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State Estimation of Petri Nets by Transformation

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Abstract— **In this paper we propose four transformation rules to estimate the marking of a net, discrete or continuous, satisfying the following assumptions: the set of transitions is partitioned into observable and unobservable transitions; the net structure and the initial marking is known. For each rule we derive a set of linear algebraic constraints that characterize the set of markings of the original net that are consistent with the observed firing sequence.**

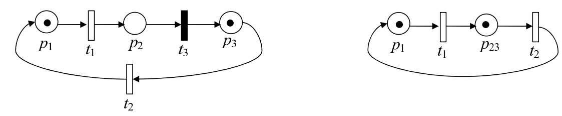

Fig. 1. A motivational example.

I. INTRODUCTION

This paper presents an original approach for the state estimation of Petri nets based on net transformations.

As in other works we assume that the set of transitions of the net is partitioned into two subsets: observable transitions whose firing can be detected by an external observer, and unobservable transitions whose firing cannot be detected. The initial marking of the net is assumed to be known.

Problems of this kind have been addressed by several authors. Benasser [1] has studied the possibility of defining the set of markings reached firing a "partially specified" set of transitions using logical formulas, without having to enumerate this set. Ramírez *at al.* [2] have discussed the problem of estimating the marking of a Petri net using a mix of transition firings and place observations. In the context of continuous nets observability has been studied by Mahulea [3].

In a previous work two of us have formally proved that —under some technical assumptions on the structure of the unobservable subnet— the set of markings consistent with the observed word can be represented by a linear system with a fixed structure that does not depend on the length of the observed word [4].

In this paper we present two new contributions:

1) we study the observability problem by means of net transformations;
2) we generalize our approach so that it can be applied to both discrete and continuous nets (untimed).

A classical Petri net analysis technique, called *analysis by transformation*, is based on the definition of *reduction rules* that preserve the properties of interest, while simplifying the structure of the net. Examples of this technique were presented by Berthelot [5] and include *place transformations*, that reduce the net structure by eliminating redundant places but do not modify the state space; *transition transformations*, which by fusing transitions reduce both structure and state space of the net. Another approach has been developed by Silva and *et al.* and is based on the determination of implicit places (see [6] for a review).

In this paper we propose to use transformation techniques for the state estimation of nets with unobservable transitions. The idea is that of removing the unobservable transitions and merging their input-output places so as to create new places, without influencing the rest of the net. The transformed net only contains observable transitions and its marking (including the marking of the new places) can be easily updated after each observable transition firing.

To reconstruct the marking of the original net it is necessary to determine the markings of the merged places. These markings can be expressed as the solution of a linear system that expresses their dependence from the marking of the new places plus eventually a set of additional constraints that keep track of the information on the initial marking.

As an example, consider the net in Fig. 1(a) where the occurrence of transitions t_1 and t_2 can be observed while transition t_3 is not observable. We may transform this net removing transition t_3 and merging its input/output places p_2 and p_3 to obtain the net in Fig. 1(b), that contains the new place p_{23}. The transformed net contains only observable transitions and its marking is known. It is also possible to reconstruct the possible markings of the original net (i.e., the markings of the merged places) given the marking of the transformed net and the observed sequence σ as follows:

$$\begin{cases} m_2 + m_3 & = & m_{23}(\sigma) \\ m_3 & \geq & 1 - \sigma_2 \\ m_2, m_3 & \geq & 0 \end{cases}$$

where m_2 and m_3 are the (unknown) markings of places p_2 and p_3, $m_{23}(\sigma)$ is the (known) marking of place p_{23} and σ_2 is the (observed) firing quantity of transition t_2. The first equation specifies that the sum of the markings of p_2 and p_3 must be equal to the marking of p_{23}. The second equation specifies that the marking of place p_3, that initially contains one token, cannot be less than $m_{0,3}$ minus the tokens removed by the firing of t_2: its marking can, however, be greater than this quantity because the unobservable transition t_3 may have fired without being observed.

Note, finally that the same approach can be used for untimed discrete nets and for untimed continuous nets. If the net is discrete then σ_2, m_2 and m_3 must be non-negative

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integers. If the net is continuous then σ_2, m_2 and m_3 must be non-negative real numbers.

An approach that is similar in spirit with the one we propose has been recently presented by Gourcuff *et al.* in [7], [8]. In these papers the authors propose a technique to reduce the number of variables in a PLC program by exploiting algebraic relationships between them. The goal is that of simplifying the subsequent formal verification phase.

As a final remark, if we compare the approach based on transformation with the approach presented in [4] we observe that in both approaches the set of markings consistent with the observed sequence is given by the solutions of a set of linear inequalities that depend on a set of parameters: the marking of new places in the former approach and the so-called *basis markings* in the latter. However, the computation of the marking of the new places is easier than the computation of the basis marking. This is the main advantage of the proposed technique.

In the rest of the paper a series of transformation rules are described. So far the rules we present can only be applied to classes of nets more restrictive than those considered in [4]. We believe, however, that the approach can be extended to richer classes of nets and this will be the goal of our future work.

II. Background on Untimed ContPN

Definition 1: A contPN system is a pair $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$, where:

- $\mathcal{N} = \langle P, T, \boldsymbol{Pre}, \boldsymbol{Post} \rangle$ is the net structure with two disjoint sets of places P and transitions T; pre and post incidence matrices $\boldsymbol{Pre}, \boldsymbol{Post} : P \times T \to \mathbb{R}_{\geq 0}$, denote the weight of the arcs from transitions to places (respectively, places to transitions);
- $\boldsymbol{m}_0 : P \to \mathbb{R}_{\geq 0}$ is the initial marking. ■

The input and output set of a node $x \in P \cup T$ is denoted by ${}^\bullet x$ and $x^\bullet$, respectively. The token load of a place p_i at the marking $\boldsymbol{m}$ is denoted by $\boldsymbol{m}[p_i]$ or simply by m_i.

A transition $t_j \in T$ is enabled at a marking $\boldsymbol{m}$ iff $\forall p_i \in {}^\bullet t_j$, $\boldsymbol{m}[p_i] \geq 0$ and the enabling degree of t_j at $\boldsymbol{m}$ is:

$$enab(t_j, \boldsymbol{m}) = \min_{p_i \in {}^\bullet t_j} \frac{m_i}{\boldsymbol{Pre}[p_i, t_j]} \tag{1}$$

When a transition t_j is enabled at a marking $\boldsymbol{m}$ it can be fired. The main difference with respect to discrete Petri nets is that in the case of contPNs it can be fired in any real amount α, with $0 \leq \alpha \leq enab(t_j, \boldsymbol{m})$ and it is not limited only to a natural number. Such a firing yields to a new marking $\boldsymbol{m'} = \boldsymbol{m} + \alpha \boldsymbol{C}[\cdot, t_j]$, where $\boldsymbol{C} = \boldsymbol{Post} - \boldsymbol{Pre}$ is the *token flow matrix* (or *incidence matrix*). This firing is also denoted $\boldsymbol{m}[t_j(\alpha)\rangle \boldsymbol{m'}$.

If a marking $\boldsymbol{m}$ is reachable from the initial marking through a firing sequence $\sigma = t_{r1}(\alpha_1) t_{r2}(\alpha_2) \cdots t_{rk}(\alpha_k)$, and we denote by $\boldsymbol{\sigma} : T \to \mathbb{R}_{\geq 0}$ the *firing count vector* whose component associated to a transition t_j is:

$$\sigma_j = \sum_{h \in H(\sigma, t_j)} \alpha_h$$

where $H(\sigma, t_j) = \{h = 1, \ldots, k | t_{r_h} = t_j\}$, then we can write $\boldsymbol{m} = \boldsymbol{m}_0 + \boldsymbol{C} \cdot \boldsymbol{\sigma}$, which is called the *fundamental equation* or *state equation*.

Note that a discrete net can be seen as a particular case of this model where $\boldsymbol{m}, \boldsymbol{Pre}, \boldsymbol{Post}, \alpha$ take only integer values.

III. Problem Statement

We propose a set of transformation rules to estimate the marking of a net, discrete or continuous, satisfying the following assumptions:

- the set of the transitions is partitioned into $T = T_o \cup T_u$, where T_o is the set of observable transitions and T_u is the set of unobservable transitions;
- the structure of the net and its initial marking $\boldsymbol{m}_0$ is known.

We provide a certain number of constructive rules to determine a transformed net and a set of linear algebraic constraints that characterize the set $\mathcal{C}(\sigma)$ of $\sigma-$consistent markings, i.e., the set of markings in which the original net can be after we have observed firing sequence σ.

One of the main features of the proposed approach is that the number of constraints depends on the structure of the original net and on the number of its unobservable transitions, but it is independent on the actual observation. Therefore, when a new observation occurs, we need to update certain parameters that define the set of constraints, while the structure of the constraints remains the same.

In this paper the presentation will be kept at an informal level. Moreover, in order to provide simpler and more intuitive explanations, we deal only with ordinary nets. A more formal and general derivation of the approach will be the object of our future work.

IV. The Observer Design

In this section we present four different structures of the original net $\langle \mathcal{N}, \boldsymbol{m}_0 \rangle$, and let us discuss the rules to construct the reduced net $\langle \mathcal{N}^R, \boldsymbol{m}_0^R \rangle$, and to derive the relative sets of constraints characterizing the set of consistent markings. Then, in the next section we will discuss a numerical example that clarifies how to combine the different rules given in this section when different structures appear simultaneously in the original net.

A. Rule 1: join unobservable transition

Let us consider an unobservable transition t satisfying the following assumptions.

(i) It is contact-free with other unobservable transitions, i.e., it does not share input and output places with other unobservable transitions. Thus, $\forall \bar{t} \in T_u \setminus \{t\}$, it holds ${}^\bullet t^\bullet \cap {}^\bullet \bar{t}^\bullet = \emptyset$.

(ii) It is conflict-free[1] and attribution-free[2] with observable transitions[3], i.e., $\forall \bar{t} \in T_o$, it holds ${}^\bullet t \cap {}^\bullet \bar{t} = \emptyset$ and $t^\bullet \cap \bar{t}^\bullet = \emptyset$.

(iii) It only has one output place, i.e., $|t^\bullet| = 1$.

As an example, let us consider the net in Fig. 2(a) where we have only one unobservable transition (t_{k+1}) with k input places (p_1 to p_k).

Note that for simplicity, we only consider one input transition to each place $p_1, \ldots, p_k$ (namely $t_1, \ldots, t_k$,

[1]Two transitions are conflict-free if they do not share a common input place.

[2]Two transitions are attribution-free if they do not share a common output place.

[3]Note that assumption (i) implies that it is also conflict-free with other unobservable transitions.

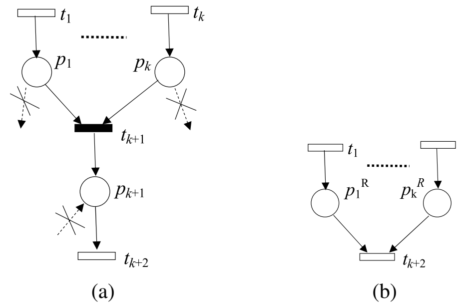


Fig. 2. Join unobservable transition.

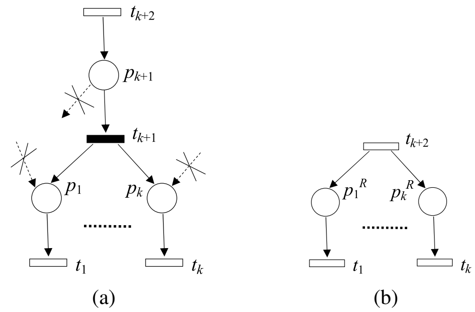


Fig. 3. Fork unobservable transition.

respectively), but the approach can be trivially generalized to the case of more input observable transitions. Analogously, we assume that p_{k+1} has only one output transition (namely t_{k+2}).

The reduced net is shown in Fig. 2(b) and it has been obtained by simply removing the unobservable transition t_{k+1} and merging places p_1–p_{k+1}, …, p_k–p_{k+1}. Thus, for any observation σ, the marking of the reduced net is given by:

$$\left\{\begin{array}{l} m_1^R(\sigma) = m_1 + m_{k+1}, \\ \quad\vdots \\ m_k^R(\sigma) = m_k + m_{k+1}. \end{array}\right. \tag{2}$$

Obviously, in the reduced net transition t_{k+2} is an output transition from all places p_1^R, …, p_k^R.

Let us observe that (2) is an application "one to many". In fact, it only has k equations in $k+1$ unknowns (m_1, …, m_k, m_{k+1}). Therefore, to avoid spurious solutions, we need one additional constraint that keeps track of the initial marking of the original net, that is known by assumption.

As an example, we can can consider as additional constraint

$$m_{k+1} \geq m_{0,k+1} - \sigma_{k+2}$$

where $\sigma_{k+2} = \sigma_{k+2}(\sigma)$ is the amount transition t_{k+2} has fired during the whole observation σ. Note that we put the symbol $\geq$ instead of $=$ because p_{k+1} has also one input unobservable transition (t_{k+1}) whose flow cannot be measured.

Note that we may alternatively assume as additional constraint, any of the following constraints

$$m_i \leq m_{0,i} + \sigma_i, \qquad i = 1, \ldots, k.$$

Summarizing, given a generic observation σ, it holds:

$$\left\{\begin{array}{lcl} m_1^R(\sigma) & = & m_{0,1}^R + \sigma_1 - \sigma_{k+2} \\ & = & m_{0,1} + m_{0,k+1} + \sigma_1 - \sigma_{k+2}, \\ & \vdots & \\ m_k^R(\sigma) & = & m_{0,k}^R + \sigma_k - \sigma_{k+2} \\ & = & m_{0,k} + m_{0,k+1} + \sigma_k - \sigma_{k+2}, \end{array}\right.$$

where σ_j is the total amount transition t_j has fired during the observation σ.

The set of markings consistent with a generic σ is thus given by:

$$\mathcal{C}(\sigma) = \left\{\begin{array}{ll} m_1 + m_{k+1} = m_1^R(\sigma) & (1) \\ \quad\vdots & \\ m_k + m_{k+1} = m_k^R(\sigma) & (k) \\ m_{k+1} \geq m_{0,k+1} - \sigma_{k+2} & (k+1) \\ m_1, \ \ldots, \ m_k, \ m_{k+1} \geq 0 & \end{array}\right. \tag{3}$$

where, as previously discussed, constraint $(k+1)$ is crucial to avoid spurious solutions.

Remark 2: Constraint $(k+1)$ in (3) is active until $\sigma_{k+2} = m_{0,k+1}$; after that it becomes redundant. ■

B. Rule 2: fork unobservable transition

We now consider an unobservable transition t satisfying the following three assumptions.

(i) It is contact-free with other unobservable transitions.
(ii) It is conflict-free and attribution-free with observable transitions.
(iii) It only has one input place, i.e., $|^\bullet t| = 1$.

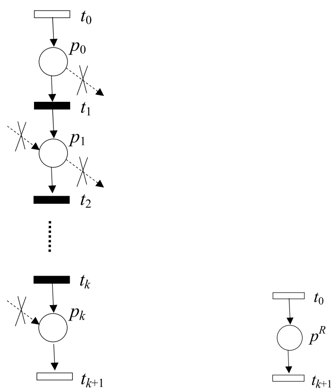
An example of this is given in Fig. 3(a) where we have one unobservable transition (t_{k+1}) with k output places (p_1 to p_k). Note that for simplicity, we assumed that p_{k+1} has only one input observable transition; however, all the results presented below can be trivially extended to the case of more input observable transitions to p_{k+1}. Analogously, we assumed that each place p_1 to p_k has only one output observable transition, but this not a requirement.

Using the same reasoning as in the above case, it is easy to obtain the reduced net sketched in Fig. 3(b) where the unobservable transition t_{k+1} has been removed, and the generic place p_i^R ($i = 1, \ldots, k$) has been obtained by merging places p_i and p_{k+1}.

For any observation σ, the set of consistent markings can be written as follows, where once again we have k equality constraints in $k+1$ unknowns (p_1, …, p_k, p_{k+1}), plus an additional inequality constraint (the $(k+1)$-th) that keeps track of the initial marking of the original net and avoids spurious solutions:

$$\mathcal{C} = \left\{\begin{array}{ll} m_1 + m_{k+1} = m_1^R(\sigma) & (1) \\ \quad\vdots & \\ m_k + m_{k+1} = m_k^R(\sigma) & (k) \\ m_{k+1} \leq m_{0,k+1} + \sigma_{k+2} & (k+1) \\ m_1, \ \ldots, \ m_k, \ m_{k+1} \geq 0 & \end{array}\right. \tag{4}$$

where



(a) (b)

Fig. 4. Series of unobservable transitions.

$$\begin{cases} m_1^R(\sigma) & = & m_{0,1}^R + \sigma_{k+2} - \sigma_1 \\ & = & m_{0,1} + m_{0,k+1} + \sigma_{k+2} - \sigma_1, \\ \vdots & & \\ m_k^R(\sigma) & = & m_{0,k}^R + \sigma_{k+2} - \sigma_k \\ & = & m_{0,k} + m_{0,k+1} + \sigma_{k+2} - \sigma_k. \end{cases}$$

We finally remark that the constraint $(k+1)$ in (4) can be replaced by any of the following constraints:

$$m_i \geq m_{0,i} - \sigma_1, \qquad i = 1, \ldots, k.$$

The constraint becomes redundant as soon as $\sigma_i = m_{0,i}$ (see Remark 2).

C. Rule 3: series of unobservable transitions

A series of transitions is defined as a set of transitions $\{t_1, \ldots, t_k\}$ such that

$$\begin{array}{l} {}^\bullet t_1 = \{p_0\}, \\ t_i^\bullet = {}^\bullet t_{i+1} = \{p_i\}, \quad i = 1, \ldots, k-1 \\ t_k^\bullet = \{p_k\}. \end{array}$$

We now consider a series of unobservable transitions that satisfies the following assumption.

(i) All transitions of the series are contact-free with other unobservable transitions.

(ii) It holds $p_i^\bullet = \{t_{i+1}\}$ for $i = 0, \ldots, k-1$, and ${}^\bullet p_i = \{t_i\}$ for $i = 1, \ldots, k$, i.e., no other transition may input in the places of the series (except for the initial one p_0) or may output from the places of the series (except for the final one p_k).

As an example, let us consider the series of k unobservable transitions in Fig. 4(a), where for simplicity we only considered one input observable transition to p_0 and one output observable transition from p_k.

The reduced net is reported in Fig. 4(b) and it has been obtained by simply merging places p_0 to p_k, thus getting the new place p^R. Here p^R has as input flow the flow coming from t_0. Finally, p^R has only one output transition that coincides with t_{k+1}.

Using the same notation as in the previous subsections, the set of markings consistent with the generic observation σ can be written as follows:

$$\mathcal{C}(\sigma) = \begin{cases} \sum_{i=0}^k m_i = m^R(\sigma) & (1) \\ m_k \geq m_{0,k} - \sigma_{k+1} & (2) \\ \sum_{i=k-1}^k m_i \geq \sum_{i=k-1}^k m_{0,i} - \sigma_{k+1} & (3) \\ \quad \vdots & \\ \sum_{i=2}^k m_i \geq \sum_{i=2}^k m_{0,i} - \sigma_{k+1} & (k) \\ \sum_{i=1}^k m_i \geq \sum_{i=1}^k m_{0,i} - \sigma_{k+1} & (k+1) \\ m_0 \ m_1, \ldots, \ m_k \geq 0 & \end{cases} \tag{5}$$

where

$$m^R(\sigma) = m_0^R + \sigma_0 - \sigma_{k+1} = \sum_{i=0}^k m_{0,i} + \sigma_0 - \sigma_{k+1}$$

Note that in such a case we have $k+1$ unknowns (namely the marking of places p_0, p_1, $\ldots$, p_k), while in (5) we only have one equality constraint (the first one) plus k inequality constraints that keep track of the initial marking of the original net. The physical meaning of the inequality constraints can be easily deduced using the same considerations as in the previous subsections.

D. Rule 4: free-choice conflict of observable and unobservable transitions

We now consider the case of a free-choice conflict of unobservable and observable transitions. In particular, we denote as $T_c = p^\bullet$ the set of transitions that are in conflict, where p is a given place in P. In the following we call T_c the *conflict set*.

We assume that transitions in T_c satisfy:

(i) They do not share output places, i.e., $\forall t, \bar{t} \in T_c$ it holds $t^\bullet \cap \bar{t}^\bullet = \emptyset$.

(ii) They only have one input place, i.e., $\forall t \in T_c$, it holds ${}^\bullet t = \{p\}$.

(iii) Unobservable transitions in T_c are contact-free with other unobservable transitions not in T_c, i.e., $\forall t \in T_c \cap T_u$ and $\forall \bar{t} \in T_u \setminus T_c$ it holds ${}^\bullet t^\bullet \cap {}^\bullet \bar{t}^\bullet = \emptyset$.

(iv) There is no self-loop involving unobservable transitions in T_c.

An example of this conflict is given in Fig. 5(a) where $T_c = \{t_1, \ldots, t_k, t_{k+1}\}$, $t_1, \ldots, t_k \in T_u$ and $t_{k+1} \in T_o$.

Note that here for simplicity we assumed that p_1 has only one input transition, and all places in $T_c^\bullet$ have only one output observable transition. Clearly, the results that follow can be easily generalized to the case of an arbitrarily large number of observable transitions entering and/or exiting place p_1 and places in $T_c^\bullet$.

The reduced net is reported in Fig. 5(b). To get this net we removed unobservable transitions t_1 to t_k, and merge places p_1–$p_{1,1}$, $\ldots$, p_1–p_{1,r_1}, $\ldots$ p_1–$p_{k,1}$, $\ldots$, p_1–p_{k,r_k}, thus obtaining places $p_{1,1}^R$, $\ldots$, p_{1,r_1}^R, $\ldots$, $p_{k,1}^R$, $\ldots$, p_{k,r_k}^R.

We assume that a fraction γ_i of the total flow that has entered p_1 —and that has not been removed by t_{k+1}— is reserved for the firing of transition t_j. Thus we assign the same weight to arcs t_{k+2}–$p_{1,1}^R$, $\ldots$, t_{k+2}–p_{1,r_1}^R (namely γ_1), $\ldots$, and to arcs t_{k+2}–$p_{k,1}^R$, $\ldots$, t_{k+2}–p_{k,r_k}^R (namely γ_k).

When transition t_{k+2} fires we have a flow entering place p_1 in the original net, and consequently places $p_{1,1}^R, \ldots, p_{1,r_1}^R$, $\ldots$, $p_{k,1}^R$, $\ldots$, p_{k,r_k}^R in the reduced net. In particular, if no unobservable transition fires, the same flow enters $p_{j,1}^R$, $\ldots$,

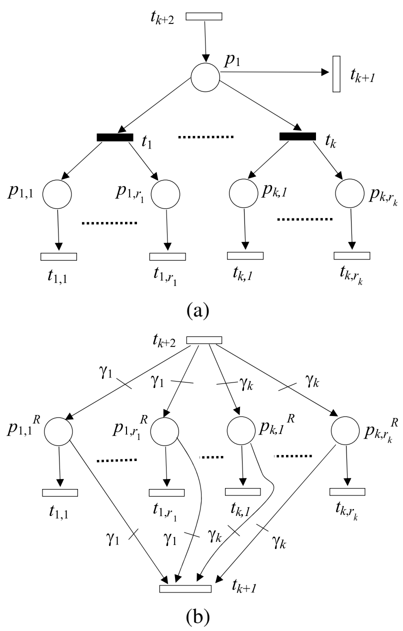


Fig. 5. Free-choice conflict of observable and unobservable transitions.

p^R_{j,r_j}, for $j = 1, \ldots, k$. On the contrary, if transition t_j fires the same flow enters places $p^R_{j,1}$, …, p^R_{j,r_j}, for $j = 1, \ldots, k$.

The weight of arcs $p^R_{1,1}$–t_{k+1}, …, p^R_{1,r_1}–t_{k+1}, …, $p^R_{k,1}$–$t_{k+1}, \ldots, p^R_{k,r_k}$–$t_{k+1}$ originates from the fact that the reduced net is representative of the original one if and only if the firing of $t_{k+1}(\alpha)t_{k+2}(\alpha)$, for any $\alpha \geq 0$, is such that no flow is accumulated in places $p^R_{1,1}$, …, p^R_{1,r_1}, …, $p^R_{k,1}$, …, p^R_{k,r_k}.

Note that for simplicity of presentation we only assumed one observable transition in T_c. However, in general cases, we can have $q > 1$ observable transitions in T_c. In such a case we simply have to add one arc from each place $p^R_{j,i}$, $j = 1, \ldots, k$, $i = 1, \ldots, r_j$, to each observable transition in T_c. The weight of the generic arc going from $p^R_{j,i}$ to the generic transition in T_c will be equal to γ_j.

The set of consistent markings is

$$\mathcal{C}(\sigma) = \left\{ \begin{array}{l} \left. \begin{array}{l} \gamma_1 \cdot m_1 + m_{1,1} = m^R_{1,1}(\sigma) \\ \quad \vdots \\ \gamma_1 \cdot m_1 + m_{1,r_1} = m^R_{1,r_1}(\sigma) \end{array} \right\} r_1 \\ \qquad \vdots \\ \left. \begin{array}{l} \gamma_k \cdot m_1 + m_{k,1} = m^R_{k,1}(\sigma) \\ \quad \vdots \\ \gamma_k \cdot m_1 + m_{k,r_k} = m^R_{k,r_k}(\sigma) \end{array} \right\} r_k \\ \sum_{j=1}^k \gamma_j = 1 \\ \left. \begin{array}{l} m_{1,1} \geq m_{0,1,1} - \sigma_{1,1} \\ \quad \vdots \\ m_{k,1} \geq m_{0,k,1} - \sigma_{k,1} \end{array} \right\} k \\ m_{j,1}, \ldots, m_{j,r_j} \geq 0, \quad j = 1, \ldots, k \\ \gamma_j \geq 0, \quad j = 1, \ldots, k \end{array} \right. \tag{6}$$

where

$$\begin{array}{rcl} m^R_{j,i}(\sigma) & = & m^R_{0,j,i} + \gamma_j \cdot \sigma_{k+2} - \sigma_{j,i} \\ & = & m_{0,1} + m_{0,j,i} + \gamma_j \cdot \sigma_{k+2} - \sigma_{j,i} \end{array}$$

and $j = 1, \ldots, k$, $i = 1, \ldots, r_j$.

The above constraints characterizing $\mathcal{C}(\sigma)$ are clearly nonlinear. However, they can be easily linearized by defining k dummy variables:

$$x_j = \gamma_j \cdot m_1, \qquad j = 1, \ldots, k.$$

In particular, they can be rewritten as:

$$\mathcal{C}(\sigma) = \left\{ \begin{array}{l} \left. \begin{array}{l} x_1 + m_{1,1} = m^R_{1,1}(\sigma) \\ \quad \vdots \\ x_1 + m_{1,r_1} = m^R_{1,r_1}(\sigma) \end{array} \right\} r_1 \\ \qquad \vdots \\ \left. \begin{array}{l} x_k + m_{k,1} = m^R_{k,1}(\sigma) \\ \quad \vdots \\ x_k + m_{k,r_k} = m^R_{k,r_k}(\sigma) \end{array} \right\} r_k \\ \sum_{j=1}^k x_j = m_{0,1} \cdot m_1 + \sigma_{k+2} \cdot m_1 \\ \left. \begin{array}{l} m_{1,1} \geq m_{0,1,1} - \sigma_{1,1} \\ \quad \vdots \\ m_{k,1} \geq m_{0,k,1} - \sigma_{k,1} \end{array} \right\} k \\ m_{j,1}, \ldots, m_{j,r_j} \geq 0, \quad j = 1, \ldots, k \\ x_j \geq 0, \quad j = 1, \ldots, k \end{array} \right. \tag{7}$$

where $m^R_{j,i}(\sigma)$, $j = 1, \ldots, k$ and $i = 1, \ldots, r_j$, are defined as already specified above.

Note that here we have $\sum_{j=1}^k r_j + 1 + k$ unknowns, i.e., the marking of the output places of unobservable transitions (namely $m_{j,i}$ for $j = 1, \ldots, k$, $i = 1, \ldots, r_j$), the marking of p_1, and x_j for $j = 1, \ldots, k$. The number of constraints is still equal to $\sum_{j=1}^k r_j + 1 + k$, where the first $\sum_{j=1}^k r_j$ constraints are equality constraints, while the remaining $k + 1$ are inequality constraints that keep track of the initial marking of the original net.

An important remark needs to be done. The last k inequality constraints in (6) (or equivalently in (7)) have been written by looking at the marking of places $p_{1,1}$, …, $p_{k,1}$. However, we can replace such constraints with any other set of k constraints of the same form, where each constraint is relative to an (arbitrarily) selected output place of a different unobservable transition.

V. A NUMERICAL EXAMPLE

Let us consider the net in Fig. 6(a) where the initial marking is $\boldsymbol{m}_0 = [m_{0,1}\ m_{0,2}\ m_{0,3}\ m_{0,4}\ m_{0,5}\ m_{0,6}\ m_{0,7}\ m_{0,8}\ m_{0,9}\ ]^T = [1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]^T$ and the observable transitions are $T_o = \{t_i\}$ for $i = 1, \ldots, 5$. We want to reduce this net with the rules defined in the previous section. In the first step shown in Fig. 6(b), according to Rule 3, we substitute the series $p_5 - t_9 - p_7 - t_{11} - p_9$ with the place p_{10}. In the second step shown in Fig. 6(c), according to Rule 3, we substitute the series $p_4 - t_8 - p_6 - t_{10} - p_8$ with the place p_{11}. Finally, in the third step reported in Fig. 6(d), according to Rule 4, we reduced the free-choice conflict of observable transition t_1 and unobservable transitions t_6 and t_7 with places p_{12} and p_{13}.

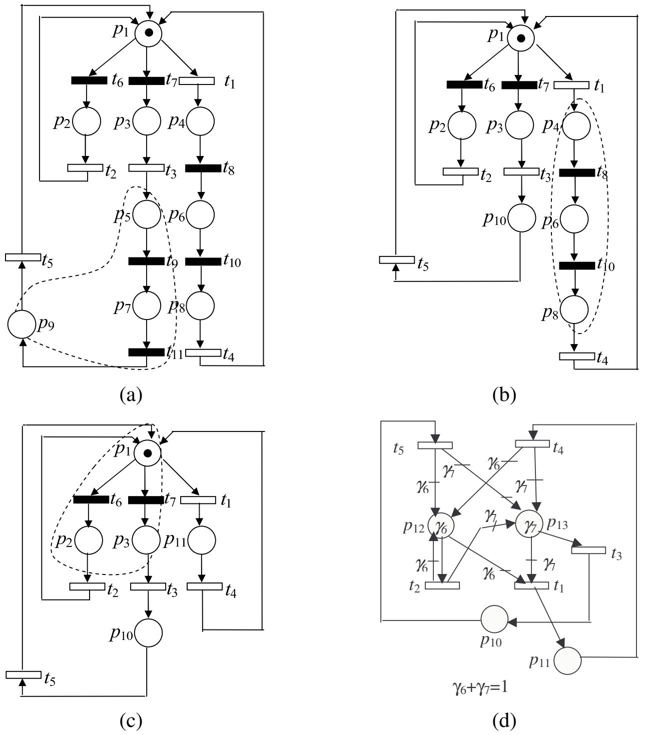


Fig. 6. The numerical example in Section V.

Note that the reduced net obtained, shown in Fig. 6(d), is a Petri net with a parameterized structure,i.e., the marking of places p_{12} and p_{13}, equal respectively to γ_6 and γ_7, are unknown variables, not numbers.

The σ-consistent markings, i.e., all the markings that are consistent with the observed firing sequence are all solutions of the following system:

$$\mathcal{C}(\sigma)=\begin{cases} \left.\begin{array}{l} m_5+m_7+m_9=m_{10}(\sigma) \\ m_7+m_9 \geq m_{0,7}+m_{0,9}-\sigma_5 \\ m_9 \geq m_{0,9}-\sigma_5 \end{array}\right\} \text{step 1} \\ \left.\begin{array}{l} m_4+m_6+m_8=m_{11}(\sigma) \\ m_6+m_8 \geq m_{0,6}+m_{0,8}-\sigma_4 \\ m_8 \geq m_{0,8}-\sigma_4 \end{array}\right\} \text{step 2} \\ \left.\begin{array}{l} \gamma_6 m_1+m_2=m_{12}(\sigma) \\ \gamma_7 m_1+m_3=m_{13}(\sigma) \\ m_2 \geq m_{0,2}-\sigma_2 \\ m_3 \geq m_{0,3}-\sigma_3 \\ \gamma_6+\gamma_7=1 \\ m_1+m_2+m_3 \geq 1-\sigma_1-\sigma_2-\sigma_3 \end{array}\right\} \text{step 3} \\ \gamma_6,\gamma_7,\boldsymbol{m} \geq 0 \end{cases}$$

where

$$\begin{cases} m_{10}(\sigma) = m_{0,10}+\sigma_3-\sigma_5 \\ \quad = m_{0,5}+m_{0,7}+m_{0,9}+\sigma_3-\sigma_5 \\ m_{11}(\sigma) = m_{0,11}+\sigma_1-\sigma_4 \\ \quad = m_{0,4}+m_{0,6}+m_{0,8}+\sigma_1-\sigma_4 \\ m_{12}(\sigma) = m_{0,12}+\gamma_6(\sigma_4+\sigma_4+\sigma_5-\sigma_1)-\sigma_2 \\ \quad = m_{0,1}+m_{0,2}+\gamma_6(\sigma_4+\sigma_4+\sigma_5-\sigma_1)-\sigma_2 \\ m_{13}(\sigma) = m_{0,13}+\gamma_7(\sigma_2+\sigma_4+\sigma_5-\sigma_1)-\sigma_3(\tau) \\ \quad = m_{0,1}+m_{0,3}+\gamma_7(\sigma_2+\sigma_4+\sigma_5-\sigma_1)-\sigma_3(\tau) \end{cases}$$

The 12th constraint, i.e., $m_1+m_2+m_3 \geq 1-\sigma_1-\sigma_2-\sigma_3$, is added to tackle the initial marking in the original net. This constraint will become redundant when the sum of the firings of transitions t_1, t_2 and t_3 will be greater than 1.

We linearize the above constraints introducing the dummy variables: $x=\gamma_6 m_1$, $y=\gamma_7 m_1$. We substitute the value of the initial marking previously introduced and we obtain:

$$\mathcal{C}(\sigma)=\begin{cases} m_5+m_7+m_9=0 \\ m_7+m_9 \geq -\sigma_5 \\ m_9 \geq -\sigma_5 \\ m_4+m_6+m_8=0 \\ m_6+m_8 \geq -\sigma_4 \\ m_8 \geq -\sigma_4 \\ x+m_2=\gamma_6 \\ y+m_3=\gamma_7 \\ m_2 \geq -\sigma_2 \\ m_3 \geq -\sigma_3 \\ x+y=m_1 \\ m_1+m_2+m_3 \geq 1-\sigma_1-\sigma_2-\sigma_3 \\ x,y,\gamma_6,\gamma_7,\boldsymbol{m} \geq 0 \end{cases}$$

VI. Conclusions and Future Work

In this paper we have provided a solution to the problem of estimating the marking of a net, based on four transformation rules. For each rule we have given a set of linear algebraic constraints that characterize the set of markings of the original net that are consistent with the observed firing sequence. An interesting application of these results can be in fault detection when the faulty behavior is modeled by unobservable transitions [9]. Since here the computational effort of obtaining the set of consistent markings with an observed firing sequence is smaller than in [9] we can expect to obtain better results. We plan to extend this technique to more general classes of Petri nets and also to timed Petri nets.

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