Time augmented Petri nets

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Outline

- Introduction
- Interpreted Petri nets
- Stochastic Petri Nets and CTMC-based analysis
- Bibliography

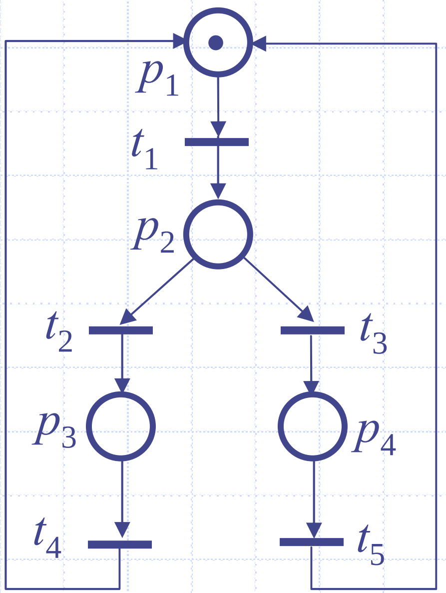
Introduction

- **Formalism**: conceptual framework suited for a given purpose
- **Life cycle**: all phases, from preliminary design, detailed design, implementation, tuning…
- Different goals in each phase $\rightarrow$
  $\rightarrow$ different formalisms
- Family of formalisms: PARADIGM

Introduction

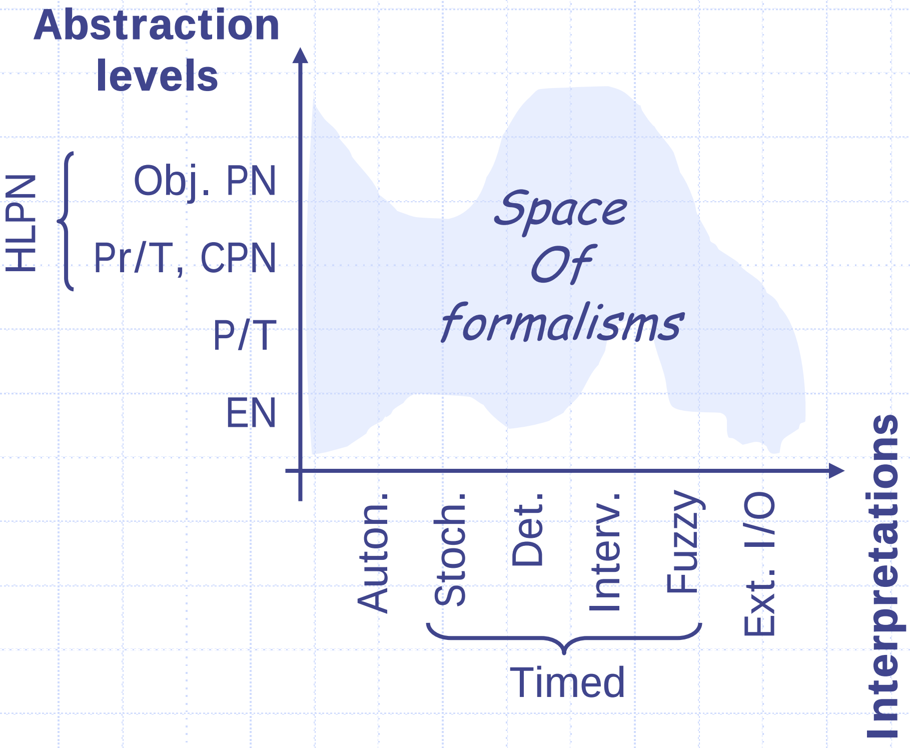
- Why time augmenting the formalism?
  - Autonomous Petri nets
    - **Non-determinism** with respect to
      - **Which** enabled transition will fire?
      - **When** will it fire?

      - duration of activities and
      - routing
    - Not valid for performance evaluation (quantitative analysis: throughput, response time, average marking)



Introduction

- Formalism suitable for system life cycle. Two characteristics:
  - Different and interrelated abstraction levels
  - Different interpretations



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Interpreted Petri nets

Abstract formalism $\leftrightarrow$ Reality

- Generic meaning:
  - Place = state variable
  - Marking = value of variable
  - Transition = transformation of state
  - Firing = event that produces transformation
- Particular meanings (annotations):
  - Place (and marking)
    - State of subsystem S_i
    - Condition C_j is true
    - Resource R_k is available
    - Stock of parts in a store…
  - Transition (and firing)
    - Subsystem S_i evolves
    - End of activity Aj
    - A customer arrives
    - A fail occurs…

Interpreted Petri nets

Interpretation

(relation with the environment)

$\Downarrow$

Constraints over the evolution

(imposed by the environment)

$\Downarrow$

Reduction of non-determinism

- Synchronization with signals (from the environment)
- Time constraints

- Typical interpretations:
  - Marking diagrams (and Grafcet)
  - Timed interpretations (time augmented Petri nets)

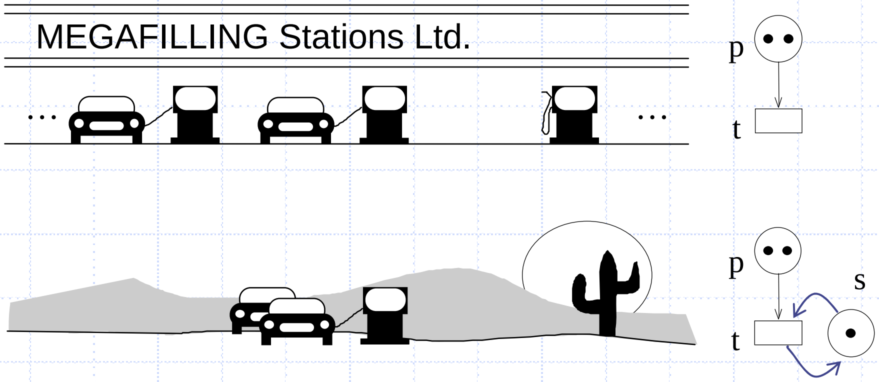
Interpreted Petri nets

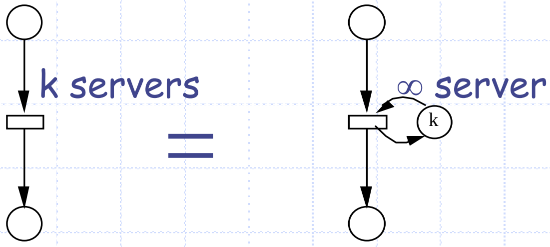
- Timed interpretations
  - Specification of activities and servers
    - sensibilization → start of activity
    - firing → end of activity

      delay
    - transition → service station (# servers)

Specification of:

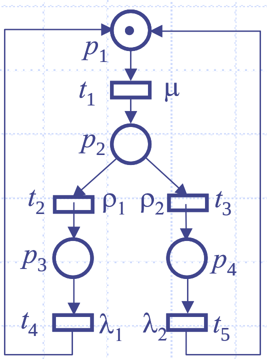
- delay
- # servers (multi-sensibilization: single, multiple, or infinite)



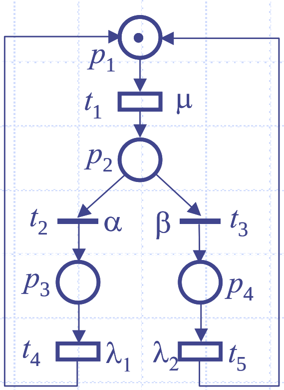


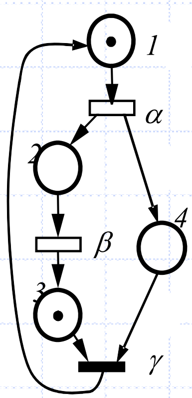
Interpreted Petri nets

- Specification of resolution of conflicts
  - race policy (race between timed enabled transitions)
  - preselection (random or deterministic choice)



- Immediate transitions
  - Modelling of synchronizations or routing
  - Zero delay $\Rightarrow$ higher priority in case of conflict





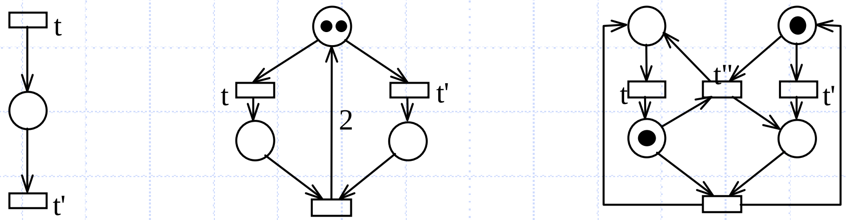
Interpreted Petri nets

- Reduction of the non-determinism
  - Define duration of activities
    (elapsed time from enabling to firing of a transitions)
    - Constant → *Timed* Petri nets (TPN, Ramchandani, 1974)
    - Interval → *Time* Petri nets (TPN, Merlin and Faber, 1976)
    - Random (exponentially distrib.) → *Stochastic* Petri nets (SPN, Symons, 1978; Natkin, 1980; Molloy, 1981)
    - Random or immediate → *Generalized Stochastic* Petri nets (GSPN, Ajmone Marsan, Balbo, Conte, 1984)
  - Define server semantics (single/multiple/infinite)
  - Define routing at conflicts
    - Race between stochastically timed transitions
    - Preselection (probabilistic or deterministic choice)

Interpreted Petri nets

Interpretation and logic properties

- An interpretation restricts possible behaviour
  - Some reachable markings are not reachable anymore
  - Analysis of qualitative properties of the autonomous model can be non conclusive



unbounded? total deadlock? live?

  - In general, a marking does not define a state
  - In a SPN:
    - The same reachable makings than autonomous model (support of r.v. = [0,∞) and race policy gives positive probabilities to all possible outcomes of conflicts)
    - A marking does define a state (memoryless property)

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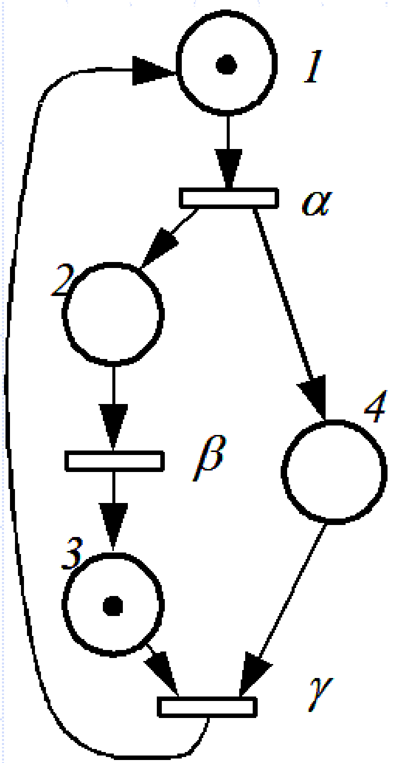
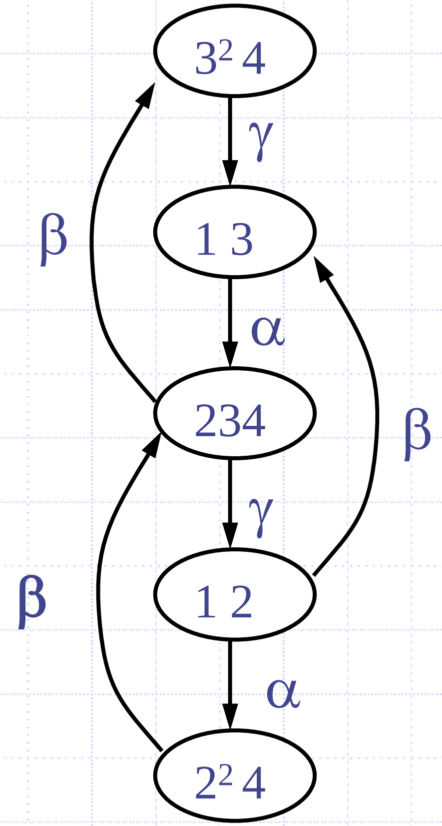
Stochastic Petri nets and CTMC-based analysis

- Time interpretation of Petri nets:
  - Duration of activities: exponentially distributed random variables
  - Single server semantics at each transition
  - Conflicts resolution: race policy

The reachability graph of the SPN is isomorphic to a Continuous Time Markov Chain

Stochastic Petri nets and CTMC-based analysis





The reachability graph of the SPN is isomorphic to a Continuous Time Markov Chain

Stochastic Petri nets and CTMC-based analysis

- The CTMC associated with a (bounded) SPN is obtained:
  - The state space $S = \{s_i\}$ of the CTMC is equal to the reachability set $RS(m_0)$ of the underlying PN ($m_i \leftrightarrow s_i$)
  - The transition rate from state s_i (corresponding to marking m_i) to state s_j (m_j) is obtained as the sum of the service rates of transitions enabled in m_i whose firing leads to marking m_j.
- If transitions have single-server semantics and marking independent rates, the components of Q are:

$$q_{ij} = \begin{cases} \displaystyle\sum_{T_k \in e_j(m_i)} w_k, & \text{si} \quad i \neq j \\ -q_i, & \text{si} \quad i = j \end{cases}$$

where

$$q_i = \sum_{T_k \in e(m_i)} w_k$$

$$e_j(m_i) = \{T_h \mid T_h \in e(m_i) \wedge m_i \xrightarrow{T_h} m_j\}$$

Stochastic Petri nets and CTMC-based analysis

- Let $\pi(m_i,\tau)$ be the probability for the SPN to be at the state m_i at instant τ.

- The Kolmogorov differential equation for the associated CTMC is:

$$\frac{d\pi(m_i,\tau)}{d\tau} = \sum_{T_k \in T} q_{kj}\pi(m_k,\tau)$$

in matrix form: $$\frac{d\pi(\tau)}{d\tau} = \pi(\tau)Q$$

and its solution can be expressed as: $$\pi(\tau) = \pi(0)e^{Q\tau}$$

where $\pi(0)$ is the initial probability distribution

(usually $\pi_i(0) = 1$ if $m_i = m_0$ and $\pi_i(0) = 0$ otherwise)

Stochastic Petri nets and CTMC-based analysis

- The steady-state "solution" of an SPN is based on the study of the probability distribution of the set of reachable markings

$$\pi = (\pi_1, \ldots, \pi_{|RS|})$$

- The limit behaviour of that distribution

$$\pi = \lim_{\tau \to \infty} \pi(\tau)$$

is computed by solving the following system of linear equations

$$\begin{cases} \pi\, Q = \mathbf{0} \\ \pi\, \mathbf{1}^{\mathrm{T}} = 1 \end{cases}$$

where $\mathbf{0}$ and $\mathbf{1}^{\mathrm{T}}$ are vectors of the size of π with all the components equal to 0 and 1 respectively

Stochastic Petri nets and CTMC-based analysis

- The steady-state distribution π is used for the computation of performance indices of interest
- Performance indices can be expressed from *reward functions* defined over the markings of the SPN, the average reward is computed as average value of the reward of the steady-state distribution

$$R = \sum_{m_i \in RS(m_0)} r(m_i)\pi_i$$

where $r(m)$ represents a given reward function

Stochastic Petri nets and CTMC-based analysis

- To compute the probability of a given condition $\Gamma(m)$ in the SPN
  - First, we define the reward function:

$$r(m) = \begin{cases} 1, & \text{if } \Gamma(m) = true \\ 0, & \text{otherwise} \end{cases}$$

  - Then, the desired probability is computed as:

$$P\{\Gamma\} = \sum_{m_i \in RS(m_0)} r(m_i)\pi_i = \sum_{m_i \in A} \pi_i$$

where

$$A = \{m_i \in RS(m_0) \mid \Gamma(m_i) = true\}$$

Stochastic Petri nets and CTMC-based analysis

- Example: mean number of tokens at place p_j
  - The reward function is

$$r(m) = n \quad \text{if and only if} \quad m(p_j) = n$$

  - Then the average marking of place:

$$\overline{\mu}(p_j) = \sum_{m_i \in RS(m_0)} r(m_i)\pi_i = \sum_{n>0} n\, P\{A(j,n)\}$$

where $A(j,n) = \{m_i \in RS(m_0) : m_i(p_j) = n\}$ and the sum is constrained to $n \leq k$ if place is k-bounded

Stochastic Petri nets and CTMC-based analysis

- Other example: throughput of transition T_j (average number of firings per time unit)
  - A transition can fire only if it is enabled, thus the reward function is

$$r(m) = \begin{cases} w_j, & \text{si } T_j \in e(m) \\ 0, & \text{en otro caso} \end{cases}$$

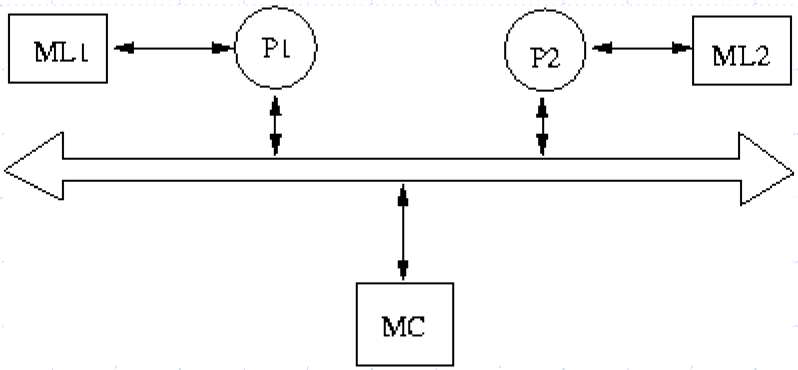
  - Then the throughput of T_j is

$$\chi_j = \sum_{m_i \in RS(m_0)} r(m_i)\pi_i = \sum_{m_i \in A_j} w_j \pi_i$$

where $A_j = \{m_i \in RS(m_0) : T_j \in e(m_i)\}$

Stochastic Petri nets and CTMC-based analysis

- Shared memory multiprocessor



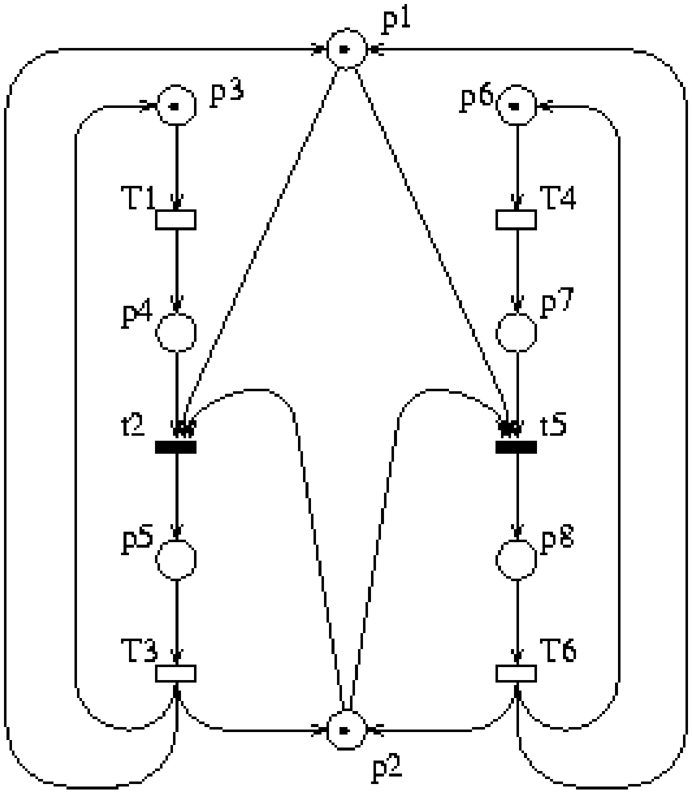
Both processors behave in a similar way:

- A cyclic sequence of: local activity, then
- an access request to the shared memory, and then
- accessing the shared memory

Stochastic Petri nets and CTMC-based analysis

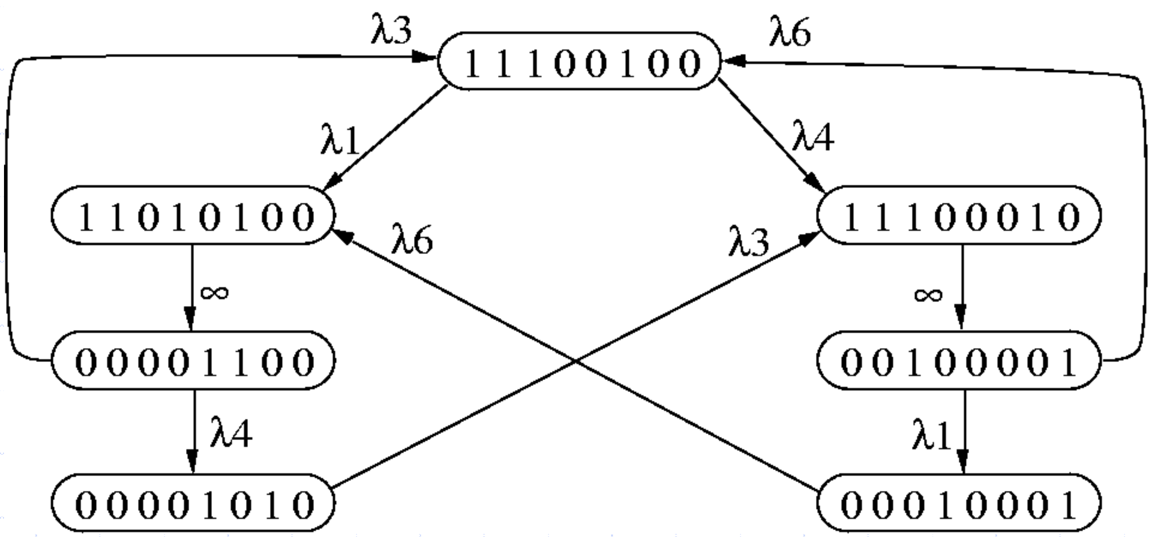
- All transitions have exponentially distributed durations, except for t2 and t5,
  access request to the shared memory (immediate)

  → GSPN



Stochastic Petri nets and CTMC-based analysis

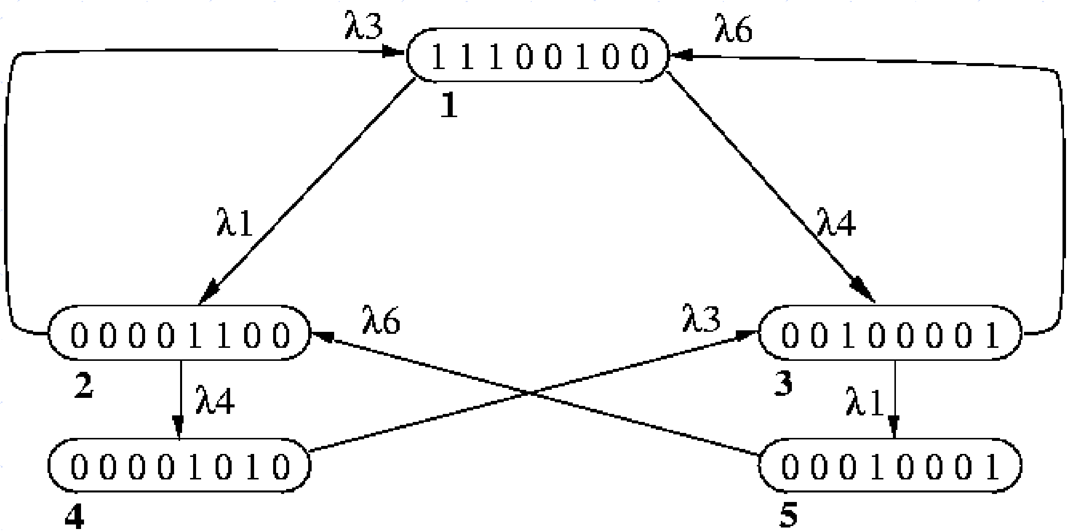
- Reachability graph



It is not isomorphic to a Continuous Time Markov Chain (infinite rates are not allowed in CTMCs)

Stochastic Petri nets and CTMC-based analysis

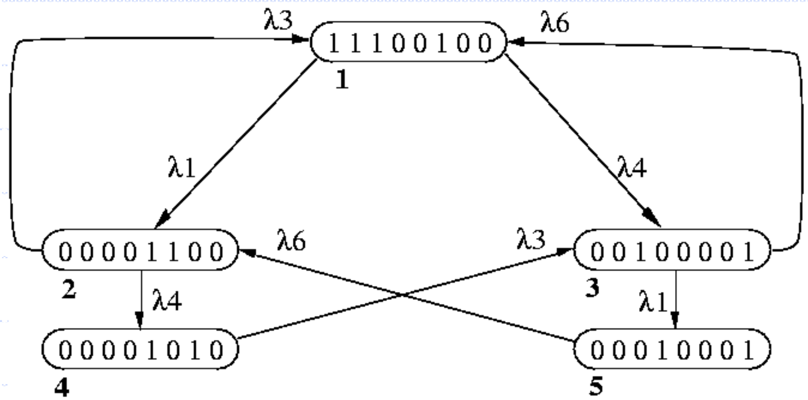
- Tangible reachability graph



It is isomorphic to a Continuous Time Markov Chain

Stochastic Petri nets and CTMC-based analysis

- Infinitesimal generator matrix of the CTMC



$$\begin{bmatrix} -(\lambda_1+\lambda_4) & \lambda_1 & \lambda_4 & 0 & 0 \\ \lambda_3 & -(\lambda_3+\lambda_4) & 0 & \lambda_4 & 0 \\ \lambda_6 & 0 & -(\lambda_1+\lambda_6) & 0 & \lambda_1 \\ 0 & 0 & \lambda_3 & -\lambda_3 & 0 \\ 0 & \lambda_6 & 0 & 0 & -\lambda_6 \end{bmatrix}$$

Stochastic Petri nets and CTMC-based analysis

- The stationary distribution can be computed (steady state probability of each state)

$$(\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\pi}_3, \boldsymbol{\pi}_4, \boldsymbol{\pi}_5) \cdot \begin{bmatrix} -(\lambda_1+\lambda_4) & \lambda_1 & \lambda_4 & 0 & 0 \\ \lambda_3 & -(\lambda_3+\lambda_4) & 0 & \lambda_4 & 0 \\ \lambda_6 & 0 & -(\lambda_1+\lambda_6) & 0 & \lambda_1 \\ 0 & 0 & \lambda_3 & -\lambda_3 & 0 \\ 0 & \lambda_6 & 0 & 0 & -\lambda_6 \end{bmatrix} = \mathbf{0}$$

$$\boldsymbol{\pi}_1 + \boldsymbol{\pi}_2 + \boldsymbol{\pi}_3 + \boldsymbol{\pi}_4 + \boldsymbol{\pi}_5 = 1$$

- And from here, compute, for instance, *utilization rate of shared memory*
  - In this case, it is equal to 1 minus the steady-state probability of the unique state with p_2 (shared memory is free) marked

$$1-\overline{\mu}[p_2] = 1-\pi_1$$

Stochastic Petri nets and CTMC-based analysis

- Other example, *processing power*
  - Average number of processors effectively (locally) working
  - We define the reward function

$$r_P(m) = m[p_3] + m[p_6]$$

  - Then:

$$P = \sum_{m_i \in RS(m_0)} r_P(m_i)\pi_i = 2\pi_1 + \pi_2 + \pi_3$$

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- M. Silva, E. Teruel: **A systems theory perspective of discrete event dynamic systems: the Petri net paradigm**. *Procs. of the Symposium on Discrete Events and Manufacturing Systems, CESA '96 IMACS Multiconference*, P.Borne, J.C. Gentina, E.Craye, S.El Khattabi (ed.), pp. 1-12, Lille, France, July 1996. Download here.
- M. Silva, E. Teruel: **DEDS along their life-cycle: interpreted extension of Petri nets**. *Procs. of IEEE International Conference on Systems, Man and Cybernetics (SMC'98)*, San Diego, USA, 1998. Download here.
- Balbo, G. and Silva, M. (Eds.), *Performance Models for Discrete Event Systems with Synchronizations: Formalisms and Analysis Techniques*, Editorial KRONOS, Zaragoza, Spain, 1998.
- M. Ajmone Marsan, G. Balbo, G. Conte, S. Donatelli, G. Franceschinis: **Modelling with Generalized Stochastic Petri Nets**. Wiley Series in Parallel Computing, John Wiley and Sons, 1995 (out of print). Download here (a revised version).