Properties and Bounds on P/T Nets

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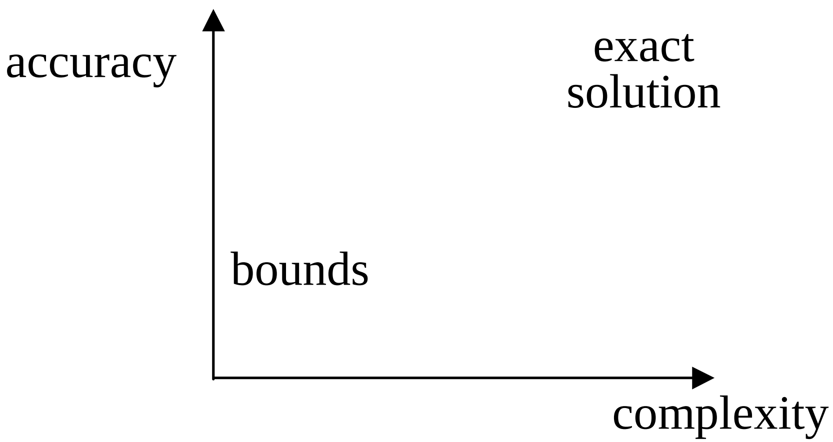
Preliminary comments (1)

- Interest of bounding techniques
  - preliminary phases of design
    - many parameters are not known accurately
    - quick evaluation and rejection of those clearly bad



Preliminary comments (2)

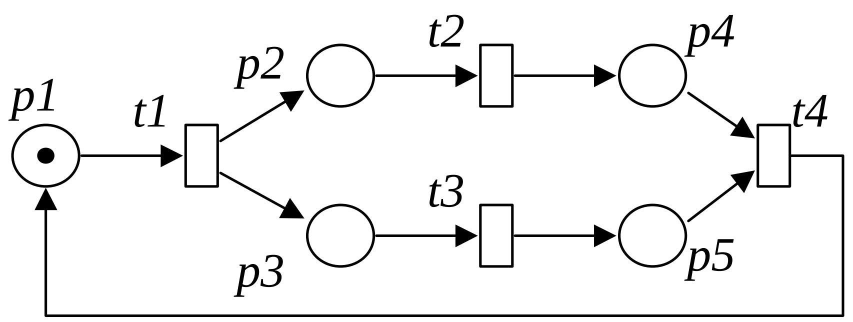
- Net-driven solution technique
  - stressing the intimate relationship between qualitative and quantitative aspects of PN's
  - structure theory of net models

  → efficient computation techniques

Outline

- Introducing the ideas: Marked Graphs case
- Generalization: use of visit ratios
- Improvements of the bounds
- A general linear programming statement

Introducing ideas: MG's case (1)



generally distributed service times
(random variables X_i with mean $\bar{\mathbf{s}}[t_j]$)

we assume **infinite-server semantics**

exact cycle time (random variable): $X = X_1 + \max\{X_2, X_3\} + X_4$

average cycle time: $\Gamma = \bar{\mathbf{s}}[t_1] + \mathrm{E}[\max\{X_2, X_3\}] + \bar{\mathbf{s}}[t_4]$

but (non-negative variables): $X_2, X_3 \leq \max\{X_2, X_3\} \leq X_2 + X_3$

therefore:

$$\bar{\mathbf{s}}[t_1] + \max\{\bar{\mathbf{s}}[t_2], \bar{\mathbf{s}}[t_3]\} + \bar{\mathbf{s}}[t_4] \leq \Gamma \leq \bar{\mathbf{s}}[t_1] + \bar{\mathbf{s}}[t_2] + \bar{\mathbf{s}}[t_3] + \bar{\mathbf{s}}[t_4]$$

Introducing ideas: MG's case (2)

Thus, the lower bound for the average cycle time is computed looking for the slowest circuit

$$\Gamma \geq \max_{\substack{C \in \{\text{circuits} \\ \text{of the net}\}}} \left(\sum_{t_i \in C} \bar{\mathbf{s}}[t_i] \Big/ \#\,\text{tokens in } C \right)$$

Interpretation:

an MG may be built synchronising circuits, so we look for the bottleneck

Introducing ideas: MG's case (3)

- Computation:

$$
\begin{aligned}
\Gamma \geq \ \text{maximum} \quad & \mathbf{y} \cdot \mathbf{Pre} \cdot \bar{\mathbf{s}} \\
\text{subject to} \quad & \mathbf{y} \cdot \mathbf{C} = \mathbf{0} \\
& \mathbf{y} \cdot \mathbf{m_0} = 1 \\
& \mathbf{y} \geq \mathbf{0}
\end{aligned}
$$

($\bar{\mathbf{s}}$ is the vector of average service times)

(the proof of this comes later for a more general case)

solving a linear programming problem
(**polynomial complexity** on the net size)

Introducing ideas: MG's case (4)

- Even if naïf, the bounds are tight!
- Lower bound for the average cycle time

$$\max\{\bar{\mathbf{s}}[t_2], \bar{\mathbf{s}}[t_3]\} \leq \mathrm{E}[\max\{X_2, X_3\}]$$

  – it is exact for deterministic timing
  – it cannot be improved using only mean values of r.v. (it is reached in a limit case for a family of random variables with arbitrary means and variances)

Introducing ideas: MG's case (5)

$$X_{m,s}(a) = \begin{cases} ma & \text{with probability } 1-e \\ m\left(a + \dfrac{1-a}{e}\right) & \text{with probability } e \end{cases} \qquad e = \frac{m^2(1-a)^2}{m^2(1-a)^2 + s^2}$$

$(0 \le a \le 1)$

$$\mathrm{E}\left[X_{m,s}(a)\right] = m \;;\; \mathrm{Var}\left[X_{m,s}(a)\right] = s^2$$

$$\lim_{a \to 1} \mathrm{E}\left[\max\left(X_{m,s}(a), X_{m',s'}(a)\right)\right] = \max\left(m, m'\right)$$

$$\mathrm{E}\left[X_{m,s}(a) + X_{m',s'}(a)\right] = m + m', \;\; \forall \;\; 0 \le a < 1$$

they behave "as deterministic" for the 'max' and '+' operators in the limit (α→1)

Introducing ideas: MG's case (6)

- Upper bound for the average cycle time

$$\Gamma \leq \sum_{t \in T} \bar{\mathbf{s}}[t]$$

  – it cannot be improved for 1–live MG's using only mean values of r.v. (it is reached in a limit case for a family of random variables with arbitrary means)

Introducing ideas: MG's case (7)

$$X_m^i(e) = \begin{cases} 0 & \text{with probability } 1 - e^i \\ \dfrac{m}{e^i} & \text{with probability } e^i \end{cases}$$

$(0 < e < 1)$

$$\mathrm{E}\left[X_m^i(e)\right] = m \ ; \ \mathrm{E}\left[X_m^i(e)^2\right] = \frac{m^2}{e^i}$$

If $X_j = X_{\bar{\mathbf{s}}[t_j]}^{j-1}(e), \ \forall t_j \in T$, then for varying (decreasing) values of e:

$$\boxed{\mathrm{E}[\max(X_i, X_j)] = \bar{\mathbf{s}}[t_i] + \bar{\mathbf{s}}[t_j] + o(e)}$$

Generalization: visit ratios (1)

- Visit ratios = relative throughput

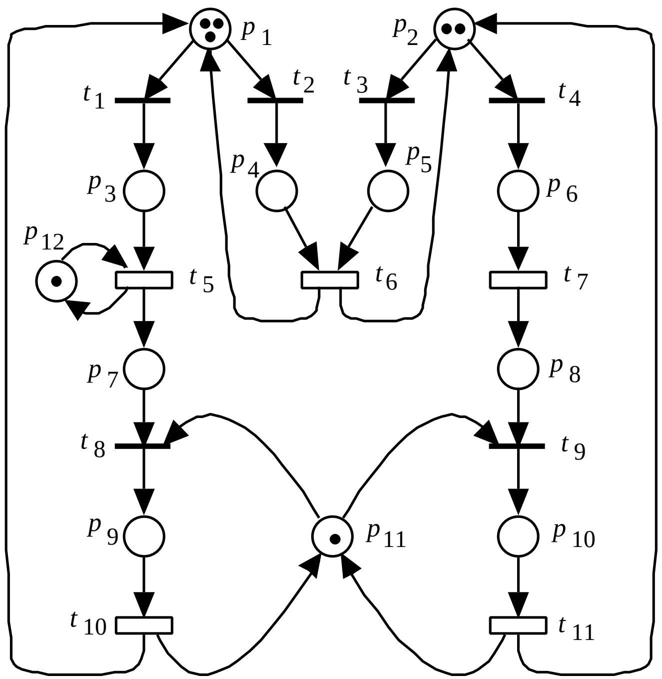
  (number of visits to t_i per each visit to t_1)

$$\mathbf{v}[t] = \frac{c[t]}{c[t_1]} = \Gamma[t_1]\ c[t]$$

$\Gamma[t_1]$ → average interfiring time of t_1

Generalization: visit ratios (2)

- For some net classes **v** can be computed as:



$$\mathbf{C} \cdot \mathbf{v} = \mathbf{0};$$
$$r_1 \mathbf{v}[t_2] = r_2 \mathbf{v}[t_1];$$
$$r_3 \mathbf{v}[t_4] = r_4 \mathbf{v}[t_3];$$
$$\mathbf{v}[t_1] = 1$$

Generalization: visit ratios (3)

- Little's law ($L = lW$) applied to a place p:

$$\bar{m}[p] = (\mathbf{Pre}[p,T] \cdot c)\ \bar{\mathbf{r}}[p]$$

Assume that timed transitions are never in conflict (**conflicts are modelled with immediate transitions**), then either all output transitions of p are immediate or p has a unique output transition, say t_1, and t_1 is timed, thus:

$$\bar{m}[p] = (\mathbf{Pre}[p,T] \cdot c)\ \bar{\mathbf{r}}[p] = \mathbf{Pre}[p,t_1]\ \ c[t_1]\ \ \bar{\mathbf{r}}[p]$$

$$\geq \mathbf{Pre}[p,t_1]\ \ c[t_1]\ \ \bar{\mathbf{s}}[t_1] = \sum_{j=1}^{m} \mathbf{Pre}[p,t_j]\ \ c[t_j]\ \ \bar{\mathbf{s}}[t_j]$$

Generalization: visit ratios (4)

Then: $\Gamma[t_1]\ \overline{m}[p] \geq \sum_{j=1}^{m} \mathbf{Pre}[p,t_j]\ \Gamma[t_1]\ \ c[t_j]\ \ \bar{\mathbf{s}}[t_j] = \sum_{j=1}^{m} \mathbf{Pre}[p,t_j]\ \ \mathbf{v}[t_j]\ \ \bar{\mathbf{s}}[t_j]$

Hence: $\Gamma[t_1]\ \ \overline{m} \geq \mathbf{Pre} \cdot \overline{\mathbf{D}}$ where $\overline{\mathbf{D}}[t] = \bar{\mathbf{s}}[t]\mathbf{v}[t]$ is the average service demand of t

Premultiplying by a P–semiflow $\mathbf{y}$ $(\mathbf{y} \cdot \mathbf{C} = \mathbf{0},\ \mathbf{y} \geq \mathbf{0},$ thus $\mathbf{y} \cdot \overline{m} = \mathbf{y} \cdot \mathbf{m_0})$,

$$\Gamma[t_1] \geq \text{maximum } \frac{\mathbf{y} \cdot \mathbf{Pre} \cdot \overline{\mathbf{D}}}{\mathbf{y} \cdot \mathbf{m_0}} \quad \text{subject to } \mathbf{y} \cdot \mathbf{C} = \mathbf{0},\ \mathbf{1} \cdot \mathbf{y} > 0,\ \mathbf{y} \geq \mathbf{0}$$

$\Longleftrightarrow$

$$\Gamma[t_1] \geq \text{maximum } \frac{\mathbf{y} \cdot \mathbf{Pre} \cdot \overline{\mathbf{D}}}{q} \quad \text{subject to } \mathbf{y} \cdot \mathbf{C} = \mathbf{0},\ \mathbf{1} \cdot \mathbf{y} > 0,\ q = \mathbf{y} \cdot \mathbf{m_0},\ \mathbf{y} \geq \mathbf{0}$$

Generalization: visit ratios (5)

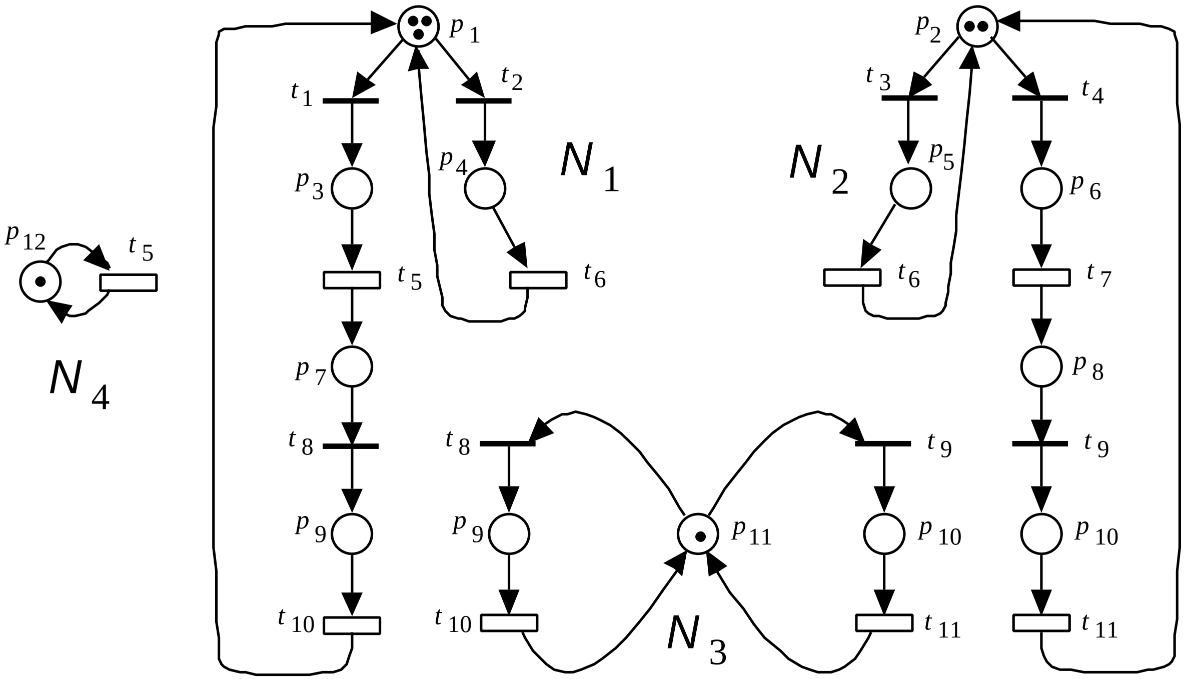
Since $\mathbf{y} \cdot \mathbf{m_0} > 0$ (live system), we change $\mathbf{y}/q$ to $\mathbf{y}$ and we obtain ($\mathbf{1} \cdot \mathbf{y} > 0$ is removed because $\mathbf{y} \cdot \mathbf{m_0} = 1$ implies $\mathbf{1} \cdot \mathbf{y} > 0$):

$$
\Gamma[t_1] \geq \quad \text{maximum} \quad \mathbf{y} \cdot \mathbf{Pre} \cdot \overline{\mathbf{D}}
$$
$$
\text{subject to} \quad \mathbf{y} \cdot \mathbf{C} = \mathbf{0}
$$
$$
\mathbf{y} \cdot \mathbf{m_0} = 1
$$
$$
\mathbf{y} \geq \mathbf{0}
$$

again, a linear programming problem
(polynomial complexity on the net size)

Generalization: visit ratios (6)

Interpretation: **slowest subsystem generated by *P*–semiflows, in isolation**



minimal *P*–semiflows

$\mathbf{y_1} = (1,0,1,1,0,0,1,0,1,0,0,0)$

$\mathbf{y_2} = (0,1,0,0,1,1,0,1,0.1,0,0)$

$\mathbf{y_3} = (0,0,0,0,0,0,0,0,1,1,1,0)$

$\mathbf{y_4} = (0,0,0,0,0,0,0,0,0,0,0,1)$

$$\Gamma[t_1] \geq \max\{ \; (\bar{\mathbf{s}}[t_5]+\bar{\mathbf{s}}[t_6]+\bar{\mathbf{s}}[t_{10}])/3, \; (\bar{\mathbf{s}}[t_6]+\bar{\mathbf{s}}[t_7]+\bar{\mathbf{s}}[t_{11}])/2, \; \bar{\mathbf{s}}[t_{10}]+\bar{\mathbf{s}}[t_{11}], \; \bar{\mathbf{s}}[t_5] \; \}$$

Generalization: visit ratios (7)

- Upper bound for the average interfiring time

$$\Gamma[t_1] \leq \sum_{t \in T} \mathbf{v}[t]\ \bar{\mathbf{s}}[t] = \sum_{t \in T} \overline{\mathbf{D}}[t]$$

remember the marked graphs case (**v** = **1**): $\Gamma \leq \sum_{t \in T} \bar{\mathbf{s}}[t]$

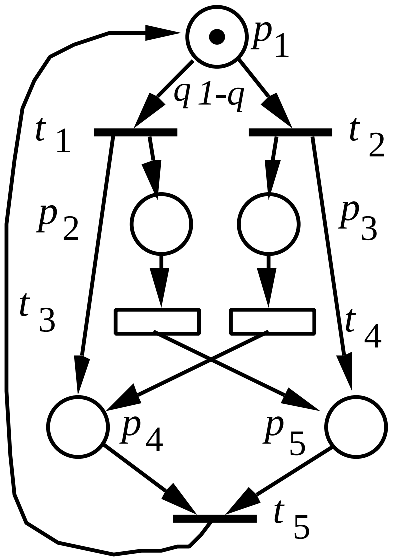
Improvements of the bounds

- Structural improvements

  bounds still based only on the mean values (not on higher moments of r.v., **insensitive** bounds)

  - lower bound for the average interfiring time: use of **implicit places** to increase the number of minimal *P*–semiflows
  - upper bound for the average interfiring time: use of **liveness bound of transitions** to improve the bound for some net subclasses

Use of implicit places (1)



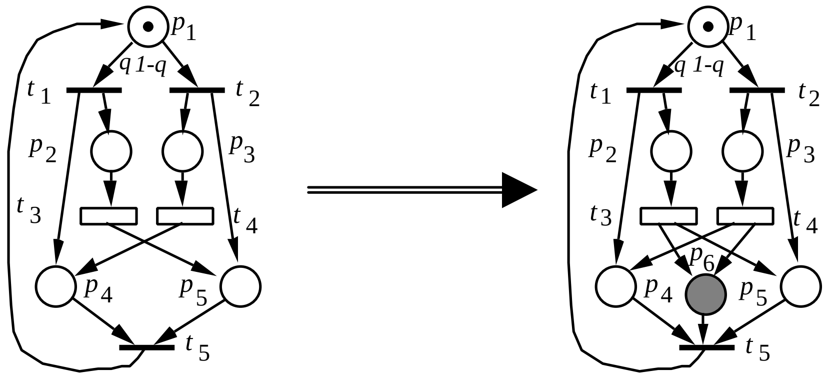
$$\Gamma[t_5] = q\bar{\mathbf{s}}[t_3] + (1-q)\bar{\mathbf{s}}[t_4]$$

$$\Gamma[t_1] \geq \begin{array}{ll} \text{maximum} & \mathbf{y} \cdot \mathbf{Pre} \cdot \bar{\mathbf{D}} \\ \text{subject to} & \mathbf{y} \cdot \mathbf{C} = \mathbf{0} \\ & \mathbf{y} \cdot \mathbf{m_0} = 1 \\ & \mathbf{y} \geq \mathbf{0} \end{array}$$

$$\Downarrow$$

$$\Gamma[t_5] \geq \max\left\{q\bar{\mathbf{s}}[t_3], (1-q)\bar{\mathbf{s}}[t_4]\right\}$$

Use of implicit places (2)



$$\Gamma[t_1] \geq \quad \text{maximum} \quad \mathbf{y}\cdot\mathbf{Pre}\cdot\overline{\mathbf{D}}$$
$$\text{subject to} \quad \mathbf{y}\cdot\mathbf{C}=\mathbf{0}$$
$$\mathbf{y}\cdot\mathbf{m_0}=1$$
$$\mathbf{y}\geq\mathbf{0}$$

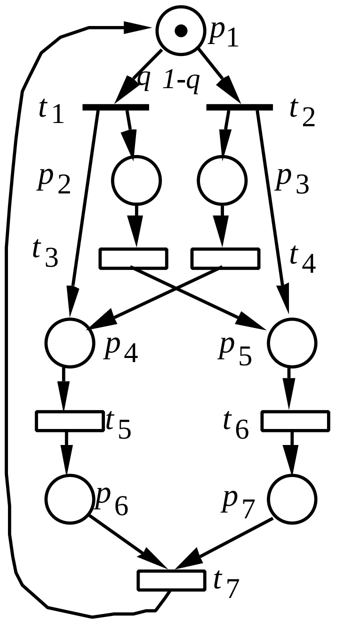
$$\Gamma[t_5] = q\bar{\mathbf{s}}[t_3]+(1-q)\bar{\mathbf{s}}[t_4]$$

$$\Gamma[t_5] \geq \max\left\{ q\bar{\mathbf{s}}[t_3],\ (1-q)\bar{\mathbf{s}}[t_4],\ \underline{q\bar{\mathbf{s}}[t_3]+(1-q)\bar{\mathbf{s}}[t_4]} \right\}$$

in this case, we get the exact value!

Use of implicit places (3)

in general…



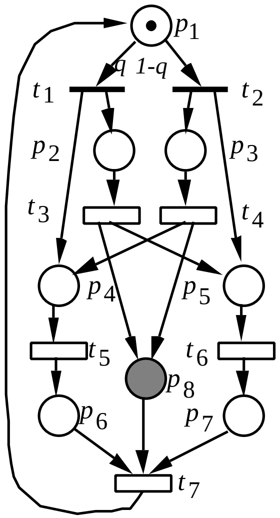
$$\Gamma[t_1] \geq \quad \text{maximum} \quad \mathbf{y} \cdot \mathbf{Pre} \cdot \overline{\mathbf{D}}$$
$$\text{subject to} \quad \mathbf{y} \cdot \mathbf{C} = \mathbf{0}$$
$$\mathbf{y} \cdot \mathbf{m_0} = 1$$
$$\mathbf{y} \geq \mathbf{0}$$

$$\Downarrow$$

$$\Gamma[t_7] \geq \max \{ \quad q\bar{\mathbf{s}}[t_3] + \bar{\mathbf{s}}[t_6] + \bar{\mathbf{s}}[t_7],$$
$$(1-q)\bar{\mathbf{s}}[t_4] + \bar{\mathbf{s}}[t_5] + \bar{\mathbf{s}}[t_7] \quad \}$$

Use of implicit places (4)

in general, the bound is non-reachable



$$\Gamma[t_7] \geq \max\{\ q\bar{\mathbf{s}}[t_3]+\bar{\mathbf{s}}[t_6]+\bar{\mathbf{s}}[t_7],\ (1-q)\bar{\mathbf{s}}[t_4]+\bar{\mathbf{s}}[t_5]+\bar{\mathbf{s}}[t_7],\ q\bar{\mathbf{s}}[t_3]+(1-q)\bar{\mathbf{s}}[t_4]+\bar{\mathbf{s}}[t_7]\ \}$$

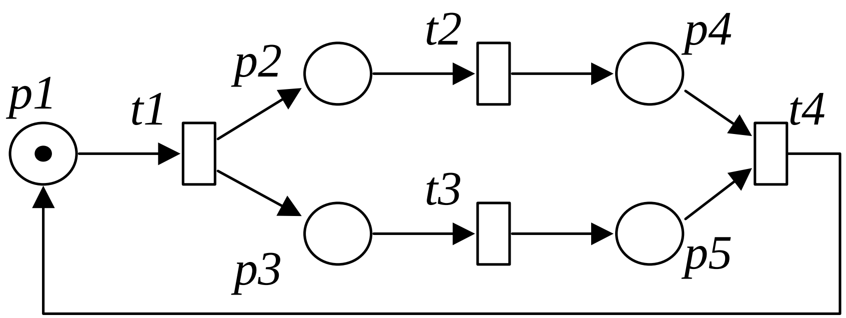
$$\Gamma[t_7] = q\max\{\bar{\mathbf{s}}[t_5],\bar{\mathbf{s}}[t_3]+\bar{\mathbf{s}}[t_6]\}+(1-q)\max\{\bar{\mathbf{s}}[t_4]+\bar{\mathbf{s}}[t_5],\bar{\mathbf{s}}[t_6]\}+\bar{\mathbf{s}}[t_7]$$

$$= \max\{\ q\bar{\mathbf{s}}[t_3]+\bar{\mathbf{s}}[t_6]+\bar{\mathbf{s}}[t_7],\ (1-q)\bar{\mathbf{s}}[t_4]+\bar{\mathbf{s}}[t_5]+\bar{\mathbf{s}}[t_7],\ q\bar{\mathbf{s}}[t_3]+(1-q)\bar{\mathbf{s}}[t_4]+(1-q)\bar{\mathbf{s}}[t_5]+q\bar{\mathbf{s}}[t_6]+\bar{\mathbf{s}}[t_7],\ q\bar{\mathbf{s}}[t_5]+(1-q)\bar{\mathbf{s}}[t_6]+\bar{\mathbf{s}}[t_7]\ \}$$

(deterministic timing)

Use of liveness bounds (1)

- upper bound for the average interfiring time:

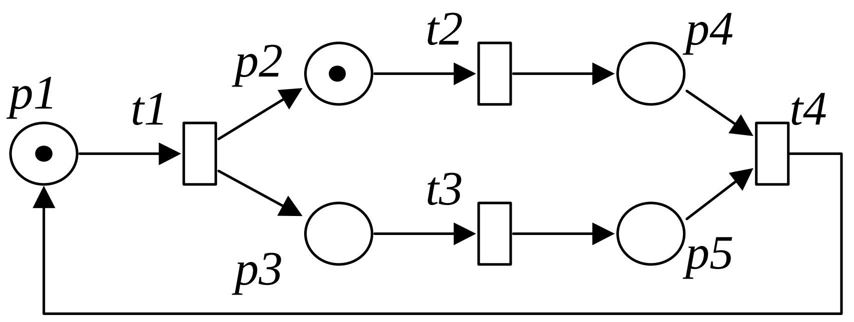


$$\Gamma \leq \sum_{t \in T} \bar{\mathbf{s}}[t]$$

reachable for 1-live marked graphs, but…

Use of liveness bounds (2)

it can be improved for *k*–live marked graphs



$$\Gamma \leq \bar{\mathbf{s}}[t_1] + \frac{\bar{\mathbf{s}}[t_2]}{2} + \bar{\mathbf{s}}[t_3] + \bar{\mathbf{s}}[t_4]$$

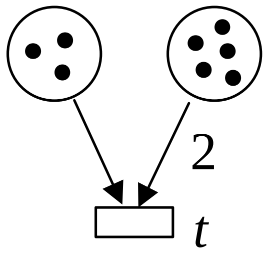
liveness bound of t_2

Use of liveness bounds (3)

- Definitions of enabling degree, enabling bound, structural enabling bound, and liveness bound

  – instantaneous enabling degree of a transition at a given marking

$$\mathbf{e}[t](\mathbf{m}) = \sup\left\{k \in \mathrm{N} : \ \forall p \in {}^{\bullet}t, \ \mathbf{m}[p] \geq k \ \mathbf{Pre}[p,t]\right\}$$

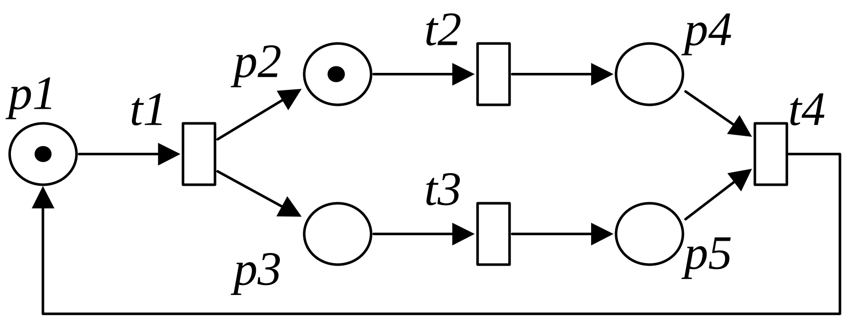


$\mathbf{e}[t](\mathbf{m}) = 2$

Use of liveness bounds (4)

– enabling bound of a transition in a given system: maximum among the instantaneous enabling degree at all reachable markings

$$\mathbf{eb}[t] = \sup\left\{ k \in \mathrm{N} :\ \exists \mathbf{m_0} \xrightarrow{s} \mathbf{m},\ \forall p \in {}^{\bullet}t,\ \mathbf{m}[p] \geq k\ \mathbf{Pre}[p,t] \right\}$$

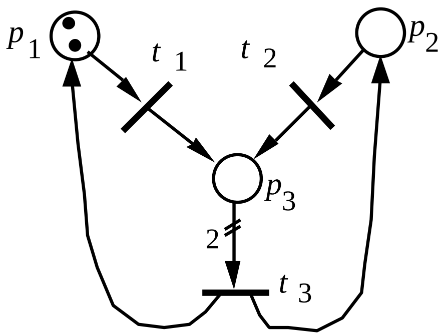


$\mathbf{eb}[t_2] = 2$

Use of liveness bounds (5)

– liveness bound of a transition in a given system:
number of servers available in *t* in steady state

$$\mathbf{lb}[t] = \sup\left\{ k \in \mathrm{N} : \forall \mathbf{m}', \mathbf{m_0} \xrightarrow{s} \mathbf{m}', \exists \mathbf{m}, \mathbf{m}' \xrightarrow{s'} \mathbf{m} \wedge \forall p \in {}^\bullet t, \mathbf{m}[p] \geq k \; \mathbf{Pre}[p,t] \right\}$$



$\mathbf{lb}[t_1] = 1 < 2 = \mathbf{eb}[t_1]$

Use of liveness bounds (6)

– structural enabling bound of a transition in a given system: structural counterpart of the enabling bound (substitute reachability condition by $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot s$; $\mathbf{m}, s \bullet \mathbf{0}$)

$$
\begin{aligned}
\mathbf{seb}[t] = \ & \text{maximum} \quad k \\
& \text{subject to} \quad \mathbf{m_0}[p] + \mathbf{C}[p,T] \cdot s \geq k\ \mathbf{Pre}[p,t], \ \forall p \in P \\
& \qquad\qquad\quad s \geq 0
\end{aligned}
$$

Property: For any net system $\mathbf{seb}[t] \geq \mathbf{eb}[t] \geq \mathbf{lb}[t]$, for all t.

Property: For live and bounded free choice systems,

$\mathbf{seb}[t] = \mathbf{eb}[t] = \mathbf{lb}[t]$, for all t.

Use of liveness bounds (7)

improvement of the bound for live and bounded free choice systems:

$$\Gamma[t_1] \leq \sum_{t \in T} \frac{\mathbf{v}[t]\ \bar{\mathbf{s}}[t]}{\mathbf{seb}[t]} = \sum_{t \in T} \frac{\overline{\mathbf{D}}[t]}{\mathbf{seb}[t]}$$

this bound cannot be improved for marked graphs (using only the mean values of service times)

A general LP statement (1)

- The idea

a linear function

maximize [or minimize] $f(\overline{m}, c)$

subject to any linear constraint that we are able to state for $\overline{m}$, c, and other needed additional variables

linear operational laws

A general LP statement (2)

- A set of linear constraints:

$$\overline{m} = \mathbf{m_0} + \mathbf{C} \cdot s \qquad \textit{(state equation)}$$

$$\sum_{t \in {}^\bullet p} c[t]\ \mathbf{Post}[p,t] \geq \sum_{t \in p^\bullet} c[t]\ \mathbf{Pre}[p,t], \quad \forall p \in P$$

$$\sum_{t \in {}^\bullet p} c[t]\ \mathbf{Post}[p,t] = \sum_{t \in p^\bullet} c[t]\ \mathbf{Pre}[p,t], \quad \forall p \in P \text{ bounded } \textit{(flow balance equation)}$$

$$\frac{c[t_i]}{r_i} = \frac{c[t_j]}{r_j}, \quad \forall t_i, t_j \in T: \text{ behavioural free choice}$$

(e.g. $\mathbf{Pre}[P, t_i] = \mathbf{Pre}[P, t_j]$)

$\ldots \qquad \ldots$

A general LP statement (3)

$$c[t]\ \bar{\mathbf{s}}[t] \leq \frac{\overline{m}[p]}{\mathbf{Pre}[p,t]}, \qquad \forall t \in T,\ \forall p \in {}^{\bullet}t \qquad \textit{(maximum throughput law)}$$

$$c[t]\ \bar{\mathbf{s}}[t] \geq \frac{\overline{m}[p] - \mathbf{Pre}[p,t] + 1}{\mathbf{Pre}[p,t]}, \qquad \forall t \in T \text{ persistent, age memory or immediate} : {}^{\bullet}t = \{p\} \qquad \textit{(minimum throughput law)}$$

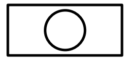
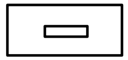
$\ldots$ $\ldots$

$$\overline{m},\ \ c,\ \ s \geq \mathbf{0}$$

A general LP statement (4)

- It can be improved using second order moments
- It can be extended to well-formed coloured nets

A general LP statement (5)

- It is implemented in *GreatSPN*
  - select place (transition) object  ()
  - click right mouse button and select “show”
  - click again right mouse button and select “ Average M.B.” (“LP Throughput Bounds”)
  - click left mouse button for upper bound
  - click middle mouse button for lower bound

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More technical details

→ look at the bibliography