p_1.

The steady-state markings we are interested *to obtain* (reference markings for the control loop) are strictly positive (if the marking of a place is zero then the flows of its output transitions are zero, meaning total inactivity of the machines or processors being controlled). These markings can be reached in finite time in the timed model. Let us first prove the following Lemma:

Lemma 13 *If all transitions are controllable, any fireable sequence in the untimed model that does not empty any place in the process, can be fired in the timed-controlled model in finite time.*

Proof: The same sequence can be reproduced, for example firing one transition each time and stopping the others. Since the firings do not empty any place, they take finite time. □

Proposition 14 *If all transitions are controllable:*

1) if $\boldsymbol{m}$ is reachable in the timed-controlled model then it is reachable in the untimed model (i.e., $RS_{timed} \subseteq RS_{untimed}$)

2) if $\boldsymbol{m}$ is reachable in the untimed model then it is lim-reachable in the timed-controlled model (i.e., $RS_{untimed} \subseteq$ lim-RS_{timed});

3) If $\boldsymbol{m} > 0$ is reachable in the untimed model, it can be reached in finite time in the timed-controlled model (i.e., $RS^{+}_{untimed} \subseteq RS_{timed}$).

Proof: 1) If $\boldsymbol{m}$ is reachable in the timed-controlled model then, according to (6), $\boldsymbol{m}(\tau) = \boldsymbol{m_0} + \int_0^\tau \boldsymbol{C} \cdot \varphi(\theta) \cdot d\theta = \boldsymbol{m_0} + \boldsymbol{C} \cdot \int_0^\tau \varphi(\theta) \cdot d\theta$. Let $\boldsymbol{\sigma} = \int_0^\tau \varphi(\theta) \cdot d\theta \geq 0$. Since $\boldsymbol{m}$ is reachable in the timed model no trap can be empty at $\boldsymbol{m}$ in $\mathcal{N}_{\boldsymbol{\sigma}}$ (a marked place cannot be emptied). Moreover, since $\boldsymbol{\sigma}$ is fireable in the timed model, a sequence with the same support can be fired in the untimed model. Hence $\boldsymbol{m}$ is reachable in the untimed net according to Prop. 1.

2) If $\boldsymbol{m}$ is reachable in the untimed model from $\boldsymbol{m_0}$, there exists a sequence $\sigma = \alpha_1 t_1 \alpha_2 t_2 \cdots \alpha_k t_k$ that leads from $\boldsymbol{m_0}$ to $\boldsymbol{m}$. This sequence is equivalent to an infinite sequence $\sigma^1 \sigma^2 \cdots$ defined as:

$$
\begin{array}{rcl}
\sigma^i & = & (\beta_{i,1}\alpha_1)t_1(\beta_{i,2}\alpha_2)t_2 \cdots (\beta_{i,k}\alpha_k)t_k \\
\beta_{1,j} & = & 1/2^j, (j = 1, \ldots, k) \\
\beta_{i,1} & = & 1/2^i,\ (i = 1, 2, \ldots) \\
\beta_{i,j} & = & \frac{1}{2}\left(\sum_{l=1}^{i} \beta_{l,j-1} - \sum_{l=1}^{i-1} \beta_{l,j}\right), (i \geq 2\ j \geq 2).
\end{array}
$$

It is proved in [**?**] that $\sigma^1\sigma^2 \cdots$ converges to σ. By construction, no place is emptied while firing this sequence, therefore it can be fired in the timed-controlled model (Lemma 13). Observe that $\boldsymbol{m}$ is reached at the limit, being an infinite sequence.

3) Let $\boldsymbol{m} > 0$ be such that a sequence $\sigma = \alpha_1 t_1 \cdots \alpha_i t_i \cdots \alpha_j t_j \cdots \alpha_k t_k$ exists that leads from $\boldsymbol{m_0}$ to $\boldsymbol{m}$ in the untimed model. If the firing of σ does not empty any place, Lemma 13 can be applied. Otherwise, we prove that a control law exists that brings the timed model from $\boldsymbol{m_0}$ to a marking $\boldsymbol{m'} > 0$ such that $\boldsymbol{m}$ can be reached from $\boldsymbol{m'}$ using other control law.

To construct σ', let us assume without loss of generality that when firing t_i and t_j while firing σ, at least one place in ${}^\bullet t_i$ and one place in ${}^\bullet t_j$ become empty. Define $\sigma' = \alpha_1 t_1 \cdots \frac{1}{2}\alpha_i t_i \cdots \frac{1}{4}\alpha_j t_j \cdots \frac{1}{4}\alpha_k t_k$. This sequence can be fired in the timed-controlled model, since the amounts in which t_i and t_j are fired ensure that no place is emptied.

The obtained marking is $\boldsymbol{m'} = \boldsymbol{m_0} + \boldsymbol{C} \cdot \boldsymbol{\sigma'} > \boldsymbol{0}$, where $\boldsymbol{\sigma'}$ is the firing count vector corresponding to the sequence σ'. The desired marking is reachable from $\boldsymbol{m'}$ according to Prop. 1: exists a firing count vector $\boldsymbol{\sigma''} = \boldsymbol{\sigma} - \boldsymbol{\sigma'} \geq 0$ with $\boldsymbol{m} = \boldsymbol{m'} + \boldsymbol{C} \cdot \boldsymbol{\sigma''}$; $\boldsymbol{m'} > 0$ implies the existence of a firing sequence with the same support as $\boldsymbol{\sigma''}$ since all the transitions of the net system are fireable; and $\boldsymbol{m} > 0$ means that no empty trap exists at $\boldsymbol{m}$.

Now, the control law to go from $\boldsymbol{m'}$ to $\boldsymbol{m}$ is constructed. Since $\boldsymbol{m} > 0$ and $\boldsymbol{m'} > 0$, the sequence $\sigma'' = \sigma - \sigma' = \frac{1}{2}\alpha_i t_i \cdots \frac{3}{4}\alpha_j t_j \cdots \frac{3}{4}\alpha_k t_k$ can be reordered in such a way that no place is emptied. For example, imagine

σ'' starts firing $\frac{1}{2}\alpha_i t_i$. This would empty a place $p_h \in {}^\bullet t_i$. However, since $\boldsymbol{m}[p_h] > 0$, at least one transition t_r puts more tokens in p_h than the amount removed by the firing of $\frac{1}{2}\alpha_i t_i$.

All the transitions that are fired before t_r can be fired "a bit less". This small amount can be defined in such way that p_h will have enough tokens to complete the firing of t_i in a second round. Then, the firing of all the other transitions until t_r can be completed.

Clearly this procedure can be extended to the case in which several places are emptied. Constructing a new firing sequence not emptying any place, Lemma 13 can be applied. $\square$

Definition 15 *Let $\boldsymbol{0} \leq \boldsymbol{u_d} \leq \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m_d}) \cdot \boldsymbol{m_d}$. Then $\boldsymbol{m_d}$ is an equilibrium point for $\boldsymbol{u_d}$ if $\dot{\boldsymbol{m}}_{\boldsymbol{d}} = \boldsymbol{0}$.*

An equilibrium point represents a state in which the system can be maintained using the defined control action. Given $\boldsymbol{m_0}$ (initial) and $\boldsymbol{m_d}$ (desired) markings, one control problem is to reach $\boldsymbol{m_d}$ and then keep it. In this section we concentrate on steady states.

Obviously, taking into account (6), $\boldsymbol{m_d} \in RS$ is a equilibrium marking if together with the control input $\boldsymbol{u_d}$ is a solution of the following system:

$$\begin{aligned} &\boldsymbol{C} \cdot (\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m_d}) \cdot \boldsymbol{m_d} - \boldsymbol{u_d}) = \boldsymbol{0} \\ &\boldsymbol{0} \leq \boldsymbol{u_d} \leq \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m_d}) \cdot \boldsymbol{m_d} \end{aligned} \tag{7}$$

Therefore, the steady-state flow of a controlled timed contPN $\boldsymbol{\varphi} = \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m_d}) \cdot \boldsymbol{m_d} - \boldsymbol{u_d}$ is a T-semiflow of the net. Notice that if the net is not consistent, some transitions should be stopped in steady-state (i.e. $\boldsymbol{\varphi}$ should contain some zero components).

Given a $\boldsymbol{u_d}$, let us denote as $\boldsymbol{M_{u_d}}$ all the equilibrium states it could maintain. That is, $\boldsymbol{M_{u_d}} = \{\boldsymbol{m} \in RS | \boldsymbol{C} \cdot (\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m}) \cdot \boldsymbol{m} - \boldsymbol{u_d}) = 0, 0 \leq \boldsymbol{u_d} \leq \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m}) \cdot \boldsymbol{m}\}$. The set $\boldsymbol{M_{u_d}}$ can have one single element (Figure 4) or an infinite number of equilibrium markings in a single configuration ($\{(p_1, t_1), (p_2, t_2), (p_3, t_3), (p_4, t_4), (p_5, t_5), (p_7, t_6)\}$ in Figures 5 and 6), or infinite equilibrium markings in several configurations ($\{(p_1, t_1), (p_4, t_2), (p_7, t_3), (p_5, t_4), (p_6, t_5), (p_8, t_6)\}$ and $\{(p_1, t_1), (p_4, t_2), (p_7, t_3), (p_5, t_4), (p_6, t_5), (p_9, t_6)\}$ in Figures 7 and 8).

Next proposition characterizes all the equilibrium points of a net system with the same control action in steady state, $\boldsymbol{u_d}$.

Proposition 16 *Let $\langle \mathcal{N}, \boldsymbol{m_0} \rangle$ be a consistent contPN system with all transitions fireable at least once. Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$ be the timed contPN system*

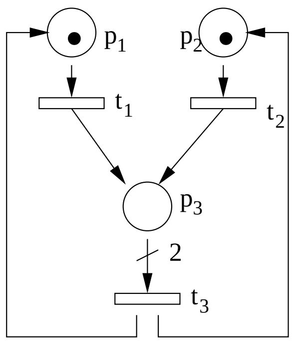

Figure 4: Timed continuous Join-Free system with $\boldsymbol{\lambda} = [1, 1, 1]^T$. It has a unique equilibrium point for a given $\boldsymbol{u_d}$ (for example $\boldsymbol{m_d} = [0.66, 0.66, 0.66]^T$ for $\boldsymbol{u_d} = [0, 0, 0]^T$).

and $\boldsymbol{m_d}$ *an equilibrium point for* $\boldsymbol{u_d}$*. Then* $\boldsymbol{m_i} \geq 0$ *is also an equilibrium point (reachable in finite time if* $\boldsymbol{m_i} > 0$*) for* $\boldsymbol{u_d}$ *iff:*

$$\left\{ \begin{array}{ll} \boldsymbol{B_y}^T \cdot (\boldsymbol{m_d} - \boldsymbol{m_i}) = 0 & (a) \\ \boldsymbol{C} \cdot \boldsymbol{\Lambda} \cdot (\boldsymbol{\Pi_d} \cdot \boldsymbol{m_d} - \boldsymbol{\Pi_i} \cdot \boldsymbol{m_i}) = 0 & (b) \\ 0 \leq \boldsymbol{u_d} \leq \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi_i} \cdot \boldsymbol{m_i} & (c) \end{array} \right. \tag{8}$$

Proof: $\Longrightarrow$ If $\boldsymbol{m_i}$ is an equilibrium point then it is a reachable marking. The system is consistent so: $\boldsymbol{B_y}^T \cdot \boldsymbol{m_i} = \boldsymbol{B_y}^T \cdot \boldsymbol{m_d}$, i.e. (8.a) is necessary.

Both markings are equilibrium points: $\boldsymbol{C} \cdot (\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi_d} \cdot \boldsymbol{m_d} - \boldsymbol{u_d}) = 0$ and $\boldsymbol{C} \cdot (\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi_i} \cdot \boldsymbol{m_i} - \boldsymbol{u_d}) = 0$. Subtracting $\boldsymbol{u_d}$ from both equations, (8.b) is obtained.

$\Longleftarrow$ Equation (8.a) ensures the reachability of $\boldsymbol{m_i}$ according to Prop. 2. The control input $\boldsymbol{u_d}$ can be applied (8.c), and using (8.b) $\boldsymbol{m_i}$ is an equilibrium marking. □

Lemma 17 *Let* $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$ *be a timed contPN system and* $\boldsymbol{m_d}$*,* $\boldsymbol{m_i}$ *two equilibrium points for* $\boldsymbol{u_d}$*. The flows at these markings are equal iff* $\boldsymbol{\Pi_d} \cdot \boldsymbol{m_d} = \boldsymbol{\Pi_i} \cdot \boldsymbol{m_i}$*.*

Proof: Flows are equal: iff $\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi_d} \cdot \boldsymbol{m_d} - \boldsymbol{u_d} = \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi_i} \cdot \boldsymbol{m_i} - \boldsymbol{u_d}$, that is iff $\boldsymbol{\Lambda} \cdot (\boldsymbol{\Pi_d} \cdot \boldsymbol{m_d} - \boldsymbol{\Pi_i} \cdot \boldsymbol{m_i}) = 0$. Since $\boldsymbol{\Lambda}$ is a full rank matrix (by definition is a diagonal matrix with diagonal elements greater than zero), this can happen iff $\boldsymbol{\Pi_d} \cdot \boldsymbol{m_d} = \boldsymbol{\Pi_i} \cdot \boldsymbol{m_i}$. □

Example 18 *For the timed contPN system depicted in Figure 9 the optimal flow* $\boldsymbol{\varphi}_{max} = [0.2, 0.2, 0.6, 0.2]^T$ *is obtained with* $\boldsymbol{u_d} = [0, 0, 0, 0]^T$

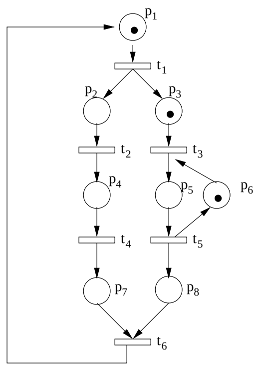

Figure 5: Timed continuous Marked Graph system with $\boldsymbol{\lambda} = [1,1,1,1,1,1]^T$ and many equilibrium points in the same configuration for a given $\boldsymbol{u_d}$.

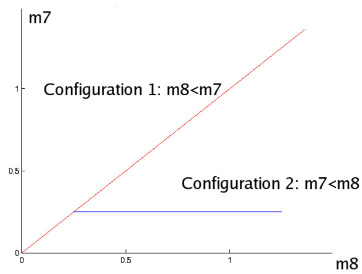

Figure 6: Equilibrium points of the timed continuous Marked Graph system in Fig. 5 for $\boldsymbol{u_d} = [0,0,0,0,0,0]^T$.

and marking $\boldsymbol{m_d} = [0.2, 0.6, 0.6, 0.6]^T$. *Marking* $\boldsymbol{m'} = [0.1, 0.3, 1.8, 0.3]^T$, *is also an equilibrium point for* $\boldsymbol{u_d} = [0,0,0,0]^T$, *and the flow is different* $\boldsymbol{\varphi'} = [0.1, 0.1, 0.3, 0.1]^T$. *Obviously, the conditions of the lemma do not hold,* $\boldsymbol{\Pi_d} \cdot \boldsymbol{m_d} \neq \boldsymbol{\Pi'} \cdot \boldsymbol{m'}$.

Let $\boldsymbol{B_x}$ be a basis of T-flows of a net (i.e. $\boldsymbol{C} \cdot \boldsymbol{B_x} = \boldsymbol{0}$).

Theorem 19 *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$ be a consistent timed contPN system with all transitions fireable at least once. In one configuration $\boldsymbol{\Pi}$ all the equilibrium points for a given $\boldsymbol{u}$ have the same flow if*

$$rank \begin{bmatrix} \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi} & | & \boldsymbol{B_x} \\ \boldsymbol{B_y}^T & | & 0 \end{bmatrix} = rank \begin{bmatrix} \boldsymbol{\Pi} \\ \boldsymbol{B_y}^T \end{bmatrix} + |T| - rank(\boldsymbol{C})$$

Proof: Let us assume that $\boldsymbol{m_a}$ and $\boldsymbol{m_b}$ are two equilibrium points under $\boldsymbol{\Pi}$ for the same control $\boldsymbol{u}$. Obviously, the flow in steady state will be a T-semiflow: $\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi} \cdot \boldsymbol{m_a} - \boldsymbol{u} = \boldsymbol{B_x} \cdot \boldsymbol{\alpha}$ ($\boldsymbol{B_x} \cdot \boldsymbol{\alpha} = \sum_i \alpha_i \cdot \boldsymbol{b_{x_i}}$), and $\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi} \cdot \boldsymbol{m_b} - \boldsymbol{u} = \boldsymbol{B_x} \cdot \boldsymbol{\beta}$. Now, subtracting both equations: $\boldsymbol{\Pi} \cdot \Delta\boldsymbol{m} - \boldsymbol{\Lambda}^{-1} \cdot \boldsymbol{B_x} \cdot \boldsymbol{\zeta} = 0$ ($\Delta\boldsymbol{m} = \boldsymbol{m_a} - \boldsymbol{m_b}$, $\boldsymbol{\zeta} = \boldsymbol{\alpha} - \boldsymbol{\beta}$). Moreover, since these markings are reachable, $\boldsymbol{B_y}^T \cdot \Delta\boldsymbol{m} = 0$.

$$\begin{bmatrix} \boldsymbol{\Pi} & | & -\boldsymbol{\Lambda}^{-1} \cdot \boldsymbol{B_x} \\ \boldsymbol{B_y}^T & | & 0 \end{bmatrix} \cdot \begin{bmatrix} \Delta\boldsymbol{m} \\ \boldsymbol{\zeta} \end{bmatrix} = 0 \tag{9}$$

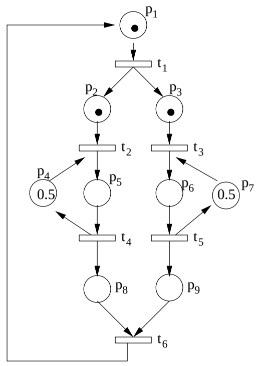

Figure 7: Timed continuous Marked Graph system with $\boldsymbol{\lambda} = [1, 1, 1, 1, 1, 1]^T$ and many equilibrium points in several configurations for a given $\boldsymbol{u_d}$.

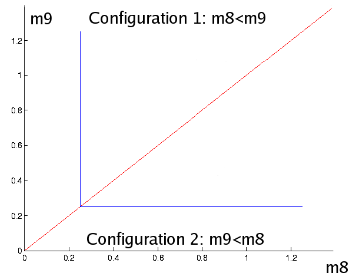

Figure 8: Equilibrium points of the timed continuous Marked Graph system in Fig. 7 for $\boldsymbol{u_d} = [0, 0, 0, 0, 0, 0]^T$.

Under the rank condition, the vectorial spaces generated by the row vectors of $[\boldsymbol{\Pi}^T|\boldsymbol{B_y}]$ and $[(\boldsymbol{\Lambda}^{-1}\boldsymbol{B_x})^T|0]$ are linearly independent. Hence the null element is the only vector that belongs to both of them, i.e., $\boldsymbol{\Lambda}^{-1}\boldsymbol{B_x} \cdot \boldsymbol{\zeta} = 0$. Moreover, $\boldsymbol{\Lambda}^{-1}$ is a diagonal matrix and the columns in $\boldsymbol{B_x}$ are linearly independent since they are a T-flows basis, and so $\boldsymbol{\zeta} = 0$. Therefore $\boldsymbol{m_a}$ and $\boldsymbol{m_b}$ have the same flow. $\square$

Example 20 *Let us consider the contPN system in Figure 10 with* $\boldsymbol{\lambda} = [2, 1, 1]^T$. *The configuration* $\{(p_4, t_1), (p_4, t_2), (p_3, t_3)\}$ *with associated matrix* $\boldsymbol{\Pi}$ *can have several equilibrium points with different flows because the conditions of Theorem 19 are not satisfied. For this system,* $\boldsymbol{\Pi} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $\boldsymbol{B_y}^T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$, $\boldsymbol{\Lambda}^{-1} \cdot \boldsymbol{B_x} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}$ *and rank* $\begin{bmatrix} \boldsymbol{\Pi} & -\boldsymbol{\Lambda} \cdot \boldsymbol{B_x} \\ \boldsymbol{B_y}^T & 0 \end{bmatrix} =$ *rank* $\begin{bmatrix} \boldsymbol{\Pi} \\ \boldsymbol{B_y}^T \end{bmatrix} = 4$. *If* $\boldsymbol{u} = [0, 0, 0]^T$, *the equilibrium markings* $\boldsymbol{m}_1 = [15.25, 1, 0.75, 0.75]^T$ *and* $\boldsymbol{m}_2 = [15.5, 0.8, 0.7, 0.7]^T$ *belonging to this configuration have the flows* $\boldsymbol{\varphi_1} = [0.75, 0.75, 0.75]^T$ *and* $\boldsymbol{\varphi_2} = [0.7, 0.7, 0.7]^T$ *respectively. Thus, any intermediate value is also possible.*

For the class of Equal Conflict contPN, if the conflicting transitions are not

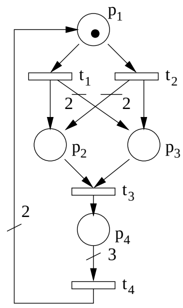

Figure 9: Conservative but not lim-live continuous EQ system with several equilibrium points for $\boldsymbol{\lambda} = [1,1,1,1]^T$, with different flow.

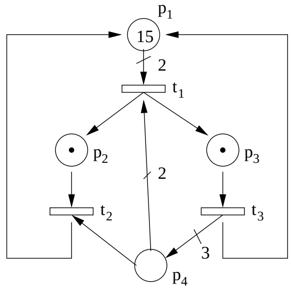

Figure 10: Bounded and lim-live contPN that has several equilibrium points with distinct flow.

controlled (otherwise the visit ratio is changed by the control), we can prove that all equilibrium points have the same flow under the same configuration.

Theorem 21 *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$ be a bounded and lim-live EQ timed contPN system. Given $\boldsymbol{u}$ in which transitions in conflict are not controlled, there exists at least one equilibrium point. If there are more than one, all of them have the same flow.*

Proof: The throughput in steady state for unforced ($\boldsymbol{u} = 0$) continuous EQ nets can be computed using a linear programming problem [?]. More precisely, the throughput is obtained looking for the slowest P-semiflow. The solution is unique with respect to the flow, but there can exist more than one marking that respect the P-semiflows and have the same associated flow.

Assume $^\bullet t = p$ (i.e. t is a non synchronizing transition) and $u[t] \neq 0$. If the steady-state marking of p is $m[p]$, we can reduce the value of $m[p]$ and transform the system into an equivalent one with the same steady-state flow with the marking $m'[p] = m[p] - \frac{\mathrm{Pre}[p,t]}{\lambda[t]} \cdot u[t]$. The flow will be: $\lambda[t] \cdot \frac{m'}{\mathrm{Pre}[p,t]} = \lambda[t] \cdot \frac{m[p]}{\mathrm{Pre}[p,t]} - u[t]$, the same as in the original system (with $u[t] \neq 0$). For every controlled transition we can apply the same technique (in the case of synchronizations we remove tokens from all input places) obtaining an equivalent system with $u = 0$. For this system all the equilibrium points have the same flow. □

Figure 4, 5 and 7 are CF (thus EQ). Therefore, this theorem ensures that all their equilibrium points have the same flow for any constant control input $\boldsymbol{u}$. The following theorem provides a sufficient condition to guarantee that the equilibrium point of a configuration is unique.

Theorem 22 *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$ be a bounded and lim-live EQ timed contPN system. If rank $\begin{bmatrix} \boldsymbol{\Pi_i} \\ \boldsymbol{B_y}^T \end{bmatrix} = |P|$ and conflict transitions are not controlled, then at most one equilibrium marking exists under $\boldsymbol{\Pi_i}$ for a given $\boldsymbol{u_d}$.*

Proof: Let *rank* $\begin{bmatrix} \boldsymbol{\Pi_i} \\ \boldsymbol{B_y}^T \end{bmatrix} = |P|$ and $\boldsymbol{m_d}$, $\boldsymbol{m_i}$ ($\boldsymbol{m_i} = \boldsymbol{m_d} + \Delta \boldsymbol{m}$) two equilibrium points under $\boldsymbol{\Pi_i}$ for $\boldsymbol{u_d}$. Using Theorem (21) all equilibrium points with the same action $\boldsymbol{u_d}$ have the same flow, i.e. $\boldsymbol{\Pi_i} \cdot \boldsymbol{m_d} = \boldsymbol{\Pi_i} \cdot \boldsymbol{m_i}$ or $\boldsymbol{\Pi_i} \cdot \Delta \boldsymbol{m} = 0$. Moreover, $\boldsymbol{B_y}^T \cdot \boldsymbol{m_i} = \boldsymbol{B_y}^T \cdot \boldsymbol{m_d}$, or $\boldsymbol{B_y}^T \cdot \Delta \boldsymbol{m} = 0$.

Under the rank assumption, the previous system has only one solution, $\Delta \boldsymbol{m} = 0$. So $\boldsymbol{m_d} = \boldsymbol{m_i}$. Hence, $\boldsymbol{\Pi_i}$ has at most one equilibrium point. □

Example 23 *Let us consider the net in Figure 7 and let*

$$\boldsymbol{\Pi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

define one configuration. One P-flow basis is:

$$\boldsymbol{B_y}^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Since rank $\begin{bmatrix} \boldsymbol{\Pi_i} \\ \boldsymbol{B_y}^T \end{bmatrix} = 8 < 9$ (the number of places) this configuration may have an infinite number of equilibrium points. In particular (Fig. 8), for $\boldsymbol{m_0} = [1, 1, 1, 0.5, 0, 0, 0.5, 0, 0]$ the configuration 1 ($\{(p_1, t_1), (p_4, t_2), (p_7, t_3), (p_5, t_4), (p_6, t_5), (p_8, t_6)\}$) has an infinite number of equilibrium points.

But, for $\boldsymbol{m_0} = [1, 0, 0, 0.5, 0, 0, 0.5, 0, 0]$ the contPN system has only one equilibrium marking. Thus, the condition in Theorem 22 is sufficient, but not necessary.

Corollary 24 *Let $\mathcal{N}$ be a conservative and consistent JF contPN. Given $\boldsymbol{u_d}$ and assuming that the conflict transitions are not controlled, only one equilibrium point exists in $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$.*

Proof: Since the net is JF, all the conflicts are topologically EQ (if $t_1, t_2 \in p^\bullet, {}^\bullet t_1 = {}^\bullet t_2 = p$), thus the net can be mapped into CF [?]. A CF net is conservative iff it is strongly-connected implying $rank(\boldsymbol{C}) = |T| - 1 = |SEQS| - 1$ [?]. A conservative and consistent contPN with $rank(\boldsymbol{C}) = |SEQS| - 1$ is lim-live and bounded [?] and Theorem 22 can be applied. $\square$

4 Optimal control for steady state

In production control, the profit function frequently depends on production sales, work in process (WIP) and amortization of investments. Under linear hypothesis for fixed machines (i.e. $\boldsymbol{\lambda}$ defined), the profit function may have the following form: $\boldsymbol{w}^T \cdot \boldsymbol{\varphi} - \boldsymbol{z}^T \cdot \boldsymbol{m} - \boldsymbol{q}^T \cdot \boldsymbol{m}_0$, where $\boldsymbol{\varphi}$ is the throughput vector, $\boldsymbol{w}^T$ a price vector w.r.t. flows, $\boldsymbol{m}$ the average marking, $\boldsymbol{z}^T$ is the WIP cost vector and $\boldsymbol{q}^T$ represents depreciations or amortization of the initial investments (over $\boldsymbol{m_0}$).

Let us consider the following linear programming problem:

$$\begin{array}{lll} \max & \boldsymbol{w}^T \cdot \boldsymbol{\varphi} - \boldsymbol{z}^T \cdot \boldsymbol{m} - \boldsymbol{q}^T \cdot \boldsymbol{m}_0 & \\ s.t. & \boldsymbol{C} \cdot \boldsymbol{\varphi} = 0,\ \boldsymbol{\varphi} \geq 0 & \text{(a)} \\ & \boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma},\ \boldsymbol{m}, \boldsymbol{\sigma} \geq 0 & \text{(b)} \\ & \varphi_i = \lambda_i \cdot \left(\frac{m_j]}{\mathrm{Pre}[p_j, t_i]} \right) - v[p_j, t_i], & \\ & \ \forall p_j \in {}^\bullet t_i,\ v[p_j, t_i] \geq 0 & \text{(c)} \end{array} \tag{10}$$

where $v[p_j, t_i]$ are *slack* variables.

The equations correspond to: (a) $\boldsymbol{\varphi}$ is a T-semiflow; (b) fundamental equation ($\boldsymbol{m}$ is a reachable marking); (c) firing law for infinite server semantics.

Theorem 25 *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$ be a timed contPN system and let $\langle \boldsymbol{\varphi}, \boldsymbol{m}, \boldsymbol{v} \rangle$ be a solution of LPP (10), then*

1. *For every transition t_i, let $u_i = \min_{p_j \in {}^\bullet t_i} v[p_j, t_i]$ be its control input. Then $\boldsymbol{u}$ is the control in steady-state for $\boldsymbol{m}$. (In the case of $|{}^\bullet t_i| = 1$ the corresponding slack variable is the same as the control input.)*

2. *If for every $u_i > 0$ transition t_i is controllable, then $\boldsymbol{u}$ is an optimal steady-state control.*

Proof: In steady state, $\varphi_i = \lambda_i \cdot \min_{p_j \in {}^\bullet t_i} \left(\frac{m_j}{\mathrm{Pre}[p_j, t_i]} \right) - u_i$. Choosing $u_i = \min_{p_j \in {}^\bullet t_i} v[p_j, t_i]$ for all transitions, the equation (10.c) is verified. If all t_i

with $u_i \neq 0$ can be controlled, the control can be applied in steady state; then the command is optimal. $\square$

For mono T-semiflow nets (conservative and consistent that have a unique minimal T-semiflow) (or nets reducible to mono T-semiflow [?]), the equation (10a) can be replaced with the equivalent one: $\boldsymbol{\varphi} = \alpha \cdot \boldsymbol{X}$ (10a') with $\boldsymbol{X}$ the minimal T-semiflow.

If the net is consistent and every transition can be fired at least once, the equation (10b) is equivalent to: $\boldsymbol{B_y}^T \cdot \boldsymbol{m} = \boldsymbol{B_y}^T \cdot \boldsymbol{m_0}$, $\boldsymbol{m} \geq 0$ (10b').

Example 26 *The solution of LPP 10 is not necessarily unique (as we mentioned in the previous section). Let us see which is the maximum throughput in steady-state for the contPN in Figure 1 with $\boldsymbol{\lambda} = [1,1,1,1]^T$ and $\boldsymbol{m_0} = [1,0,3,3,1,0]^T$. Notice that this is a marked graph net system, hence is monotone and the optimal control should be $\boldsymbol{u_d} = 0$. Indeed, LPP (10) with (10a') and (10b') leads to:*

$$
\begin{array}{ll}
\max & \varphi_1 \\
s.t. & \varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 \qquad \textit{(10a')} \\
& \left.\begin{array}{l} m_1 + m_2 = m_5 + m_6 = 1 \\ m_3 + m_4 = 6 \end{array}\right\} \textit{(10b')} \\
& \varphi_1 = m_1 - u_1 \\
& \varphi_2 = m_2 - v_{22} \\
& \varphi_2 = m_3 - v_{23} \\
& \varphi_3 = m_4 - v_{34} \\
& \varphi_3 = m_5 - v_{35} \\
& \varphi_4 = m_6 - u_4 \\
& \boldsymbol{\varphi}, \boldsymbol{m}, \boldsymbol{v} \geq 0
\end{array}
\tag{11}
$$

One optimal solution of this LPP is: $\varphi_1 = 0.5$, $\boldsymbol{m_d} = [0.5\ 0.5\ 3.5\ 2.5\ 0.5\ 0.5]^T$ and $\boldsymbol{v} = [0,0,3,2,0,0]^T$. Therefore $u_2 = \min(v_{22}, v_{23}) = 0$, $u_3 = \min(v_{34}, v_{35}) = 0$ and $\boldsymbol{u_d} = [0\ 0\ 0\ 0]^T$ is an optimal control in steady state ($\boldsymbol{u_d} = \boldsymbol{0}$ leads always to optimal flow in marked graph).

For sure the solution is not unique: all the markings that satisfy (12) are also solution of (11).

$$
\left\{
\begin{array}{l}
m_1 = m_2 = m_5 = m_6 = 0.5 \\
m_3 + m_4 = 6 \\
m_3, m_4 \geq 0.5
\end{array}
\right.
\tag{12}
$$

Up to now we have considered that all transitions are controllable. What happens when some are uncontrollable? In the extreme case, in which *all* transitions are uncontrollable (the *unforced* system), the problem to compute the optimal steady-state (maximum throughput) was addressed in [?] and can be solved using a branch and bound algorithm. Let us assume $T = T_C \cup T_N$, where T_C is the set of controllable transitions and T_N the set of the uncontrollable transitions.

When all synchronizations are controllable ($\{t \text{ s.t. } |^\bullet t| > 1\} \subseteq T_C$), the problem remains polynomial time. In fact, it is the same problem as (10) in which no slack variables is allowed for the uncontrolled transitions.

When a synchronization is not controllable, the problem may be more difficult. The corresponding slack variable cannot be used. As in [?] we can relax (10) and the flow of non controllable transitions will be upper bounded with inequalities written for every input place:

$$\begin{array}{lll}
\max & \boldsymbol{w}^T \cdot \boldsymbol{\varphi} - \boldsymbol{z}^T \cdot \boldsymbol{m} - \boldsymbol{q}^T \cdot \boldsymbol{m}_0 & \\
s.t. & \boldsymbol{C} \cdot \boldsymbol{\varphi} = 0,\ \boldsymbol{\varphi} \geq 0 & \text{(a)} \\
& \boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma},\ \boldsymbol{m}, \boldsymbol{\sigma} \geq 0 & \text{(b)} \\
& \varphi_i = \lambda_i \cdot \left(\frac{m_j}{\mathrm{Pre}[p_j, t_i]} \right) - v[p_j, t_i], & \\
& \forall p_j \in {}^\bullet t_i,\ t_i \in T_C,\ v[p_j, t_i] \geq 0 & \text{(c)} \\
& \varphi_i = \lambda_i \cdot \left(\frac{m[p]}{\mathrm{Pre}[p, t_i]} \right),\ \text{if } p = {}^\bullet t_i,\ t_i \in T_N, & \text{(d)} \\
& \varphi_i \leq \lambda_i \cdot \left(\frac{m_j}{\mathrm{Pre}[p_j, t_i]} \right), \forall p_j \in {}^\bullet t_i,\ t_i \in T_N. & \text{(e)}
\end{array} \quad (13)$$

Because of (13.e), the LPP (13) provides in general a non tight bound, i.e. the solution may be non reachable. This occurs when none of the input places of a non controllable join transition really restricts the flow of that transition. Similar to [?], a branch and bound algorithm can be used. For every non controllable join transition t_j, a number of $|^\bullet t_j|$ LPPs should be computed by adding an equation that relates the flow of t_j with the marking of each one of its input places. Thus, the algorithm in [?] can be used in this situation.

5 Approaching dynamic control: on controllability and marking invariance laws

5.1 Definition of controllability

5.1.1 Controllability with constrained inputs

Assume the systems under study are described by the equations in (6). The classical control theory for linear systems cannot be applied because we are

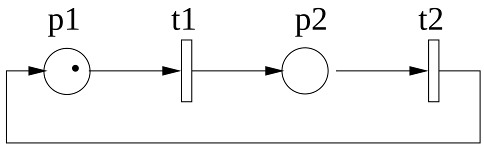

Figure 11: Join-Free timed contPN system with $\boldsymbol{\lambda} = [1,1]^T$, $t_1 \in T_N$, $t_2 \in T_C$.

working inside a polytope (not in a *vectorial space*) and our control input is bounded.

Definition 27 *Given $\Sigma = \langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$ and controlling transitions $T_C \subset T$, a marking $\boldsymbol{m_f}$ is said to be a* reachable steady-state *when there exists a constrained control action $\boldsymbol{u}(\tau)$ on T_C that is able to drive the marking from $\boldsymbol{m_0}$ to $\boldsymbol{m_f}$ (in finite or infinite time) and maintain it.*

Definition 28 *The timed contPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is* controllable *if $\forall \boldsymbol{m_0}$ and $\forall \boldsymbol{m_f} \geq 0$ such that $\boldsymbol{B_y^T} \cdot \boldsymbol{m_0} = \boldsymbol{B_y^T} \cdot \boldsymbol{m_f}$, $\boldsymbol{m_f}$ is a lim-reachable steady-state.*

Unfortunately, the controllability of all transitions is required in order to obtain a controllable contPN system.

Example 29 *Let us consider the net system in Figure 11 and assume that only t_2 is controllable. The marking $\boldsymbol{m} = [1,0]^T$ cannot be an equilibrium marking because in steady-state $\varphi_1 = \varphi_2$ so $1 = 0 - \boldsymbol{u}[t_2]$ which implies a negative command on t_2. Therefore, $\boldsymbol{m}$ cannot be maintained in the timed and controlled model. In practice, any marking $\boldsymbol{m'}$ with $\boldsymbol{m'}[p_1] > \boldsymbol{m'}[p_2]$ is non maintainable, because $\dot{\boldsymbol{m}}'[p_1] = -\boldsymbol{m'}[p_1] + \boldsymbol{m'}[p_2] - \boldsymbol{u}[t_2] \leq -\boldsymbol{m'}[p_1] + \boldsymbol{m'}[p_2] < 0$. Hence, the timed JF model of Figure 11 is not controllable.*

Proposition 30 *A pure and conservative timed contPN $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is controllable iff all transitions are controllable i.e. $T = T_C$.*

Proof: The sufficient condition is immediate: if all transitions allow a control action, the net is controllable. We can reach any desired marking (maybe at the limit) (Prop. 14) and then we stop all transitions (i.e. $\boldsymbol{u} = \boldsymbol{f}$).

Necessity: Let $\boldsymbol{m_0}$ be an initial marking that puts tokens in all the P-semiflows and let us assume t_i is not controllable. We are going to prove that a marking $\boldsymbol{m}$ satisfying $\boldsymbol{B_y^T} \cdot \boldsymbol{m_0} = \boldsymbol{B_y^T} \cdot \boldsymbol{m}$ exists that cannot be maintained.

Let

$$\beta_i = \max \left\{ b | \boldsymbol{m} \geq b \cdot \boldsymbol{Pre}[\cdot, t_i] \text{ and } \boldsymbol{B_y^T} \cdot \boldsymbol{m_0} = \boldsymbol{B_y^T} \cdot \boldsymbol{m} \right\} \tag{14}$$

In fact, β_i represents the enabling bound of t_i [**?**]. Let $\boldsymbol{m}$ be a solution of (14). Since β_i is obtained through maximization, for sure $\boldsymbol{m}[p_j] > 0, \forall p_j \in {}^\bullet t_i$.

Since the net is pure and conservative, $t_i{}^\bullet \cap (T \setminus {}^\bullet t_i) \neq \emptyset$, then at least one place in $t_i{}^\bullet$ must be empty (otherwise the enabling degree would be greater). Clearly this place cannot remain empty if t_i is not controlled. □

5.1.2 Classical approach: controllability without constrained inputs

From this point, a relaxation of the equations modeling the system is proposed, eliminating the restrictions related to the bounds of the control input. Therefore, the system under study is relaxed to the non-linear equations (this is the dynamical equation in Eq. (6)):

$$\dot{\boldsymbol{m}} = \boldsymbol{C} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m}) \cdot \boldsymbol{m} - \boldsymbol{C} \cdot \boldsymbol{u} \tag{15}$$

The goal is to better understand the behavior of contPN and interpret classical results in the contPN case. In many cases, the regulation of the system is done to a point (desired marking + desired input) that is not at the boundary. In this case, a region around it can be defined in which the constraints are not active. Basically, the number of null eigenvalues are explored, eigenvalues that introduce token conservation laws. It will be seen that some of these conservation laws are given by the net structure $\mathcal{N}$ (the P-flows, Subsection 5.2), others depend on $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ (Subsection 5.3) and others depend also on the particular marking $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$ (Subsection 5.4).

For classical linear systems *controllability* has been thoroughly studied (see the Appendix for some basic results). For contPN systems, every $\boldsymbol{\Pi}(\boldsymbol{m})$ leads to a linear and time-invariant dynamic system with controllability matrix $\mathbb{C}(\boldsymbol{m})$:

$$\mathbb{C}(\boldsymbol{m}) = -\left[\begin{array}{ccc} \boldsymbol{C} & \cdots & (\boldsymbol{C} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m}))^{n-1} \cdot \boldsymbol{C} \end{array}\right] \tag{16}$$

Proposition 31 *If all transitions are controllable, $\forall \boldsymbol{m} \in RS$, the spaces generated by the columns of $\mathbb{C}(\boldsymbol{m})$ and $\boldsymbol{C}$ are equal. Thus $rank(\mathbb{C}(\boldsymbol{m})) = rank(\boldsymbol{C}) = |P| - \dim(\boldsymbol{B_y})$.*

Proof: Since the columns of $\boldsymbol{C}$ are contained in $\mathbb{C}$, it is immediate that the space generated by the columns of $\mathbb{C}$ contains the space generated by the columns of $\boldsymbol{C}$. Thus we only need to prove that it cannot be greater. Observe that $(\boldsymbol{C} \cdot \boldsymbol{\Lambda}\boldsymbol{\Pi}(\boldsymbol{m}))^{n-1} \cdot \boldsymbol{C} = \boldsymbol{C} \cdot (\boldsymbol{\Lambda}\boldsymbol{\Pi}(\boldsymbol{m}) \cdot \boldsymbol{C})^{n-1}$.

Thus, $\mathbb{C} = \boldsymbol{C} \cdot [\boldsymbol{I} \cdots (\boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m}) \cdot \boldsymbol{C})^{n-1}]$. Notice that any P-flow of $\boldsymbol{C}$ is also a P-flow of $\mathbb{C}$. Hence, $rank(\mathbb{C}) = rank(\boldsymbol{C}) = |P| - \dim(\boldsymbol{B_y})$. $\square$

Notice that $\mathbb{C}(\boldsymbol{m})$ depends on $\boldsymbol{\Pi}(\boldsymbol{m})$, but the space generated by its columns is always the same, just that one defined by matrix $\boldsymbol{C}$. This is something that can be easily expected because all transitions have been assumed to be controllable.

5.2 Uncontrollable zero valued poles and decomposition

Token conservation laws given by the net structure (P-flows) produce non controllable contPN systems in a classical sense. This was observed in [?] and happens because the P-flows based token conservation laws make $|P| - rank(\boldsymbol{C})$ places linearly-redundant. Using a proper similitude transformation (the $\mathbb{Q}_{\mathcal{N}}$ matrix that will be given in Definition 33) it may be possible to obtain a decomposition into a controllable subsystem and an uncontrollable one (similar to the Kalman controllable canonical form). The uncontrollable subsystem has only zero valued poles and they will be called *uncontrollable (zero) valued poles.*

Example 32 *Let us consider the contPN system in Figure 1 with* $\boldsymbol{\lambda} = [\alpha, \beta, \gamma, \delta]^T$. *This net has three linearly independent token conservation laws derived from P-(semi)flows:* $m_1 + m_2 = 1, m_3 + m_4 = 6$ *and* $m_5 + m_6 = 1$. *Thus* $\dot{m}_1 + \dot{m}_2 = \dot{m}_3 + \dot{m}_4 = \dot{m}_5 + \dot{m}_6 = 0$, *which means that three uncontrollable zero valued poles will appear.*

The following transformation matrix is used to change the reference in which the marking vector is expressed. This will be useful for studying the controllability of the system. The kind of transformation matrix to be considered will have in this context a particular structure.

Definition 33 *Let* $\mathcal{N}$ *be a contPN. A transformation matrix* $\mathbb{Q}_{\mathcal{N}}$, *is formed with rows from a basis of P-flows and elementary vectors in order to build a full rank matrix.*

Example 34 *For the timed models in Figure 11 and Figure 1, P-flow basis are respectively:*

$$\boldsymbol{B_y}_1^T = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B_y}_2^T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{17}$$

Adding elementary vectors, $\mathbb{Q}$ *matrices can be, for example:*

$$\mathbb{Q}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \textit{and} \quad \mathbb{Q}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

The system described by equation (15) can be rewritten in new coordinates $\bar{\boldsymbol{m}}$, when matrix $\mathbb{Q}_{\mathcal{N}}$ is used as a state vector transformation matrix. Let $\bar{\boldsymbol{m}} = \mathbb{Q}_{\mathcal{N}} \cdot \boldsymbol{m}$.

Definition 35 *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$ be a timed contPN described by equation (15), where $\mathbb{Q}_{\mathcal{N}}$ is a transformation matrix of $\mathcal{N}$. Then*

$$\dot{\bar{\boldsymbol{m}}} = \mathbb{Q}_{\mathcal{N}} \boldsymbol{C} \boldsymbol{\Lambda} \boldsymbol{\Pi}(\boldsymbol{m}) \mathbb{Q}_{\mathcal{N}}^{-\mathbf{1}} \bar{\boldsymbol{m}} - \mathbb{Q}_{\mathcal{N}} \boldsymbol{C} \boldsymbol{u} \tag{19}$$

will be called a $\mathbb{Q}$-canonical representation of equation (15).

Theorem 36 *Let $\Sigma = \langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$ be a contPN system, then:*

1) In the linear system dynamics under $\boldsymbol{\Pi}(\boldsymbol{m})$ the number of zero valued poles is given by the dimension of the right annullers of $\boldsymbol{C} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\Pi}(\boldsymbol{m})$.

2) The number of non controllable poles is $|P| - rank(\boldsymbol{C})$ and they are zero valued.

Proof: 1) The zero eigenvalues of the matrix $\boldsymbol{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}(\boldsymbol{m})$ are:

$$\boldsymbol{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}(\boldsymbol{m}) \cdot \boldsymbol{v} = 0 \cdot \boldsymbol{v} = 0$$

2) Making the change of variables $\bar{\boldsymbol{m}} = \mathbb{Q}_{\mathcal{N}} \cdot \boldsymbol{m}$, (19) is obtained. For each P-flow of the basis one zero row appears in $\mathbb{Q}_{\mathcal{N}} \cdot \boldsymbol{C}$.

Without loss of generality, assume that the row i of $\mathbb{Q}_{\mathcal{N}} \cdot \boldsymbol{C}$ is zero, then the row i of $\mathbb{Q}_{\mathcal{N}} \boldsymbol{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}(\boldsymbol{m}) \mathbb{Q}_{\mathcal{N}}^{-1}$ is zero. Therefore the value of the state variable $\bar{\boldsymbol{m}}_i$ is never affected by other state variables, or by the input, thus $\bar{\boldsymbol{m}}_i$ is uncontrollable. Each one of these $\bar{\boldsymbol{m}}_i$ comes from a P-flow equation, a linear constraint among variables (i.e. token conservation law: $\boldsymbol{b_i}^T \cdot \bar{\boldsymbol{m}} = \boldsymbol{b_i}^T \cdot \bar{\boldsymbol{m}}_\mathbf{0}$). Thus the pole value associated to $\bar{\boldsymbol{m}}_i$ is zero and there exist $\dim(\boldsymbol{B_y})$ uncontrollable zero valued poles. According to Proposition 31, $rank(\mathbb{C}(\boldsymbol{m})) = |P| - \dim(\boldsymbol{B_y})$, then there exist no more uncontrollable poles. Otherwise stated, if there are more zero valued poles, they are controllable (as we will see in Section 5.3). □

Example 37 *Let us consider the contPN system in Figure 11. It has the following equation:*

$$\dot{\boldsymbol{m}} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \boldsymbol{m} - \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \boldsymbol{u} \tag{20}$$

The controllability matrix of this timed net is the following:

$$\mathbb{C} = \begin{bmatrix} -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix}$$

Its rank being one, it has only one controllable pole (equal to -2) and one non controllable pole (equal to 0). A transformation matrix is:

$$\mathbb{Q} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and, the corresponding $\mathbb{Q}$-canonical representation is:

$$\overset{\bullet}{\bar{\boldsymbol{m}}} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \bar{\boldsymbol{m}} - \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \boldsymbol{u}$$

5.3 Token conservation laws and controllable zero valued poles

In addition to those expressed by P-flows, other token conservation laws corresponding to zero valued poles can appear.

Example 38 *Let us consider the contPN system shown in Figure 1 with $\boldsymbol{\lambda} = [\alpha, \beta, \gamma, \delta]^T$. Clearly, $\dot{m}_1 + \dot{m}_2 = \dot{m}_3 + \dot{m}_4 = \dot{m}_5 + \dot{m}_6 = 0$ are token conservation laws that correspond to zero valued uncontrollable poles (as mentioned in Example 32).*

If we fix m_2, m_3 and m_5 as state variables then m_1, m_4 and m_6 are redundant. The linear dynamic system corresponding to the configuration $\{(p_1, t_1), (p_2, t_2), (p_5, t_3), (p_6, t_4)\}$ is:

$$\begin{cases} \dot{m}_2 = -\beta \cdot m_2 + \alpha \cdot (1 - m_2) = -(\alpha + \beta) \cdot m_2 + \alpha \\ \dot{m}_3 = -\beta \cdot m_2 + \gamma \cdot m_5 \\ \dot{m}_5 = -\gamma \cdot m_5 + \delta \cdot (1 - m_5) = -(\gamma + \delta) \cdot m_5 + \delta \end{cases}$$

Eliminating all variables in the right hand side:

$$-\frac{\beta}{\alpha + \beta} \cdot \dot{m}_2 + \frac{\gamma}{\gamma + \delta} \cdot \dot{m}_5 + \dot{m}_3 = \frac{\gamma \cdot \delta}{\gamma + \delta} - \frac{\alpha \cdot \beta}{\alpha + \beta} = q$$

Therefore, if $q = 0$ a new token conservation law appears introducing an additional zero valued pole (that it is not rooted in a P-flow): $-\frac{\beta}{\alpha+\beta} \cdot m_2 + \frac{\gamma}{\gamma+\delta} \cdot m_5 + m_3 = constant$. If $q \neq 0$, sooner or later the above configuration will be left. This is evident since at least one of the variables (m_2, m_3 or m_5) will grow (or decrease) while the system is in the configuration. This can also be deduced using the fact that the steady state flow has to be a T-semiflow of the net. Since it has only one minimal T-semiflow $[1,1,1,1]^T$, in steady state: $f_1 = f_2 = f_3 = f_4$.

$$f_1 = f_2 \Longrightarrow \alpha \cdot m_1 = \beta \cdot (1 - m_1) \Longrightarrow f_1 = \frac{\alpha \cdot \beta}{\alpha + \beta}$$

$$f_3 = f_4 \Longrightarrow \gamma \cdot m_5 = \delta \cdot (1 - m_5) \Longrightarrow f_3 = \frac{\gamma \cdot \delta}{\gamma + \delta}$$

$$f_1 = f_3 \Longleftrightarrow \frac{\alpha \cdot \beta}{\alpha + \beta} = \frac{\gamma \cdot \delta}{\gamma + \delta} \Longleftrightarrow q = 0$$

Thus, if $q \neq 0$ the configuration will not be an equilibrium configuration.

Globally speaking, this new system has the following poles: (0,0,0,-2,0,-2) (for $\boldsymbol{\lambda} = [1,1,1,1]^T$) and three linearly independent P-flows. The fourth zero valued pole appears in the configuration and is given by a new token conservation law which depends on $\boldsymbol{\lambda}$.

Obviously, the non controllable poles (P-flow related) appear in all the configurations. On the other hand, the controllable poles can have different values depending on the configuration. For example, if we consider the same system and the configuration $\{(p_1,t_1),(p_3,t_2),(p_5,t_3),(p_6,t_4)\}$, the poles are: (0,0,0,-1,-1,-2).

Example 39 *This shows that for a specific value of $\boldsymbol{\lambda}$, additional token conservation laws and zero valued poles can appear. Let us consider the net in Figure 10 with $\boldsymbol{\lambda} = [\alpha,\beta,\gamma]^T$ and let us assume the configuration $\{(p_4,t_1),(p_4,t_2),(p_3,t_3)\}$. For place p_2 we can write:*

$$\dot{m}_2 = f_1 - f_2 = \alpha \cdot \frac{m_4}{2} - \beta \cdot m_4 = m_4 \cdot \left(\frac{\alpha}{2} - \beta\right)$$

1. *$\frac{\alpha}{2} = \beta \Longrightarrow \dot{m}_2 = 0$. In this situation, a new zero valued pole introduces a new token conservation law. For example, if $\alpha = 2, \beta = 1, \gamma = 1$ the poles of this configuration are: $(0,0,0,-4)$.*

2. *$\frac{\alpha}{2} > \beta \Longrightarrow \dot{m}_2 > 0$. Now, the marking of p_2 will increase and since the net is bounded, this configuration will be left sooner or later. Moreover, a positive pole appears. If $\boldsymbol{\lambda} = [3,1,1]$ the configuration poles are: $(0,0,0.0981,-5.0981)$.*

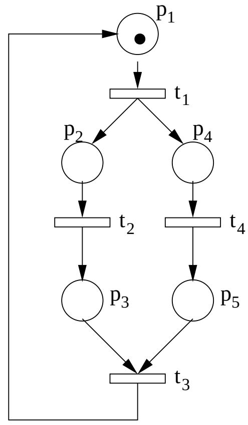

Figure 12: Choice-Free timed contPN system for Example 40.

3. *$\frac{\alpha}{2} < \beta \Longrightarrow \dot{m}_2 < 0$. The marking of p_2 will decrease and the only solution to be an equilibrium configuration is to reach the deadlock (m_4 should be 0). All the controllable poles are negative. For $\boldsymbol{\lambda} = [1, 1, 1]$ they are $(0, 0, -0.1771, -2.8229)$.*

So, other token conservation laws can appear depending on $\boldsymbol{\lambda}$ (here when $\frac{\alpha}{2} = \beta$) that introduce new zero valued but controllable poles.

5.4 Token conservation laws and controllable non zero valued poles

New token-invariant laws may appear depending on $\langle \mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{m_0} \rangle$ (i.e. depending not only on the net structure as those derived from the P-semiflows), but also on $\boldsymbol{\lambda}$ and the precise marking $\boldsymbol{m_0}$. Let us present a simple case.

Example 40 *Consider now the contPN in Figure 12 with $\boldsymbol{\lambda} = [\alpha, \beta, \delta, \gamma]^T$. There exist two P-semiflows: $m_1 + m_2 + m_3 = 1$ and $m_1 + m_4 + m_5 = 1$. Then there are only three state variables, for example m_1, m_3 and m_5. The dynamic linear system associated with the configuration $\{(p_1, t_1), (p_2, t_2), (p_3, t_3), (p_4, t_4)\}$ is:*

$$\begin{cases} \dot{m}_1 = \delta \cdot m_3 - \alpha \cdot m_1 \\ \dot{m}_3 = -\delta \cdot m_3 + \beta \cdot (1 - m_1 - m_3) \\ \dot{m}_5 = -\delta \cdot m_3 + \gamma \cdot (1 - m_1 - m_5) \end{cases}$$

Nevertheless, if $\beta = \gamma$, $\dot{m}_3 - \dot{m}_5 = -\beta \cdot (m_3 - m_5)$. Making a linear transformation in order to compute: $\bar{m}_{35} = m_3 - m_5$, then $\dot{\bar{m}}_{35} = -\beta \cdot \bar{m}_{35}$. If

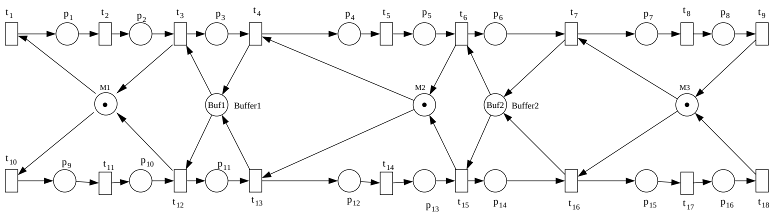

Figure 13: Continuous mono-T-semiflow reducible net system with $\boldsymbol{\lambda} = \mathbf{1}$.

$m_0[p_3] = m_0[p_5] \Longrightarrow \dot{m}_{35} = 0$. *In this case, the pole is different from 0, and depends on* $\boldsymbol{m_0}$*, thus* $m_3 = m_5$ *is a token conservation law that is not rooted in a zero valued pole.*

6 Case study

Let us consider the manufacturing system sketched in Fig. 13 which consists in three machines ($M1$, $M2$ and $M3$) and two intermediate buffers ($Buffer_1$ and $Buffer_2$). Assume that each operation takes 1 time unit. Hence, the firing rate of all the transitions is 1.

This net has 5 P-semiflows ($\boldsymbol{y_1} = p_1 + p_2 + p_9 + p_{10} + M_1$, $\boldsymbol{y_2} = p_3 + p_{11} + Buffer_1$, $\boldsymbol{y_3} = p_4 + p_5 + p_{12} + p_{13} + M_2$, $\boldsymbol{y_4} = p_6 + p_{14} + Buffer_2$, $\boldsymbol{y_5} = p_7 + p_8 + p_{15} + p_{16} + M_3$) introducing in every configuration 5 uncontrolled zero valued poles. Computing the optimal steady-state (maximum flow) for the controlled contPN, the solution is: $\boldsymbol{\varphi} = 0.2 \cdot \mathbf{1}$ and $\boldsymbol{u} = \mathbf{0}$.

This net has $\gamma = 256$ configurations and for each one Theorem 19 tells that all the equilibrium points have the same flow in steady state (i.e. 0.2) for the same control input $\boldsymbol{u} = \mathbf{0}$. Nevertheless, as expected, the equilibrium marking is not unique. For example, the configuration $\{(p_1, t_2), (p_2, t_3), (p_4, t_5), (p_5, t_6), (p_7, t_8), (p_8, t_9), (p_9, t_{11}), (p_{10}, t_{12}), (p_{12}, t_{14}), (p_{13}, t_{15}), (p_{15}, t_{17}), (p_{16}, t_{18}), (M_1, t_1), (M_1, t_{10}), (M_2, t_4), (M_2, t_{13}), (M_3, t_7), (M_3, t_{16})\}$ has:

$$rank \begin{bmatrix} \boldsymbol{\Pi} \\ \boldsymbol{B_y}^T \end{bmatrix} = 17 \qquad (21)$$

Therefore, this configuration may contain an infinite number of equilibrium markings (Theorem 22). It is easy to see that the places corresponding to the P-semiflows given by the buffers ($\boldsymbol{y_2}$ and $\boldsymbol{y_4}$) can be loaded in any quantity greater than 0.2 and an equilibrium marking is obtained.

Computing the poles of this configuration we obtain 9 zero valued poles, three of them equal with $-2+i$, other three $-2-i$ and six equal with -1. Five of these zero valued poles are uncontrollable and are given by the P-semiflows and the other four are given by some token conservation laws given by the particular value of $\boldsymbol{\lambda}$ and the considered configuration. Anyhow, these are controllable and can be moved using appropriate control law.

7 Conclusions

This work has dealt with some control problems of continuous Petri nets. Necessary conditions for the equilibrium points in steady-state are given by some easy algebraic equations. For continuous EQ nets the steady-state flow is unique, even if several steady-state markings are possible. For general contPNs, a necessary condition for the existence of several steady-state markings with different flows is presented. An optimal steady-state flow and input control problem is addressed by means of an LPP, that can be solved in polynomial time. In the last part of the paper the constraints on the actions are relaxed, and classical controllability theory of linear dynamic systems is used to provide a first interpretation to the class of systems that appear in our field. All transitions are assumed to be controllable, because otherwise it has been shown that the global system is not controllable. Controllability and control schemes are currently topics under consideration.

Appendix

Let us consider a time-invariant linear system expressed by:

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = \boldsymbol{A} \cdot \boldsymbol{x}(\tau) + \boldsymbol{B} \cdot \boldsymbol{u}(\tau) \\ \boldsymbol{y}(\tau) = \boldsymbol{S} \cdot \boldsymbol{x}(\tau) + \boldsymbol{D} \cdot \boldsymbol{u}(\tau) \end{cases} \tag{22}$$

where $\boldsymbol{x}(\tau) \in \boldsymbol{X}^n$ is the state of the system, $\boldsymbol{u}(\tau) \in \boldsymbol{U}^m$ the input control and $\boldsymbol{y}(\tau) \in \boldsymbol{Y}^l$ is the output.

Definition 41 *[?] [?] A dynamic system* (22) *is said to be* completely state controllable *if for any time τ_0, it is possible to construct an* unconstrained *control vector $\boldsymbol{u}(\tau)$ that will transfer a given initial state $\boldsymbol{x}(\tau_0)$ to a final state $\boldsymbol{x}(\tau)$ in a finite time interval.*

A very well-known controllability criterion exists which allows to decide whether a continuous linear system is controllable or not. Given a linear system (22), the *controllability matrix is* defined as:

$$\mathbb{C} = [\boldsymbol{B} \cdots \boldsymbol{A}^k \boldsymbol{B} \cdots \boldsymbol{A}^{(n-1)} \boldsymbol{B}] \tag{23}$$

Proposition 42 *[?] [?] A linear continuous-time system* (22) *is completely controllable iff $\mathbb{C}$ is full rank (i.e. $rank(\mathbb{C}) = n$). If $\mathbb{C}$ is not a full rank matrix then the controllable subspace has dimension $rank(\mathbb{C})$.*

Equation (22) corresponds to a state-space representation of the system description. Other representation is the input/output one. Applying Laplace transform to the first equation in (22) and considering null initial conditions ($\boldsymbol{x}(0) = \boldsymbol{0}$, which can always be obtained by translation) we have: $s \cdot \boldsymbol{x}(s) = \boldsymbol{A} \cdot \boldsymbol{x}(s) + \boldsymbol{B} \cdot \boldsymbol{u}(s)$ and combining with the second equation, the *transfer-matrix function* is obtained:

$$\boldsymbol{G}(s) = \frac{\boldsymbol{y}(s)}{\boldsymbol{u}(s)} = \boldsymbol{S} \cdot (s \cdot \boldsymbol{I} - \boldsymbol{A})^{-1} \cdot \boldsymbol{B} + \boldsymbol{D} = \frac{\boldsymbol{S} \cdot adj(s \cdot \boldsymbol{I} - \boldsymbol{A})\boldsymbol{B} + \boldsymbol{\Delta}(s)\boldsymbol{D}}{\boldsymbol{\Delta}(s)} \tag{24}$$

where, $adj(s \cdot \boldsymbol{I} - \boldsymbol{A})$ is the adjoint of the matrix $(s \cdot \boldsymbol{I} - \boldsymbol{A})$ and $\boldsymbol{\Delta}(s)$ the determinant of the same matrix.

The roots of the denominator of the transfer function are called *the poles* of the system and they can be obtained by solving the characteristic equation: $\boldsymbol{\Delta}(s) = det(s \cdot \boldsymbol{I} - \boldsymbol{A}) = 0$. Notice that the poles of the transfer function matrix and the eigenvalues of the matrix $\boldsymbol{A}$ are the same.

The poles play a very important role in system analysis and design. For example, if all poles have negative real part then the system is stable (for any bounded input, the output is bounded). If one pole has positive real part then the system is unstable. Zero valued poles corresponds to integrators.

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