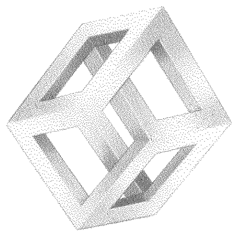
A decomposition approximation method for general closed queueing networks

Bruno BAYNAT

Associate Professor, *University Pierre & Marie Curie, Paris 6*

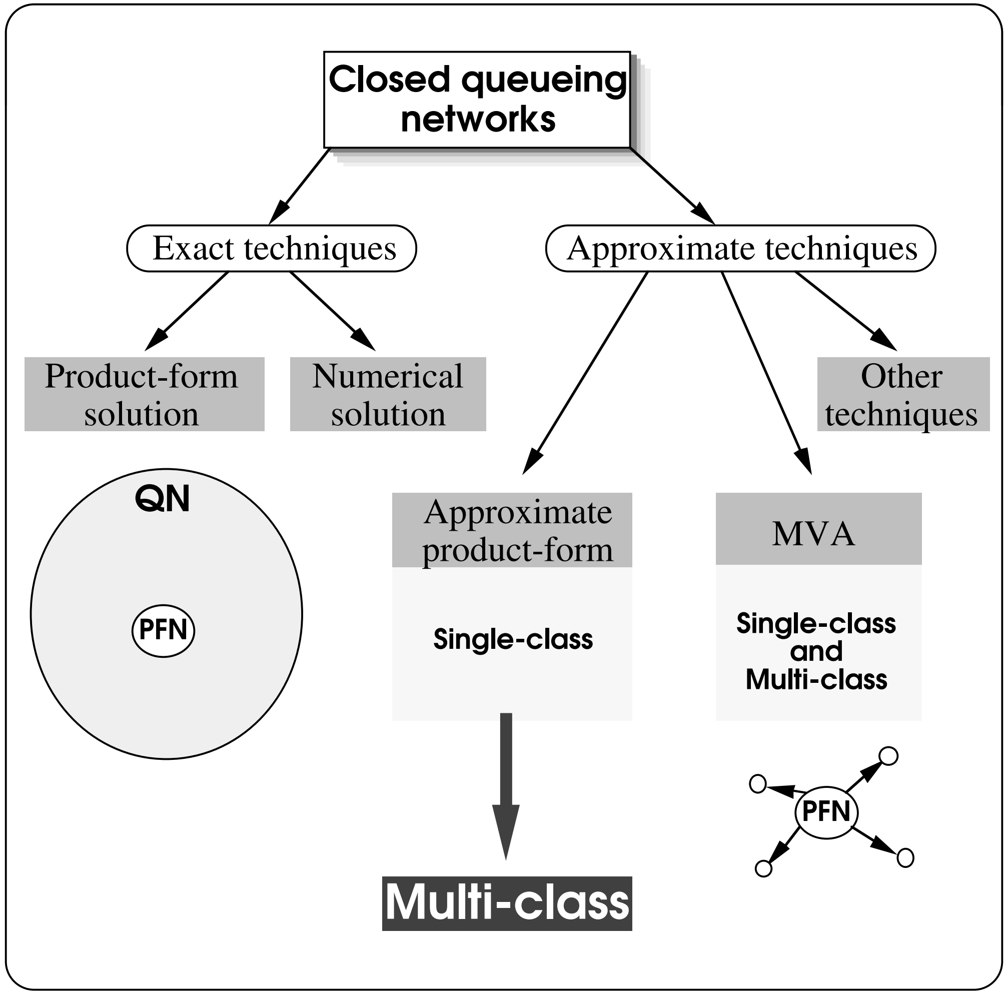
Department MASI, Team MAQ², Yves DALLERY



Visiting Scholar, Department of *IEOR, Columbia University, NYC*

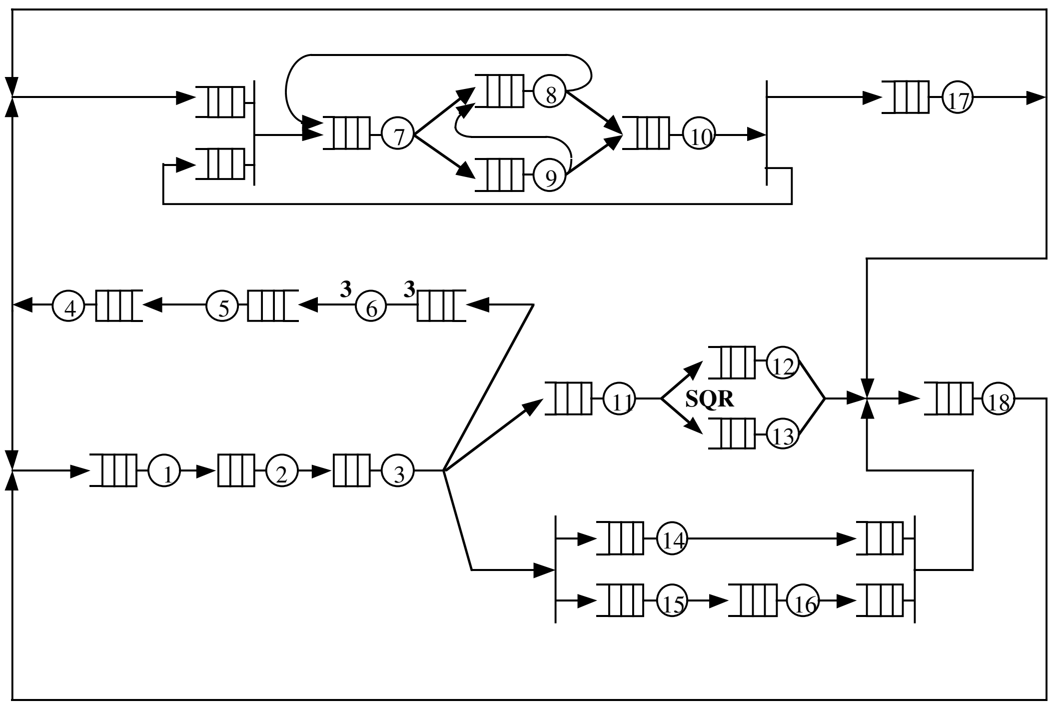
March-September 1996

Analytical techniques for closed queueing networks



General closed queueing networks

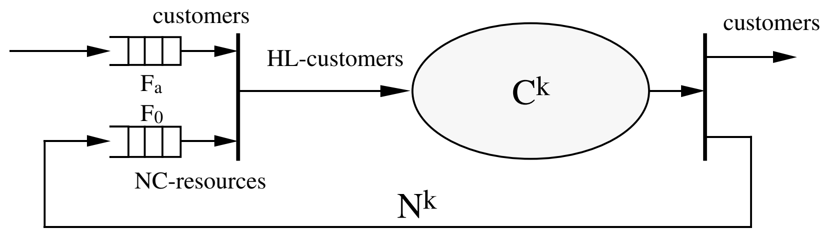
General features:
- Several classes of customers
- General service-time distributions
- Priorities
- State-dependent mecanisms
- Blocking
- Synchronizations
- Shared resources
- Population constraints
- Fork/join
- ...



Higher-Level and Lower-Level customers

Higher-Level customers (HL-customers):

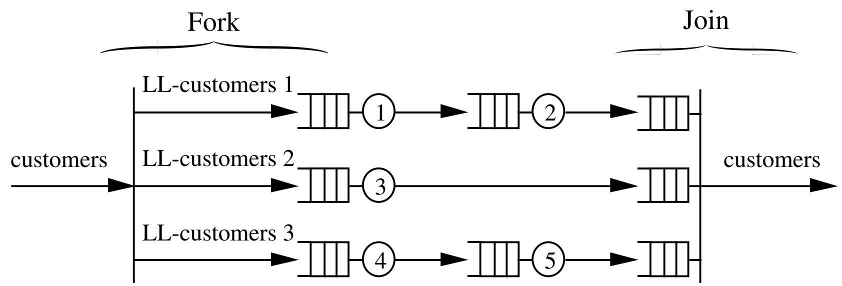
- Population constraint mechanism: Semaphore representation



F_a: external queue of subnetwork C^k
F_0: Non-Consumable resources (NC-resources) queue
N^k: total number of NC-resources

Lower-Level customers (LL-customers):

- Fork/Join mecanism

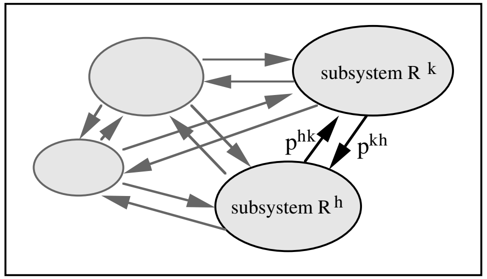


General principle of the single-class method

1 Decompose the system into K disjoint subsystems:

$(R^1, ..., R^k, ..., R^K)$

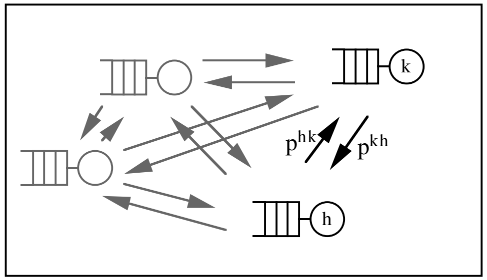
This partition has to satisfy 4 basic assumptions.



mean routing probabilities p^{hk}

$\Rightarrow$ average visit rates V^k

2 Associate with this decomposition of the system, an equivalent product-form network consisting of K exponential stations with load-dependent service rates $\mu^k(n^k)$:



$\mathbf{n} = (n^1, ..., n^k, ..., n^K)$

$$P(\mathbf{n}) = \frac{1}{G(K,N)} \prod_{k=1}^{K} \left[\frac{\left[V^k\right]^{n^k}}{\prod_{n=1}^{n^k} \mu^k(n)} \right]$$

First approximation

3 Estimate the rates $\mu^k(n^k)$ of the stations of this equivalent network

Second approximation

Assumptions on the partition

A1 All transitions between subsystems are single-step transitions.

($\Rightarrow$ bulk departures, ...)

A2 The behaviour of each subsystem R^k only depends on the state of this subsystem.

($\Rightarrow$ state dependent service rates, blocking, ...)

A3 The input/output behaviour of a customer through each subsystem is independent of the state-dependent routing decisions taken during its sojourn in the subsystem.

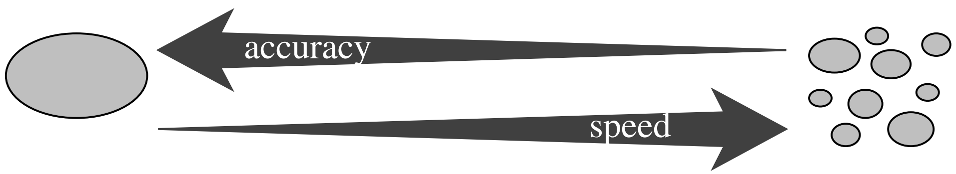
($\Rightarrow$ state dependent routing)

A4 The only transitions that may occur between subsystems are customer transitions. So, the following entities can not transit between subsystems :

- low level customers (LL-customers)
- high level customers (HL-customers)
- resources

($\Rightarrow$ semaphore mecanisms, fork/join, ...)

Partition satisfying A1 to A4 = **Feasible** partition



maximal

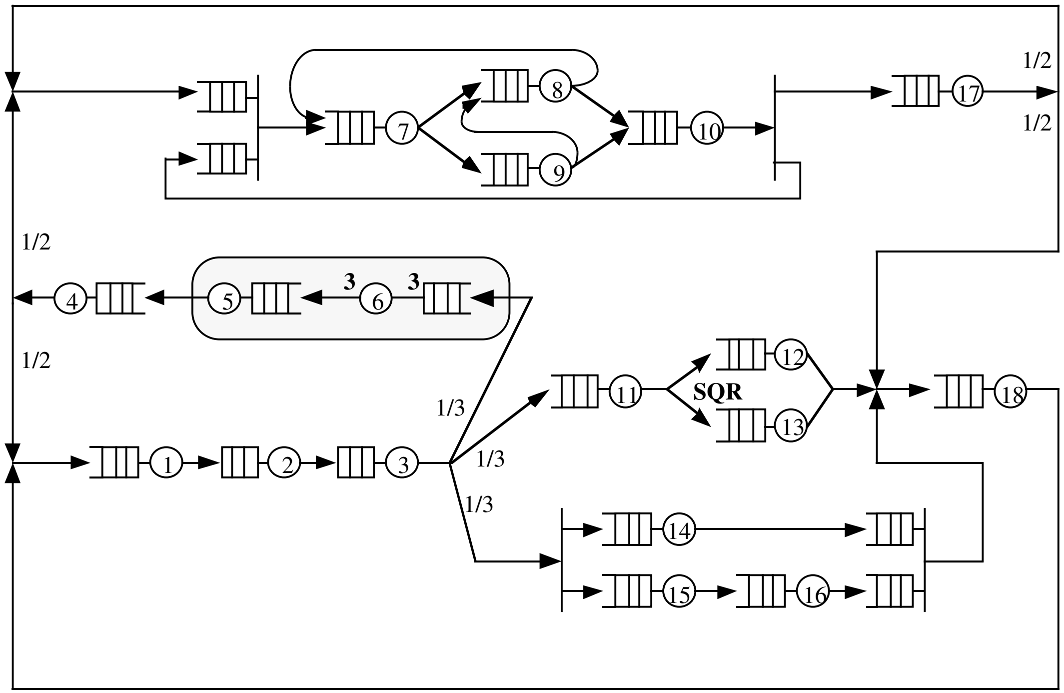
Choose of a partition =

Trade-off between accuracy and complexity

Assumptions on the partition

A1 All transitions between subsystems are single-step transitions.

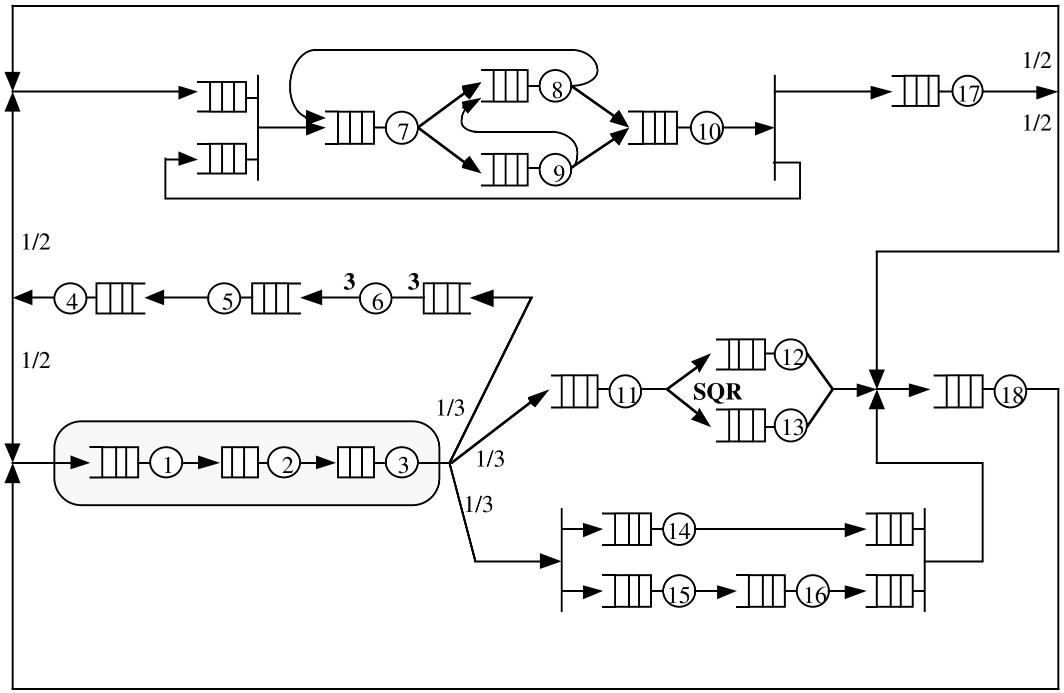
(⇒ bulk departures, ...)



Assumptions on the partition

A2 The behaviour of each subsystem R^k only depends on the state of this subsystem.

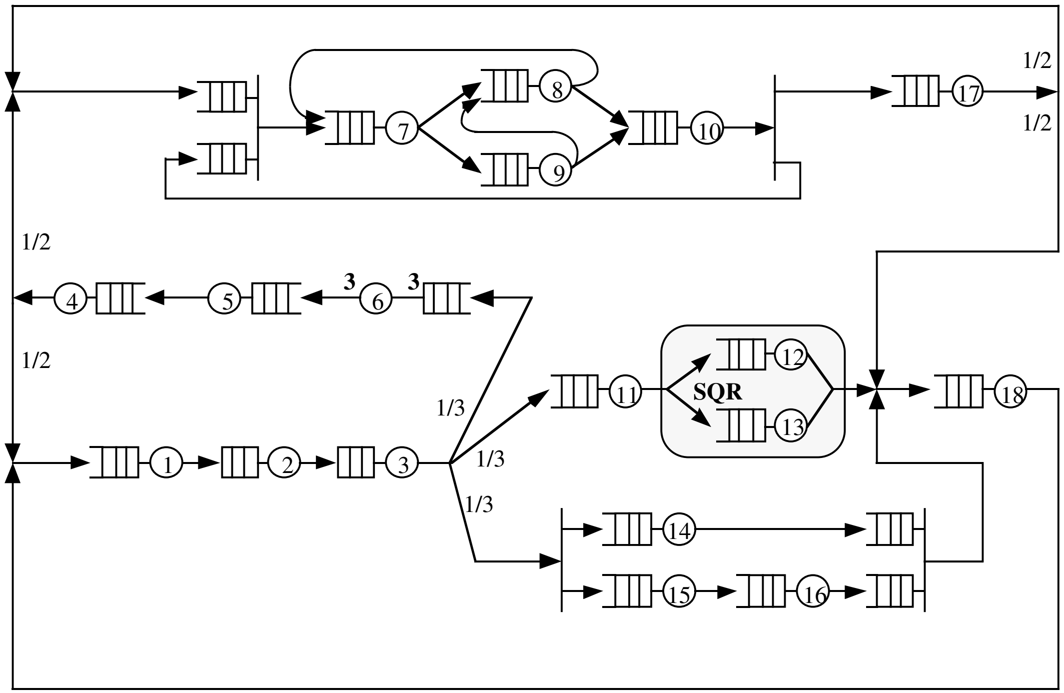
(⇒ state dependent service rates, blocking, ...)



Assumptions on the partition

A3 The input/output behaviour of a customer through each subsystem is independent of the state-dependent routing decisions taken during its sojourn in the subsystem.

(⇒ state dependent routing)

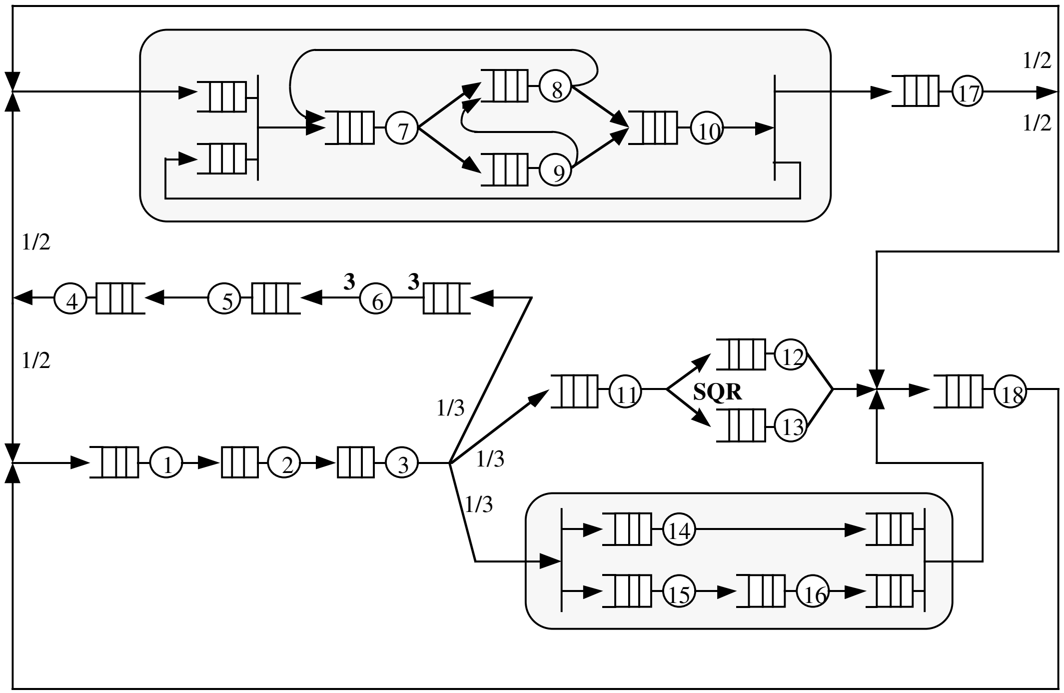


Assumptions on the partition

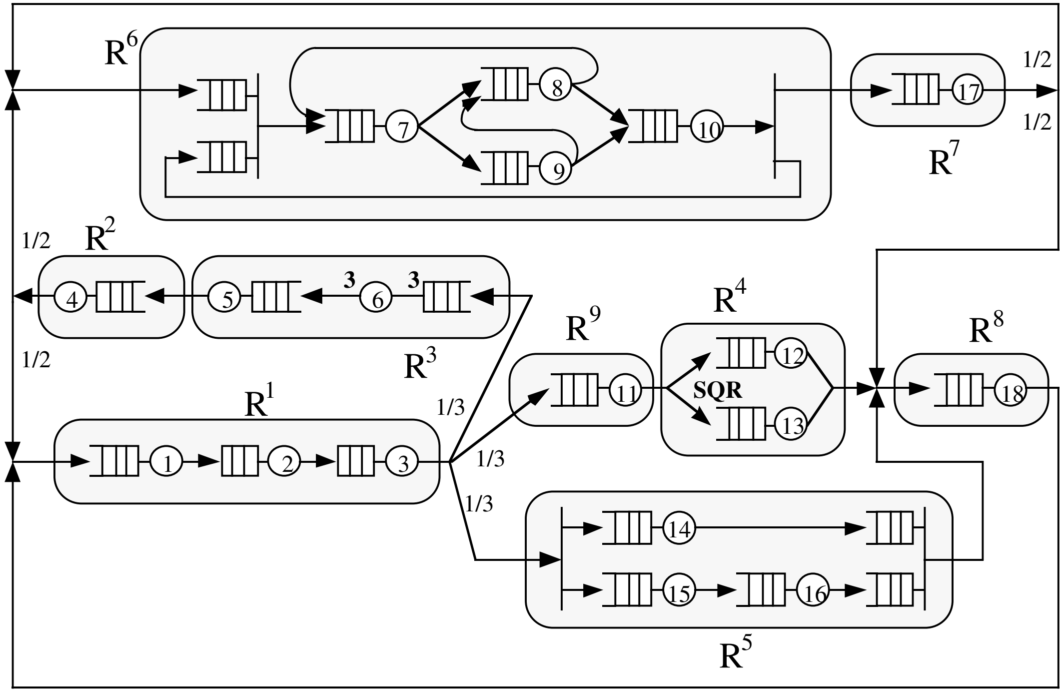
A4 The only transitions that may occur between subsystems are customer transitions. So, the following entities can not transit between subsystems:

- low-level customers (LL-customers)
- high-level customers (HL-customers)
- resources

(⇒ semaphore mecanisms, fork/join, ...)



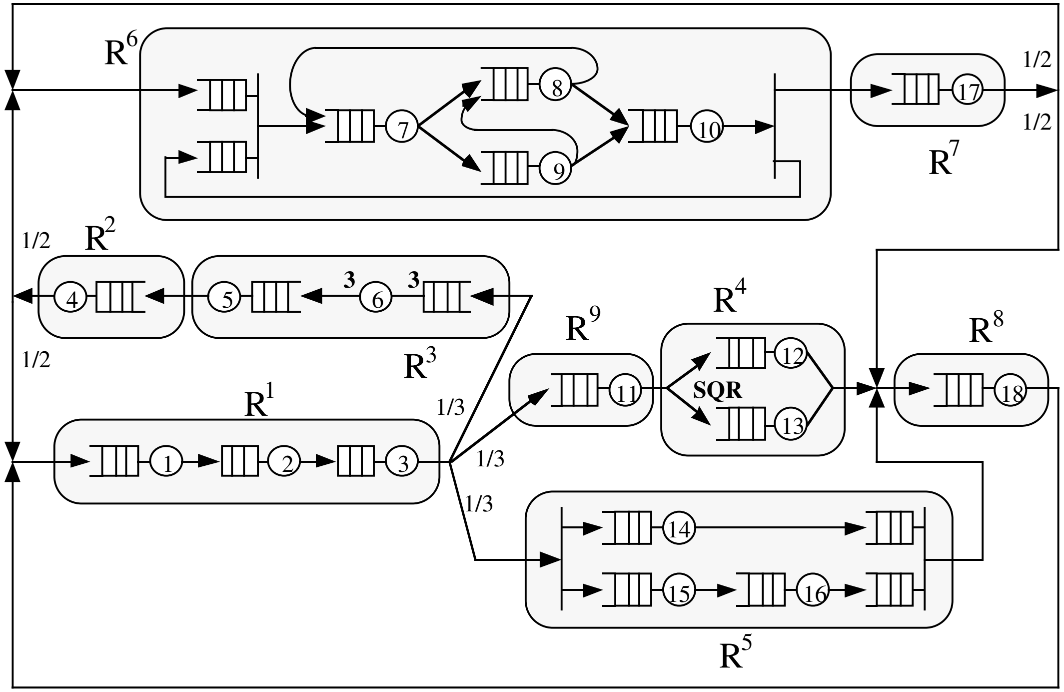
Feasible partition of the example



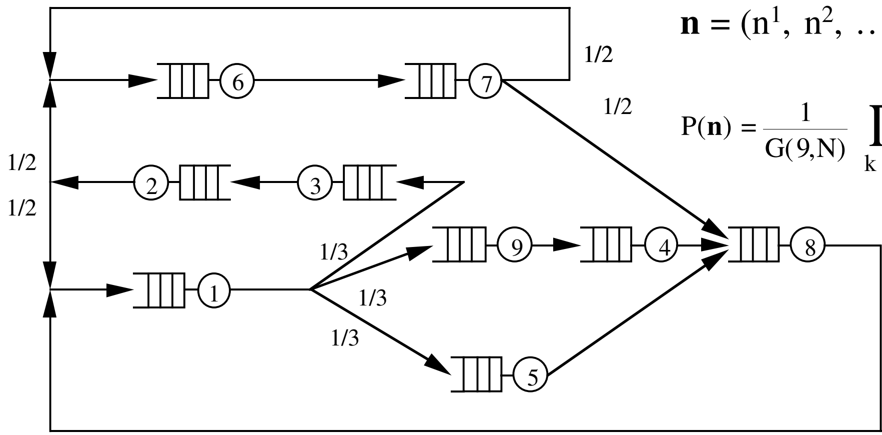
global visit rates:

$$V^1 = 1\ ;\ V^2 = \frac{1}{3}\ ;\ V^3 = \frac{1}{3}\ ;\ V^4 = \frac{1}{3}\ ;\ V^5 = \frac{1}{3}\ ;\ V^6 = \frac{1}{3}\ ;\ V^7 = \frac{1}{3}\ ;\ V^8 = \frac{5}{6}\ ;\ V^9 = \frac{1}{3}$$

Association of a product-form network



$\Downarrow$



Product-form:

$$\mathbf{n} = (n^1,\ n^2,\ \ldots,\ n^9)$$

$$P(\mathbf{n}) = \frac{1}{G(9,N)} \prod_{k=1}^{9} \left[\frac{\left[V^k\right]^{n^k}}{\prod_{n=1}^{n^k} \mu^k(n)} \right]$$

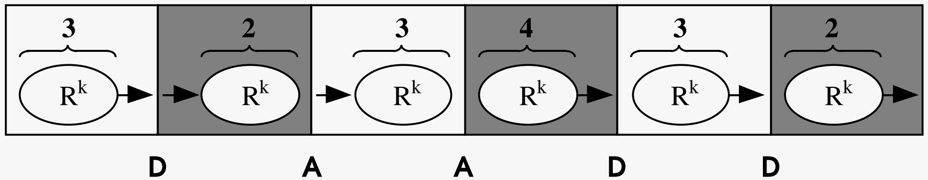
Estimation of the parameters of the equivalent network

Ideally: The service rates $\mu^k(n^k)$ should be equal to the conditional throuhgputs $\nu^k(n^k)$ of the corresponding subsystem in the initial system.

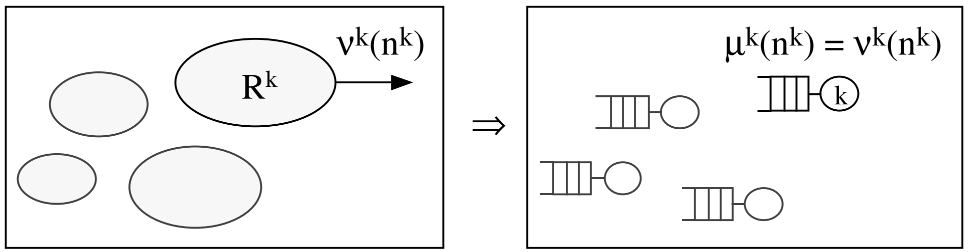
Conditional throuhgputs $\nu^k(n^k)$

$\Updownarrow$

Average flow of customers out of subsystem R^k when the state of this subsystem is n^k.



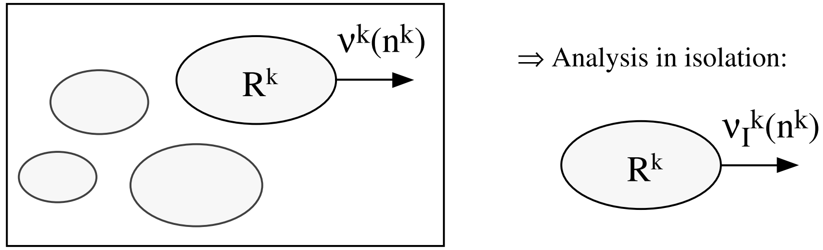
$$\nu^k(3) = \frac{D^k(3)}{T^k(3)} = \frac{2}{3}$$



Require an exact solution of the initial system !

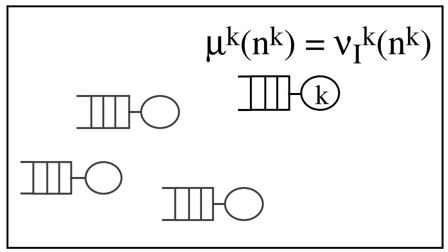
Estimation of the parameters of the equivalent network

Aim: Estimate the conditional throughputs $\nu^k(n^k)$ of the different subsystems in the initial system by the conditional throughputs $\nu_I^k(n^k)$ of the subsystems analyzed in isolation.



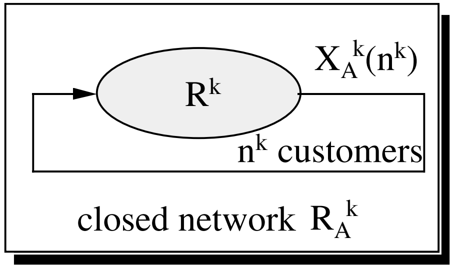
2 techniques
- the aggregation technique $\Rightarrow \nu_A^k(n^k)$
- Marie's method $\Rightarrow \nu_M^k(n^k)$

Then choose the service rates $\mu^k(n^k)$ equal to the estimated conditional throughputs.



The aggregation technique

Principle: Analyze the behaviour of subsystem R^k in isolation under a constant load of n^k customers:

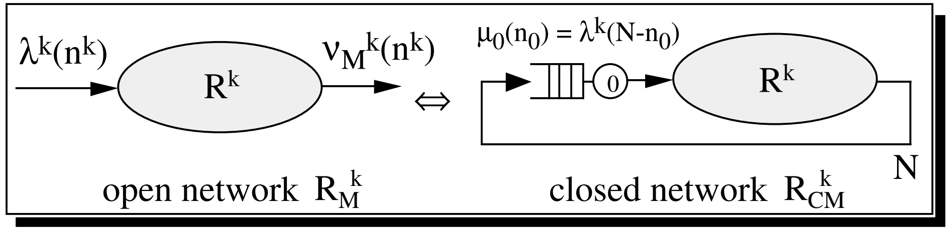


- Analyze closed network $R_A{}^k$ using any appropriate technique for a population n^k varying from 1 to N.
  - exact analysis (PF, numerical analysis)
  - approximative analysis (simulation, general method)
- Calculate the throughputs $X_A{}^k(n^k)$ of this closed network.
- $\nu_A{}^k(n^k) = X_A{}^k(n^k)$
- Estimation of the service rates of the associated exponential station:

$$\boxed{\mu^k(n^k) = X_A{}^k(n^k)}$$

Marie's method

Principle: Analyze the subsystem R^k in isolation as an open model, under a state-dependent Markovian arrival process, which rate $\lambda^k(n^k)$ depends on the total number n^k of customers currently present in the subsystem:



☞ First, if we suppose that the $\lambda^k(n^k)$ are known:

- Analyze in isolation open network $R_M{}^k$ using any appropriate technique.
- Derive the steady-state probabilities $P_I{}^k(n^k)$ of this open system.
- Calculate $\nu_M^k(n^k) = \lambda^k(n^k\text{-}1)\,\dfrac{P_I^k(n^k\text{-}1)}{P_I^k(n^k)}$ for $n^k = 1, \ldots, N$
- Estimate the service rates as : $\boxed{\mu^k(n^k) = \nu_M{}^k(n^k)}$

☞ Conversely, now that the service rates $\lambda^k(n^k)$ are obtained:

- Derive the arrival rates $\lambda^k(n^k)$ from the product-form solution.

$\mu^k(n^k)$ solutions of a fixed-point problem
$\Rightarrow$ **Iterative procedure**

Comparison Aggregation/Marie's method

Accuracy:

The conditional throughputs of a given subsystem in the initial system depend on the whole netwok.

☞ **Aggregation**: the conditional throughputs are independent of the "outside world".

☞ **Marie's method**: approximate modeling of the "outside world".

⇒ Marie's method > Aggregation

Complexity:

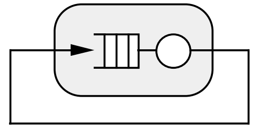
☞ **Aggregation**:
- subsystems in isolation: closed networks to be analyzed for each population from 1 to N
- non iterative

☞ **Marie's method**:
- subsystems in isolation: closed networks with an extra station to be analyzed with a single population N
- iterative

In pratice: about 10 iterations to reach convergence for Marie's method.

Analysis in isolation according to the aggregation technique

Subsystems consisting of a single station:



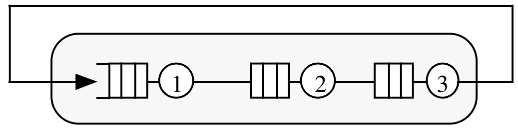
n^k

$X^k(n^k) = \frac{1}{m}$

⇒ known to be a bad approximation

Other subsystems:

n^k

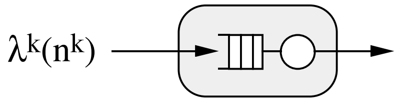


for $n^k = 1, ..., N$

⇒ N analyses in isolation

Analysis in isolation according to Marie's method

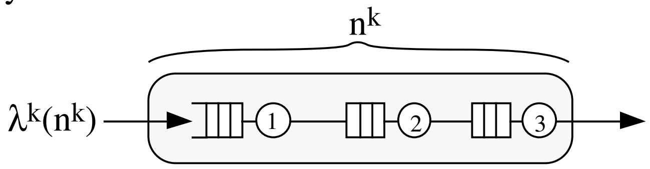
Subsystems consisting of a single station:



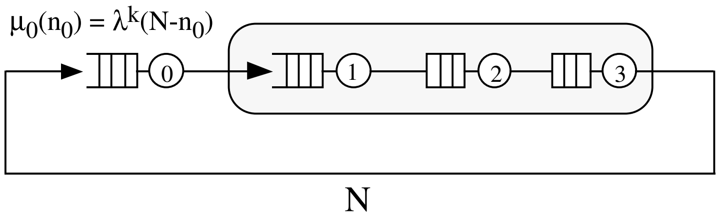
$\lambda(n)/G/1/N$ queue

$\Rightarrow$ known to be a good approximation

Other subsystems:



$\Updownarrow$



$\Rightarrow$ n_M analyses in isolation

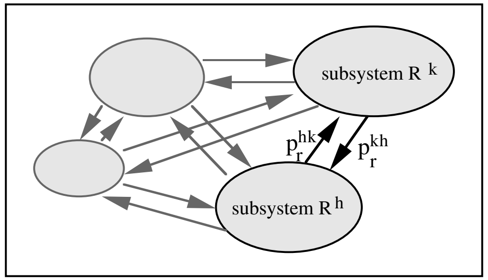
n_M = number of iterations to reach the convergence

Principle of the multi-class general method

1 Decompose the system into K disjoint subsystems:

$(R^1, ..., R^k, ..., R^K)$

This partition is obtained from the different single-class partitions.

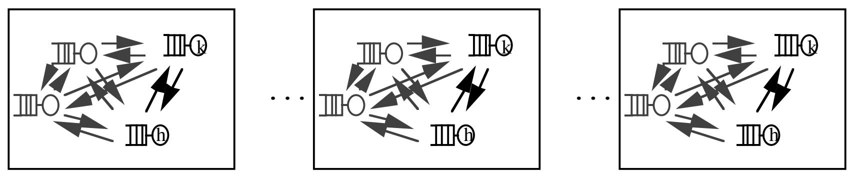


mean routing probabilities p_r^{hk}

$\Rightarrow$ average visit rates V_r^k

2 Associate with this decomposition of the system, R equivalent product-form networks of exponential stations with load-dependent service rates $\mu_r^k(n_r^k)$:

class 1 class r class R



PFN 1 **PFN r** **PFN R**

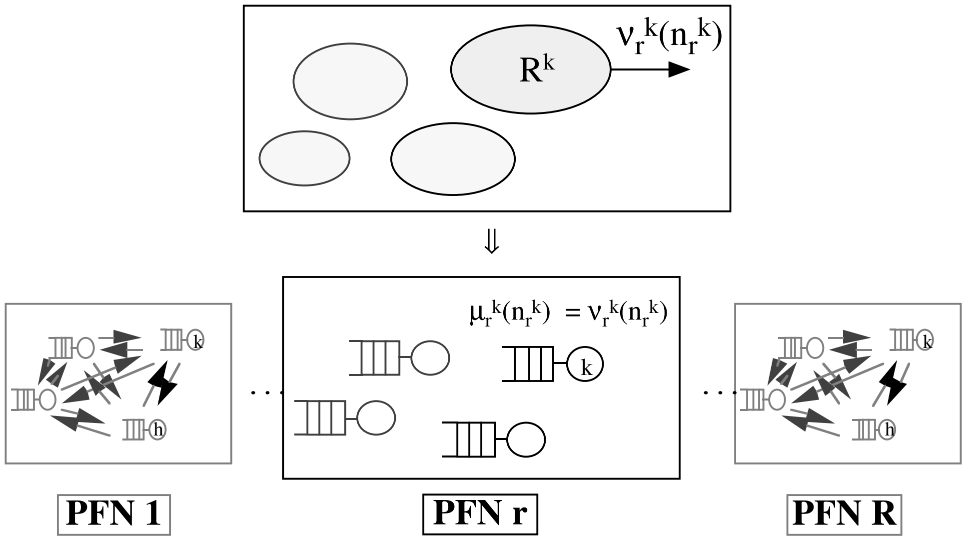
First approximation

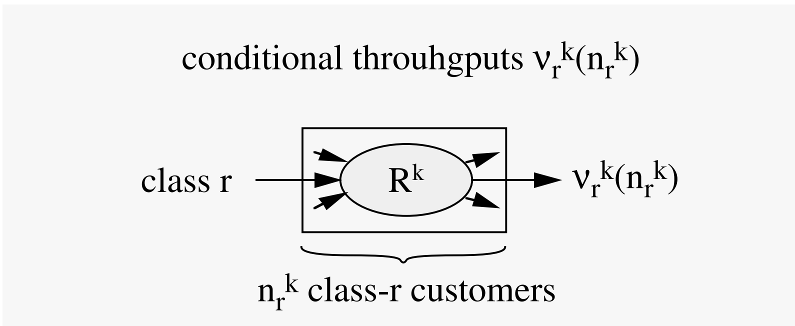
3 Estimate the rates $\mu_r^k(n_r^k)$ of the stations of the equivalent networks

Second approximation

Estimation of the parameters of the equivalent networks

Ideally: The service rates $\mu_r^k(n_r^k)$ should be equal to the conditional throuhgputs $\nu_r^k(n_r^k)$ of class-r customers out of the corresponding subsystem in the initial system.



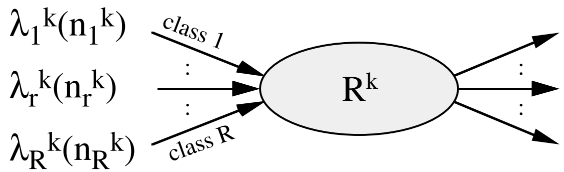


Problem: Require an exact solution of the initial system!

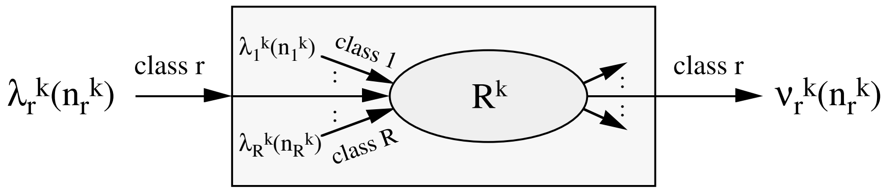
Aim: Estimate the conditional throughputs $\nu_r^k(n_r^k)$ of the different subsystems in the initial system by analyzing the subsystems in isolation.

Analysis of subsystems in isolation

Principle: Analyze the subsystem R^k in isolation as an open model, fed by Markovian arrival processes, which rates $\lambda_r^k(n_r^k)$ depend on the total number n_r^k of class-r customers currently present in the subsystem:



$\Rightarrow$ marginal probabilities $P_r^k(n_r^k)$

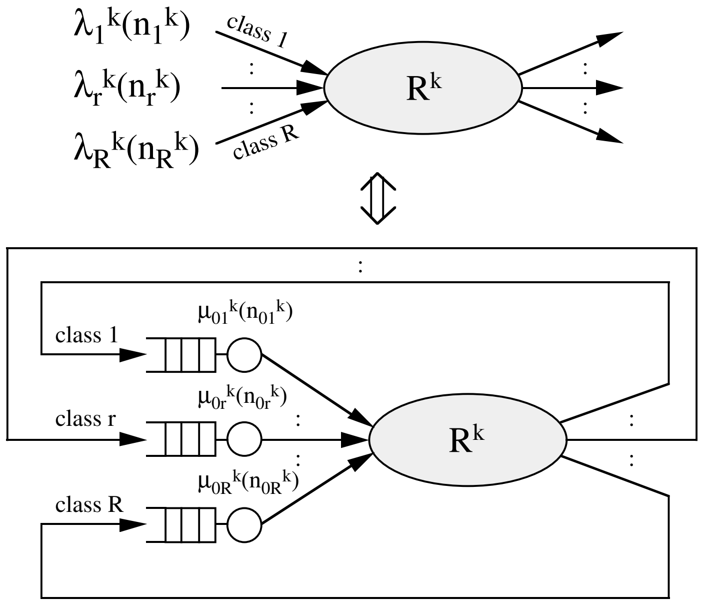


$\Rightarrow$ Estimation of the service rates: $\mu_r^k(n_r^k) = \nu_r^k(n_r^k)$

$\mu_r^k(n_r^k)$ solutions of a fixed-point problem
$\Rightarrow$ **Iterative procedure**

Analysis of subsystems in isolation

⇒ The complexity of the whole analysis is reduced to the complexity of the analyses in isolation:



Analysis of subsystem R^k in isolation

⇓

Analysis of a closed network.

Exact analysis:

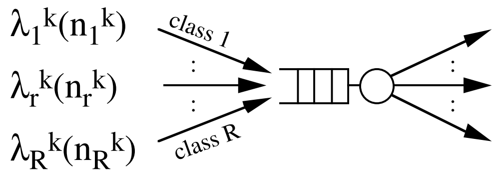
- Subsystem R^k is BCMP
- Markovian analysis

Approximate analysis:

- Simulation
- General technique
  ⇒ hierarchical analysis
- Approximate technique

Reduction of the complexity of the analyses in isolation

Subsystem consisting of a single station:



FIFO
RANDOM
PR
HOL

First idea: Modify the behavior of the subsystem.

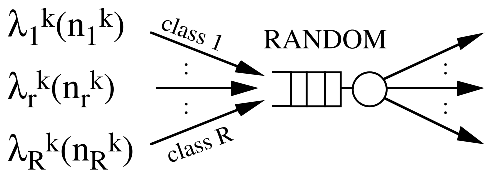
Exact Markovian analysis:

Scheduling	Service	Number of states for 4 classes and **N** = (5, 4, 3, 2)
FIFO	Exponential class-dependent	5 783 545
RANDOM	Exponential class-dependent	1 099

⇒ FIFO queues replaced by RANDOM queues.

Class-aggregation technique

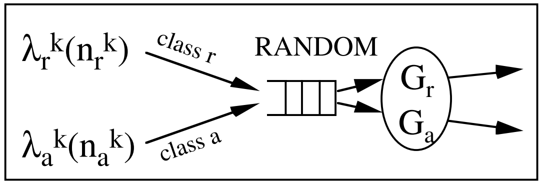
Class-aggregation:



$\Rightarrow$ Conditional throughputs:

$$\nu_{1}{}^{k}{}_{I}(n_1{}^k)$$
$$\vdots$$
$$\nu_{R}{}^{k}{}_{I}(n_R{}^k)$$

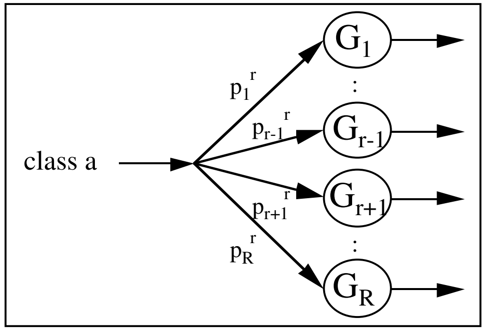
Remplaced by R analyses in isolation:



$\Rightarrow$ Conditional throughputs:

$$\nu_{r}{}^{k}{}_{I}(n_r{}^k)$$

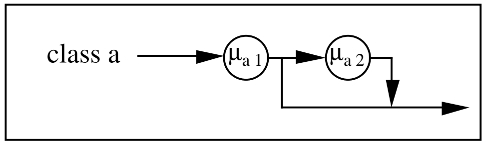
law G_a:



$\Downarrow$

Law-aggregation:

Identify G_a with a law having the same 2 ou 3 first moments:



Reduction of complexity

RANDOM queue
visited by R classes of customers and 10 customers per class.
Class-dependent Coxian-2 service-time distribution.

Number of classes: R	Number of states of the Markov chain associated with an analysis in isolation for: an exact analysis	an approximate analysis with **class-aggregation**	an approximate analysis with **class-aggregation** and **law-aggregation**
1	21	21	21
2	441	441	441
3	7 261	1 301	661
4	106 481	2 601	1 281
5	1 464 101	4 341	1 701
6	19 326 121	6 521	2 121

Application: Closed queueing networks with subnetworks having a population constraint

Queueing netwok:

- M stations
- N customers
- General service-time distributions
- C subnetworks with population constraint:

$$\underset{k=1,\dots,C}{C^k} \longrightarrow \text{capacity } N^k$$

Application:

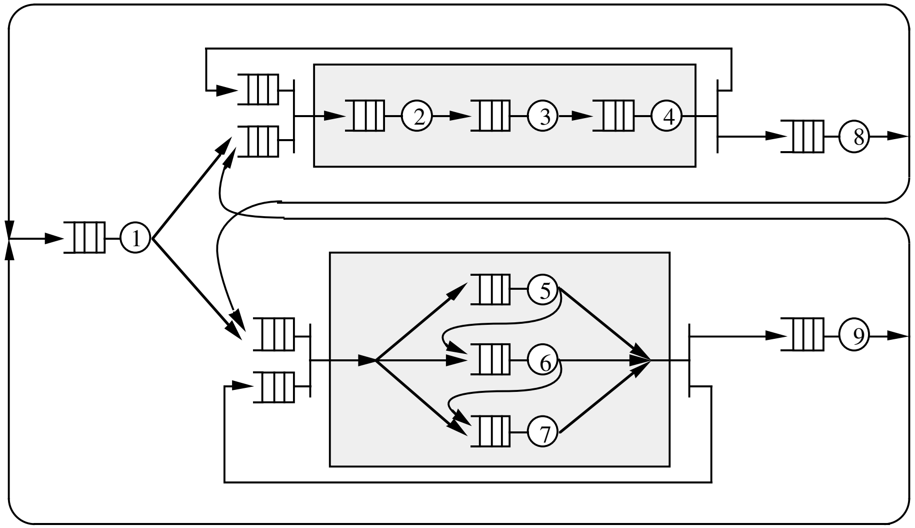
Manufacturing systems | Communication networks

The population constraint can model

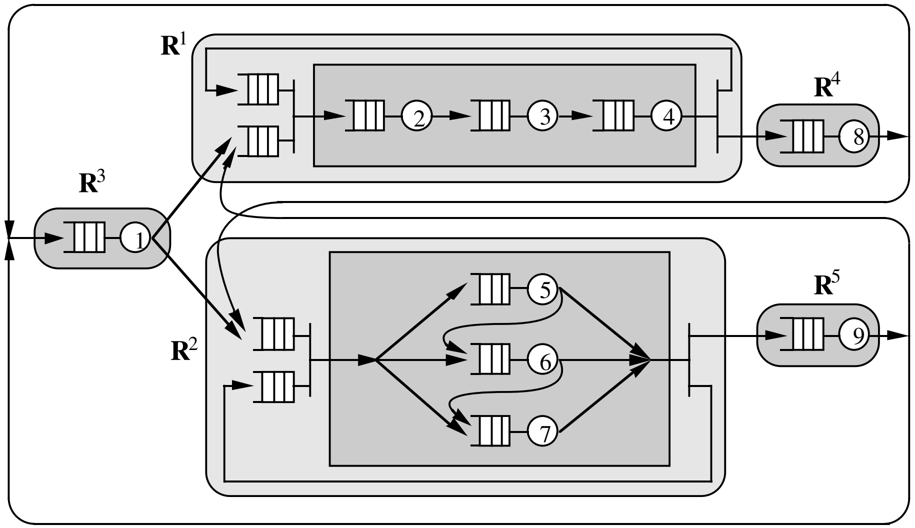
a physical constraint of the system
resources in limited quantity (pallets, buffer storages)

a control policy
restriction of the number of parts in a subpart of the manufacturing process
aim: to limit the inventory levels

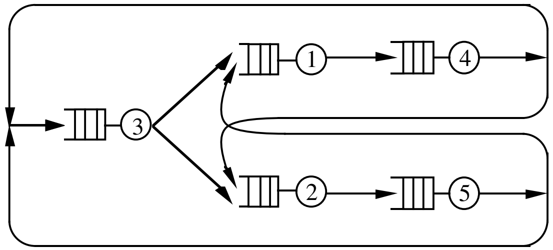
a window-flow control



Decomposition/Association of a PFN



$\Downarrow$



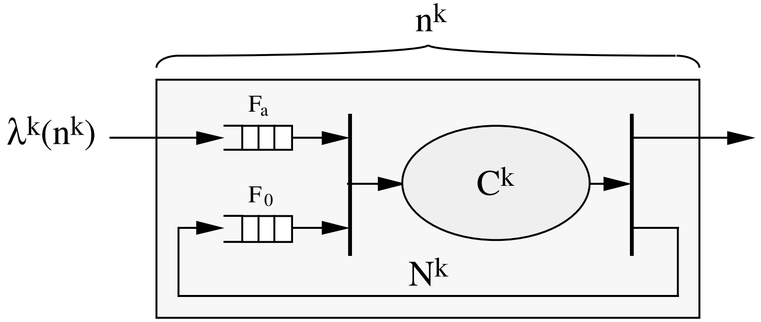
$V^1 = 1\ ;\ V^2 = 1\ ;\ V^3 = 1\ ;\ V^4 = 1\ ;\ V^5 = 1$

global state vector: $\mathbf{n} = (n^1, n^2, \dots, n^5)$

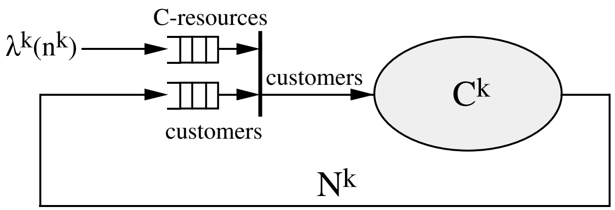
product-form: $$P(\mathbf{n}) = \frac{1}{G(5,N)} \prod_{k=1}^{5} \left[\frac{\left[V^k\right]^{n^k}}{\prod_{n=1}^{n^k} \mu^k(n)} \right]$$

Analyses in isolation

Subsystem consisting of subnetwork C^k with restricted capcity N^k (Marie's method):

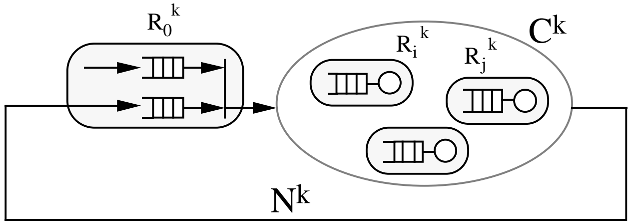


$\Updownarrow$



$\Rightarrow$ Hierchachical analysis:

Application of the general method at a local level



Application: Closed queueing networks with Fork/Join subnetworks

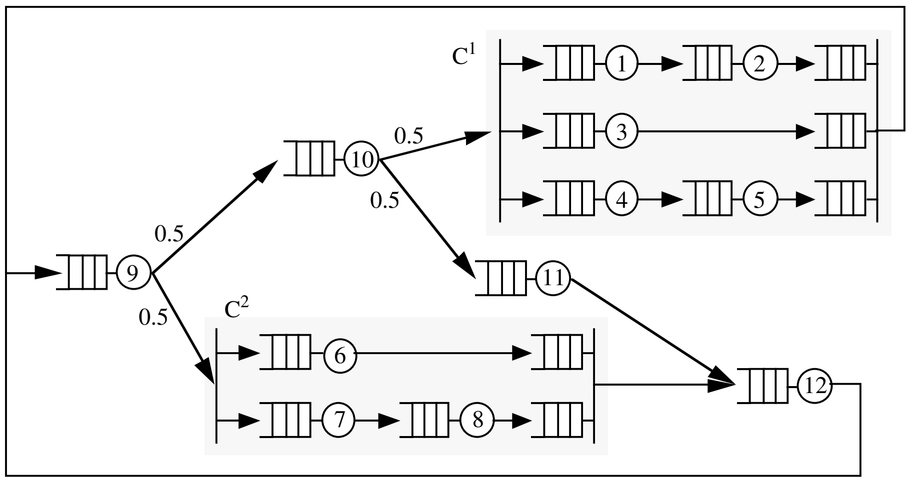
Queueing netwok:

- M stations
- N customers
- General service-time distributions
- C Fork/Join subnetworks:

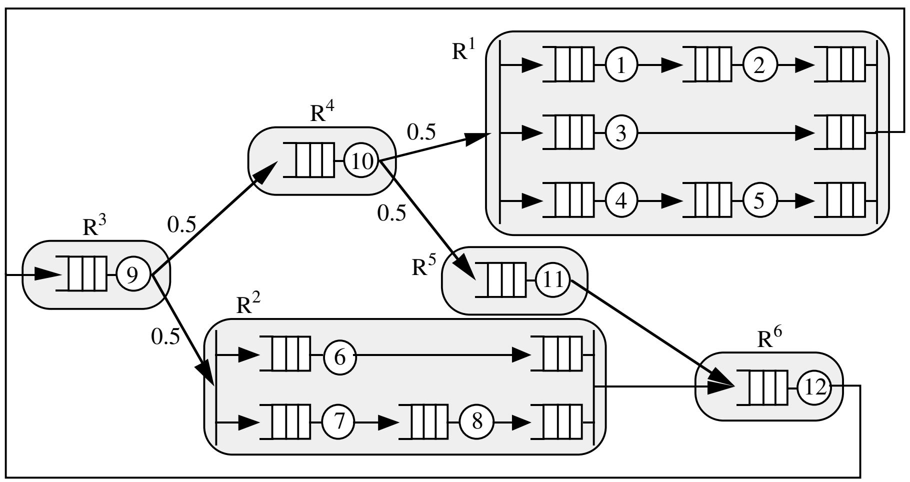
C^k, $k = 1, ..., C$ $\longrightarrow$ R^k type of lower-level customers (LL-customers)

Application:

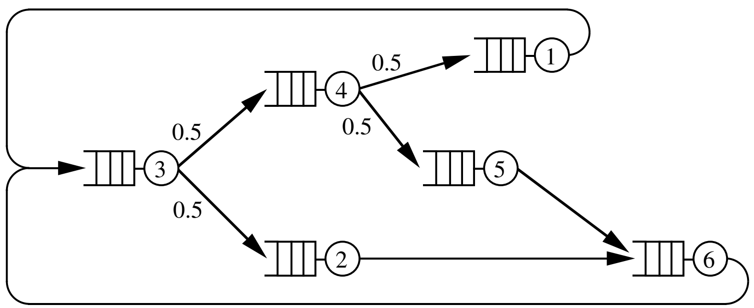
Parallel computing systems	Communication networks
$\downarrow$	$\downarrow$
A job is split into a set of tasks that can be executed independently on parallel processors. The job is completed as soon as all the tasks have been completed.	A packet is split into a set of frames that are transmitted independently through the network. The reassembly of the packet occurs when all frames have reached the receiver.



Decomposition/Association of a PFN



⇓



$V^1 = 0.25$; $V^2 = 0.5$; $V^3 = 1$; $V^4 = 0.5$; $V^5 = 0.25$; $V^6 = 0.75$

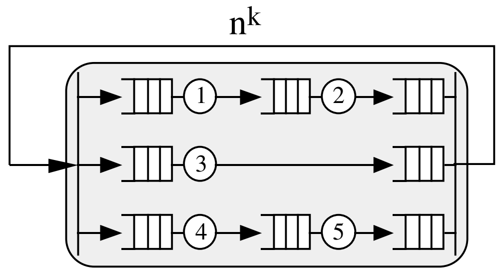
global state vector: $\mathbf{n} = (n^1, n^2, ..., n^6)$

product-form: $$P(\mathbf{n}) = \frac{1}{G(6,N)} \prod_{k=1}^{6} \left[\frac{\left[V^k\right]^{n^k}}{\prod_{n=1}^{n^k} \mu^k(n)} \right]$$

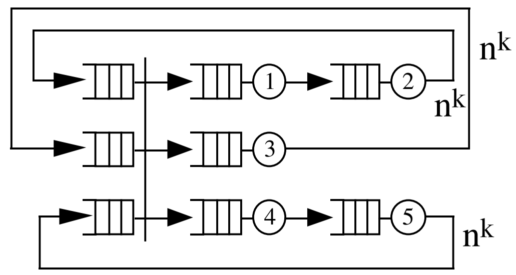
Analyses in isolation

Subsystem consisting of a Fork/Join subnetwork C^k

(Aggregation technique):



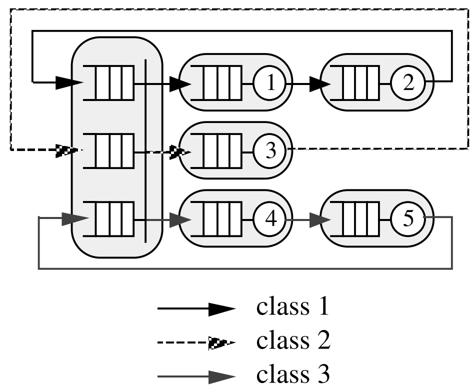
Equivalent transformation: ⇕



⇒ Synchronized closed queueing network

⇒ Hierchachical analysis:

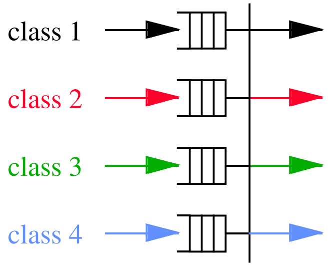
Application of the multi-class method at a local level



Application: Synchronized multi-class closed queueing networks

Queueing netwok:

- M stations
- R classes of customers
- General service-time distributions
- Synchronization stations between differents classes

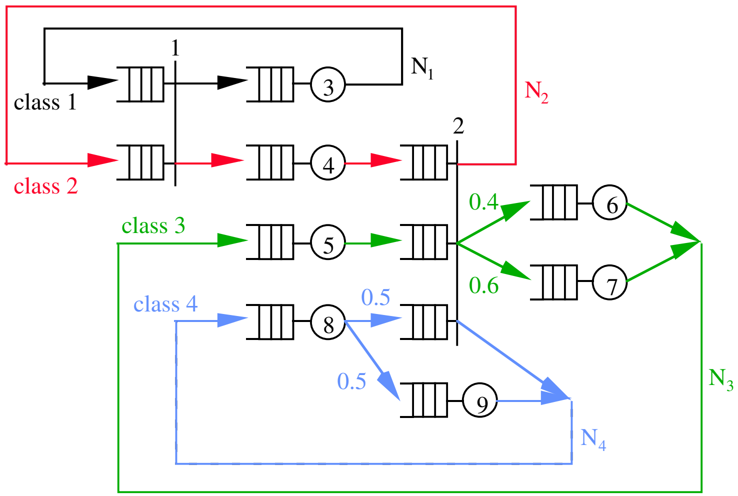


Application:

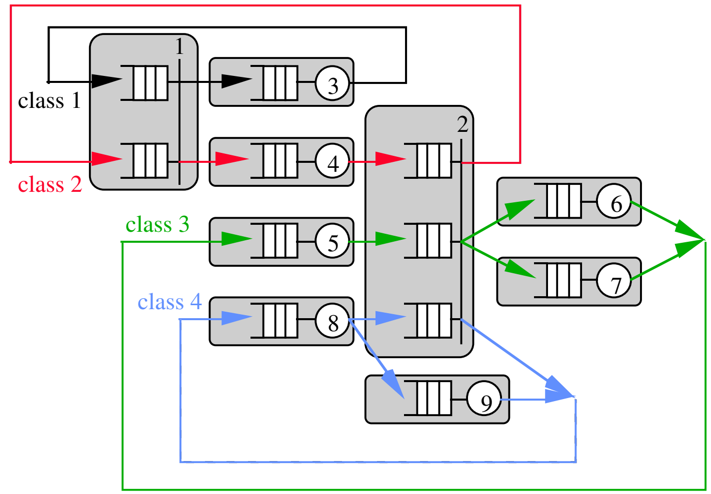
Distributed systems

Multimedia systems

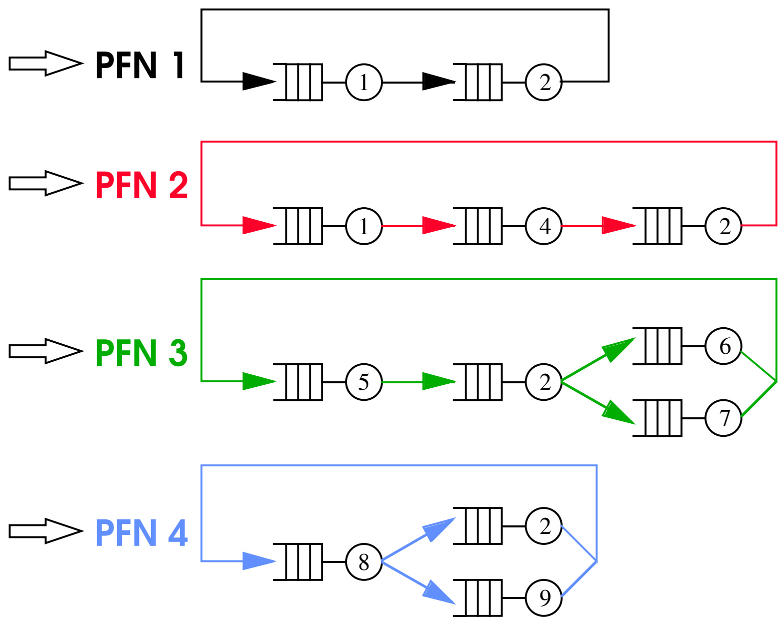
Manufacturing systems (Kanban systems)



Decomposition/Association of several PFNs

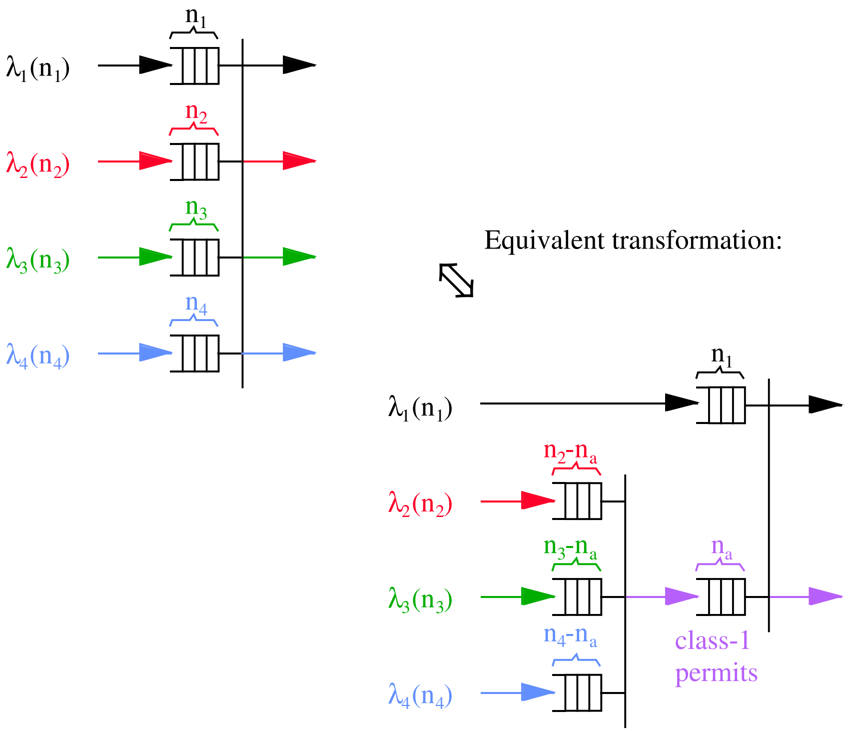


⇓



Analyses in isolation

Subsystem consisting of a synchronization station:



Class-aggregation technique: