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Effective Hausdorff Dimension

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Abstract. Lutz (2000) has recently proved a new characterization of Hausdorff dimension in terms of gales, which are betting strategies that generalize martingales. We present here this characterization and give three instances of how it can be used to define effective versions of Hausdorff dimension in the contexts of constructible, finite-state, and resource-bounded computation.

1 Introduction

Resource-bounded measure, a generalization of classical Lebesgue measure, was developed by Lutz in 1991 [Lut92] in order to investigate the internal structures of complexity classes. This line of research has proven to be very fruitful [ASM97], [Lut97], [LM01], but there are certain inherent limitations to the information that resource-bounded measure can provide. These limitations, also present in classical Lebesgue measure, come from the fact that measure cannot make quantitative distinctions inside a measure 0 set, and also from the Kolmogorov zero-one law ([Dai01], [Lut98]) that implies that for most sets of interest in computational complexity and recursion theory the measure is 0, 1 or undefined.

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Hausdorff dimension was defined as an augmentation of Lebesgue measure theory [Hau19]. Every subset of a given metric space is assigned a dimension. We are interested in the metric space being the Cantor space $\mathbf{C}$ consisting of all infinite binary strings. In this case the dimension of each set $X \subseteq \mathbf{C}$ is a real number $\dim_{\mathrm{H}}(X) \in [0, 1]$. Hausdorff dimension is monotone, $\dim_{\mathrm{H}}(\emptyset) = 0$, $\dim_{\mathrm{H}}(\mathbf{C}) = 1$, and intermediate values occur for many interesting sets. Also, if $\dim_{\mathrm{H}}(X) < 1$ then X is a measure 0 subset of $\mathbf{C}$, so Hausdorff dimension can quantitatively distinguish among measure 0 sets.

Hausdorff dimension was originally defined topologically ([Hau19], [Fal85]). Lutz [Lut00a] recently proved a new characterization of Hausdorff dimension in terms of gales or supergales, which are betting strategies that generalize martingales. The class of gales that "succeed" on the sequences in a set $X \subseteq \mathbf{C}$ determines the dimension of X. Therefore a natural generalization of Hausdorff dimension arises by restricting the class of admissible gales or supergales. This is useful when we want a nontrivial dimension inside a countable set such as the set of all decidable sequences.

In this paper we present three effectivizations of Hausdorff dimension. The first one is constructive dimension ([Lut00b], [Lut02]), defined by restricting to constructive (lower semicomputable) supergales, the second is finite-state dimension ([DLLM01]), defined by the natural finite-state effectivization, and the third one is resource-bounded dimension, in which a particular complexity time-bound is enforced on gales.

Hausdorff dimension has also been related to measures of information content. Ryabko ([Rya86], [Rya93], [Rya94]), Staiger ([Sta93], [Sta98]) and Cai and Hartmanis [CH94] have all proven results that relate Hausdorff dimension with Kolmogorov complexity. This line of thought is also present in effective dimension. Constructive dimension can be fully characterized in terms of Kolmogorov complexity ([May02]) and finite state dimension is definable in terms of sequence compressibility [DLLM01]. We present here these two results supporting the intuition that dimension is a measure of information density.

In section 2 we introduce a few preliminary definitions. Section 3 contains Lutz's characterization of Hausdorff dimension, with the key concepts of gale and supergale. Section 4 presents constructive dimension of sets and sequences. Section 5 contains finite-state dimension, and section 6 is a brief sketch of resource-bounded dimension.

2 Preliminaries

We work in the set $\{0,1\}^*$ of all (finite, binary) *strings* and in the Cantor space $\mathbf{C}$ of all (infinite, binary) *sequences*. We write $|w|$ for the length of a string $w \in \{0,1\}^*$. The empty string is denoted λ. For $S \in \mathbf{C}$ and $i, j \in \mathbb{N}$, $i \geq j$, we write $S[i..j]$ for the string consisting of the i^{th} through j^{th} bits of S, stipulating that $S[0..0]$ is the leftmost bit of S. For each $w \in \{0,1\}^*$, w is a prefix of S if $w = S\,[0..|w|-1]$ or $w = \lambda$.

We will freely identify each language $A \subseteq \{0,1\}^*$ with its characteristic sequence $\chi_A \in \mathbf{C}$.

We will refer to the low levels of the arithmetical hierarchy of sets of sequences, specifically to sets in Π^0_1, Σ^0_2, and Π^0_2. The definition can be found in [Rog67].

We will also mention sequences in Σ^0_1, that is, r.e. sequences, sequences in Π^0_1, that is, co-r.e. sequences, and sequences in Δ^0_2, that is, sequences that are decidable relative to the halting oracle.

A function $f : \{0,1\}^* \to \mathbb{R}$ is *lower semicomputable* if its lower graph $\mathrm{Graph}^-(f) = \left\{(x,s) \in \{0,1\}^* \times [0,\infty) \,\middle|\, s < f(x)\right\}$ is recursively enumerable.

A real number α is *computable* if there is a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that for all $r \in \mathbb{N}$, $|f(r) - \alpha| < 2^{-r}$. A real number α is Δ^0_2*-computable* if there is a function $f : \mathbb{N} \to \mathbb{Q}$ that is computable relative to the halting oracle such that for all $r \in \mathbb{N}$, $|f(r) - \alpha| < 2^{-r}$.

Let $t : \mathbb{N} \to \mathbb{N}$. A function $f : \{0,1\}^* \to \mathbb{R}$ is *computable in time (space)* t if there is a function $\hat{f} : \{0,1\}^* \times \mathbb{N} \to \mathbb{Q}$ that can be computed by a Turing Machine in time (space) t and such that for all $x \in \{0,1\}^*$, $r \in \mathbb{N}$, $|\hat{f}(x,r) - f(x)| < 2^{-r}$.

$\mathrm{K}(x)$, the Kolmogorov complexity of a finite binary sequence x, is the length of the shortest description of x (the full definition and properties can be found in the book by Li and Vitànyi [LV97]).

We use the logspace-uniform version of the bounded-depth circuit complexity class AC_0 ([Joh90]).

For each $f : \mathbb{N} \to \mathbb{N}$, $\mathrm{SIZE}(f)$ is the set of languages $A \subseteq \{0,1\}^*$ such that for all $n \in \mathbb{N}$ there is an n-input boolean circuit of at most $f(n)$ gates that recognizes $A \cap \{0,1\}^n$.

We define the *frequency* of a nonempty string $w \in \{0,1\}^*$ to be the ratio $\mathrm{freq}(w) = \frac{\#(1,w)}{|w|}$, where $\#(b,w)$ denotes the number of occurrences of the bit b in w. For each $\alpha \in [0,1]$, we define the set

$$\mathrm{FREQ}(\alpha) = \left\{ S \in \mathbf{C} \,\middle|\, \lim_{n\to\infty} \mathrm{freq}(S\,[0..n-1]) = \alpha \right\}.$$

The binary Shannon entropy function $\mathcal{H} : [0,1] \to [0,1]$ is defined as

$$\mathcal{H}(x) = x \log \frac{1}{x} + (1-x) \log \frac{1}{1-x},$$

with $\mathcal{H}(0) = \mathcal{H}(1) = 0$ to ensure continuity at 0 and 1.

3 Lutz Characterization of Hausdorff Dimension

This section reviews the gale characterization of classical Hausdorff dimension, which motivates our development.

Definition 1. [Lut00a] Let $s \in [0, \infty)$

1. An s-*supergale* is a function $d : \{0,1\}^* \to [0,\infty)$ that satisfies the condition
$$d(w) \geq 2^{-s}\,[d(w0) + d(w1)] \qquad (*)$$
for all $w \in \{0,1\}^*$.
2. An s-*gale* is an s-supergale that satisfies condition $(*)$ with equality for all $w \in \{0,1\}^*$.
3. A *supermartingale* is a 1-supergale.
4. A *martingale* is a 1-gale.

Intuitively, an s-supergale is a strategy for betting on the successive bits of a sequence $S \in \mathbf{C}$. For each prefix w of S, $d(w)$ is the capital (amount of money) that d has after betting on the bits w of S. When betting on the next bit b of a prefix wb of S (assuming that b is equally likely to be 0 or 1), condition $(*)$ tells us that the expected value of $d(wb)$ – the capital that d expects to have after this bet – is $(d(w0)+d(w1))/2 \leq 2^{s-1}d(w)$. If $s = 1$, this expected value is at most $d(w)$ – the capital that d has before the bet – so the payoffs are at most "fair." If $s < 1$, this expected value is less than $d(w)$, so the payoffs are "less than fair." Similarly, of $s > 1$, the payoffs can be "more than fair."

Note: We will use in section 4 the concept of β-martingale, where β is a real number in $[0,1]$. In this case when betting on the next bit

b of a prefix wb or S we assume that β is the probability of b being 1. Specifically, a *β-martingale* is a function $d : \{0,1\}^* \to [0,\infty)$ that satisfies the condition

$$d(w) = (1-\beta)\, d(w0) + \beta\, d(w1)$$

for all $w \in \{0,1\}^*$. Note that for $\beta = 1/2$ we obtain the above definition.

Of course the objective of an s-supergale is to win a lot of money.

Definition 2. [Lut00a] Let d be an s-supergale, where $s \in [0,\infty)$.

1. We say that d *succeeds* on a sequence $S \in \mathbf{C}$ if
$$\limsup_{n\to\infty} d(S\,[0..n-1]) = \infty.$$
2. The *success set* of d is
$$S^\infty[d] = \left\{ S \in \mathbf{C} \;\middle|\; d \text{ succeeds on } S \right\}.$$
3. For $X \subseteq \mathbf{C}, \mathcal{G}(X)$ is the set of all $s \in [0,\infty)$ such that there is an s-gale d for which $X \subseteq S^\infty[d]$.
4. For $X \subseteq \mathbf{C}, \widehat{\mathcal{G}}(X)$ is the set of all $s \in [0,\infty)$ such that there is an s-supergale d for which $X \subseteq S^\infty[d]$.

Note that if $s, s' \in [0,\infty)$ then for every s-supergale d, the function $d' : \{0,1\}^* \to [0,\infty)$ defined by $d'(w) = 2^{(s'-s)|w|}d(w)$ is an s'-supergale.

It was shown in [Lut00a] that the following definition is equivalent to the classical definition of Hausdorff dimension in $\mathbf{C}$.

Definition 3. The *Hausdorff dimension* of a set $X \subseteq \mathbf{C}$ is

$$\dim_{\mathrm{H}}(X) = \inf \mathcal{G}(X) = \inf \widehat{\mathcal{G}}(X).$$

See [Fal85] for a good overview of classical Hausdorff dimension, including the original topological definition based on open covers by balls of diminishing radii [Hau19].

The gale characterization of Hausdorff dimension that we have presented can be generalized by restricting the class of gales or supergales that are allowed. We will follow this idea in the rest of the paper.

Eggleston [Egg49] proved the following classical result on the Hausdorff dimension of a set of sequences with a fixed asymptotic frequency.

Theorem 1. *[Egg49] For each real number $\alpha \in [0,1]$,*

$$\dim_{\mathrm{H}}(\mathrm{FREQ}(\alpha)) = \mathcal{H}(\alpha).$$

We will reformulate this last result in the contexts of the dimensions defined in sections 4, 5, and 6.

4 Constructive Dimension

In this section we present the first effective version of of Hausdorff dimension that is defined by restricting the class of supergales to those that are lower semicomputable. We first give the definitions of constructive dimension of a set and constructive dimension of a sequence, and then we relate them and give their main properties. We finish the section summarising the main results on constructive dimension. The results in this section are mainly from [Lut00b] and [Lut02], also including those in [May02] and [Hit02].

An s-supergale is *constructive* if it is lower semicomputable. We define constructive dimension as follows.

Definition 4. [Lut02] Let $X \subseteq \mathbf{C}$.

1. $\widehat{\mathcal{G}}_{\mathrm{constr}}(X)$ is the set of all $s \in [0,\infty)$ such that there is a constructive s-supergale d for which $X \subseteq S^{\infty}[d]$.
2. The *constructive* dimension of a set $X \subseteq \mathbf{C}$ is

$$\mathrm{cdim}(X) = \inf \widehat{\mathcal{G}}_{\mathrm{constr}}(X).$$

3. The constructive dimension of an individual sequence $S \in \mathbf{C}$ is $\dim(S) = \mathrm{cdim}(\{S\})$.

By Lutz characterization of Hausdorff dimension (Definition 3), we conclude that $\mathrm{cdim}(X) \geq \dim_{\mathrm{H}}(X)$ for all $X \subseteq \mathbf{C}$. But in fact much more is true for certain classes, as Hitchcock shows in [Hit02]. For sets that are low in the arithmetical hierarchy, constructive dimension and Hausdorff dimension coincide.

Theorem 2. *[Hit02] If $X \subseteq \mathbf{C}$ is a union of Π^0_1 sets, then*

$$\dim_{\mathrm{H}}(X) = \mathrm{cdim}(X).$$

We note that Theorem 6 below yields a new proof of Theorem 2 from Theorem 5 of Staiger [Sta98].

Hitchcock also proves that this is an optimal result for the arithmetical hierarchy, since it cannot be extended to sets in Π^0_2.

For Hausdorff dimension, all singletons have dimension 0, and in fact all countable sets have Hausdorff dimension 0. The situation changes dramatically when we restrict to constructive supergales, since a singleton can have positive constructive dimension, and in fact can have any constructive dimension.

Theorem 3. *[Lut02] For every $\alpha \in [0,1]$, there is an $S \in \mathbf{C}$ such that $\dim(S) = \alpha$.*

We note that Theorem 6 below yields a new proof of Theorem 3 from Lemma 3.4 of Cai Hartmanis [CH94].

The constructive dimension of any set $X \subseteq \mathbf{C}$ is completely determined by the dimension of the individual sequences in the set.

Theorem 4. *[Lut02] For all $X \subseteq \mathbf{C}$,*

$$\mathrm{cdim}(X) = \sup_{x \in X} \dim(x).$$

There is no analogue of this last theorem for Hausdorff dimension or for any of the concepts we will define in sections 5 and 6. The key ingredient in the proof of Theorem 4 is the existence of optimal constructive supergales, that is, constructive supergales that multiplicatively dominate any other constructive supergale. This is analogous to the existence of universal tests of randomness in the theory of random sequences.

Theorem 2 together with Theorem 4 implies that the classical Hausdorff dimension of every Σ^0_2 set $X \subseteq \mathbf{C}$ has the pointwise characterization $\dim_{\mathrm{H}}(X) = \sup_{x \in X} \dim(x)$.

Theorem 4 immediately implies that constructive dimension has the *countable stability* property, which also holds for classical Hausdorff dimension.

Corollary 1. [Lut02] For all $X_0, X_1, X_2, \dots \subseteq \mathbf{C}$,

$$\mathrm{cdim}\left(\bigcup_{k=0}^{\infty} X_k\right) = \sup_{k \in \mathbb{N}} \mathrm{cdim}(X_k).$$

Let $\alpha \in [0,1]$. If we define $\mathrm{DIM}_{\leq\alpha} = \left\{ S \in \mathbf{C} \,\middle|\, \dim(S) \leq \alpha \right\}$, then this is the largest set of constructive dimension α.

Theorem 5. *[Lut02] For every $\alpha \in [0,1]$, the set $\mathrm{DIM}_{\leq\alpha}$ has the following two properties.*

1. $\mathrm{cdim}(\mathrm{DIM}_{\leq\alpha}) = \alpha$.
2. *For all $X \subseteq \mathbf{C}$, if $\mathrm{cdim}(X) \leq \alpha$, then $X \subseteq \mathrm{DIM}_{\leq\alpha}$.*

We note that Theorem 6 below yields a new proof of Theorem 5 (part 1) from Theorem 2 of Ryabko [Rya84].

The constructive dimension of a sequence can be characterized in terms of the Kolmogorov complexities of its prefixes.

Theorem 6. *([May02]) For all $A \in \mathbf{C}$,*

$$\dim(A) = \liminf_{n\to\infty} \frac{\mathrm{K}(A\,[0..n-1])}{n}$$

This latest theorem justifies the intuition that the constructive dimension of a sequence is a measure of its algorithmic information density. The relation of dimension and information content will appear again in a different context in the next section, where the finite-state dimension of a sequence (to be defined) is characterized in terms of its compressibility.

We now briefly state the main results proven so far on constructive dimension, including the existence of Δ^0_2 sequences of any Δ^0_2 dimension, the constructive version of Eggleston theorem, and the constructive dimension of sequences that are random relative to a non-uniform distribution.

An important result in the theory of random sequences is the existence of random sequences in Δ^0_2. We have the following analogue for constructive dimension.

Theorem 7. *[Lut02] For every Δ^0_2-computable real number $\alpha \in [0,1]$ there is a Δ^0_2 sequence S such that $\dim(S) = \alpha$.*

And this cannot be improved to Σ^0_1 or Π^0_1 sequences since they all have constructive dimension 0.

This is the constructive version of the classical Theorem 1 (Eggleston [Egg49]).

Theorem 8. *[Lut02] If α is Δ^0_2-computable real number in $[0,1]$ then*

$$\mathrm{cdim}(\mathrm{FREQ}(\alpha)) = \mathcal{H}(\alpha).$$

An anonymous referee has pointed out that an alternative proof of Theorem 8 can be derived from the newer Theorem 6 and earlier results of Eggleston [Egg49] and Kolmogorov [ZL70]. In fact, this approach shows that Theorem 8 holds for *arbitrary* $\alpha \in [0,1]$.

A sequence is (Martin-Löf) *random* [ML66] if it passes every algorithmically implementable test of randomness. This can be reformulated in terms of martingales as follows

Definition 5. [Sch71] A sequence $A \in \mathbf{C}$ is (Martin-Löf) *random* if there is no constructive supermartingale d such that $A \in S^\infty[d]$.

By this definition, random sequences have constructive dimension 1. For nonuniform distributions we have the concept of β-randomness, for β any real number in $(0,1)$ representing the bias.

Definition 6. [Sch71] Let $\beta \in (0,1)$. A sequence $A \in \mathbf{C}$ is (Martin-Löf) *random relative to* β if there is no constructive β-martingale d such that $A \in S^\infty[d]$.

Lutz relates randomness relative to a non-uniform distribution with Shannon information theory.

Theorem 9. *[Lut02] Let $\beta \in (0,1)$ be a computable real number. Let $A \in \mathbf{C}$ be random relative to β. Then* $\dim(A) = \mathcal{H}(\beta)$.

A more general result for randomness relative to sequences of coin-tosses is obtained in [Lut02].

Very recently Lutz [Lut02] has introduced the concept of dimension of finite binary strings, with very interesting connections with constructive dimension. We cannot cover this topic here due to lack of space.

5 Finite-State Dimension

Our second effectivization of Hausdorff dimension will be the most restrictive of those presented here, we will go all the way to the level of finite-state computation. In this section we use gales computed by multi-account finite-state gamblers to develop the finite-state dimensions of sets of binary sequences and individual binary sequences. The theorem of Eggleston is shown to hold for finite-state dimension. Every rational number in $[0,1]$ is the finite-state dimension of a sequence in the low-level complexity class AC_0. The main theorem of this section shows that the finite-state dimension of a sequence is precisely the infimum of all compression ratios achievable on the sequence by information-lossless finite-state compressors.

All the results in this section are from [DLLM01].

We start by introducing the concept of finite-state gambler that is used to develop finite-state dimension. Intuitively, a finite-state gambler is a finite-state device that places k separate binary bets on each of the successive bits of its input sequence. These bets correspond to k separate accounts into which the gambler's capital is divided. Bets are required to be rational numbers in $\mathbf{B} = \mathbb{Q} \cap [0,1]$.

Definition 7. If k is a positive integer, then a *k-account finite-state gambler* (*k-account FSG*) is a 5-tuple

$$G = \left(Q, \delta, \vec{\beta}, q_0, \vec{c}_0\right),$$

where

- Q is a nonempty, finite set of *states*,
- $\delta : Q \times \{0,1\} \to Q$ is the *transition function*,
- $\vec{\beta} : Q \to \mathbf{B}^k$ is the *betting function*,
- $q_0 \in Q$ is the *initial state*, and
- $\vec{c}_0 = (c_{0,1}, \ldots, c_{0,k})$, the *initial capital vector*, is a sequence of nonnegative rational numbers.

A *finite state gambler (FSG)* is a k-account FSG for some positive integer k.

The case $k = 1$, where there is only one account, is the model of finite-state gambling that has been considered (in essentially equivalent form) by Schnorr and Stimm [SS72], Feder [Fed91], and others.

Intuitively, if a k-account FSG $G = (Q, \delta, \vec{\beta}, q_0, \vec{c}_0)$ is in state $q \in Q$ and its current capital vector is $\vec{c} = (c_1, \ldots, c_k) \in (\mathbb{Q} \cap [0,\infty))^k$, then for each of its accounts $i \in \{1, \ldots, k\}$, it places the bet $\beta_i(q) \in \mathbf{B}$. If the payoffs are fair, then after this bet G will be in state $\delta(q,b)$ and its i^{th} account will have capital

$$2c_i[(1-b)(1-\beta_i(q))+b\beta_i(q)] = \begin{cases} 2\beta_i(q)c_i & \text{if } b=1 \\ 2(1-\beta_i(q))c_i & \text{if } b=0. \end{cases}$$

This suggests the following definition.

Definition 8. Let $G = (Q, \delta, \vec{\beta}, q_0, \vec{c}_0)$ be a k-account finite-state gambler.

1. For each $1 \le i \le k$, the i^{th} *martingale* of G is the function

$$d_{G,i} : \{0,1\}^* \to [0,\infty)$$

defined by the recursion

$$d_{G,i}(\lambda) = c_{0,i},$$
$$d_{G,i}(wb) = 2d_{G,i}(w)[(1-b)(1-\beta_i(\delta(w))) + b\beta_i(\delta(w))]$$

for all $w \in \{0,1\}^*$ and $b \in \{0,1\}$.
2. The *total martingale* (or simply the *martingale*) of G is the function

$$d_G = \sum_{i=1}^{k} d_{G,i}.$$

3. For $s \in [0,\infty)$, the s*-gale of* an FSG G is the function

$$d_G^{(s)} : \{0,1\}^* \to [0,\infty)$$

defined by

$$d_G^{(s)}(w) = 2^{(s-1)|w|} d_G(w)$$

for all $w \in \{0,1\}^*$. In particular, note that $d_G^{(1)} = d_G$.
4. For $s \in [0,\infty)$, a *finite-state* s*-gale* is an s-gale d for which there exists an FSG G such that $d_G^{(s)} = d$.

We now use finite-state gales to define finite-state dimension.

Definition 9. Let $X \subseteq \mathbf{C}$.

1. $\mathcal{G}_{\mathrm{FS}}(X)$ is the set of all $s \in [0, \infty)$ such that there is a finite-state s-gale d for which $X \subseteq S^\infty[d]$.
2. The *finite-state dimension* of set X is

$$\dim_{\mathrm{FS}}(X) = \inf \mathcal{G}_{\mathrm{FS}}(X).$$

3. The *finite-state dimension* of a sequence $S \in \mathbf{C}$ is

$$\dim_{\mathrm{FS}}(S) = \dim_{\mathrm{FS}}(\{S\}).$$

In general, $\dim_{\mathrm{FS}}(X)$ is a real number satisfying $0 \leq \dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{FS}}(X) \leq 1$. Like Hausdorff dimension, finite-state dimension has the stability property.

Theorem 10. *For all* $X, Y \subseteq \mathbf{C}$*,*

$$\dim_{\mathrm{FS}}(X \cup Y) = \max\left\{\dim_{\mathrm{FS}}(X), \dim_{\mathrm{FS}}(Y)\right\}.$$

Let us briefly digress on the role of multiple accounts in the FSG model. The proof of Theorem 10 is based on the fact that if d_1 and d_2 are finite-state s-gales then $d_1 + d_2$ is a finite-state s-gale, that is, finite-state gales are closed under nonnegative, rational, linear combinations. But multiple accounts are required for this closure property to hold, since there exist 1-account finite-state s-gales d_1 and d_2 such that $d_1 + d_2$ is not a 1-account finite-state s-gale.

Notwithstanding the usefulness of the above closure property, the question remains whether multiple accounts are strictly necessary for a theory of finite-state dimension. The next result shows that multiple accounts are not strictly necessary if we are willing to accept a large blowup in the number of states.

Theorem 11. *For each n-state, k-account FSG G and each $\epsilon \in (0, 1)$ there is an $n \cdot k^{\frac{1}{\epsilon}}$-state, 1-account FSG G' such that for all $s \in [0, 1]$,*

$$S^\infty[d_G^{(s)}] \subseteq S^\infty[d_{G'}^{(s+\epsilon)}].$$

The theorem of Eggleston [Egg49] (Theorem 1) holds for finite-state dimension.

Theorem 12. *For all* $\alpha \in \mathbb{Q} \cap [0, 1]$*,*

$$\dim_{\mathrm{FS}}(\mathrm{FREQ}(\alpha)) = \mathcal{H}(\alpha).$$

The following theorem says that every rational number $r \in [0,1]$ is the finite-state dimension of a reasonably simple sequence.

Theorem 13. *For every* $r \in \mathbb{Q} \cap [0,1]$ *there exists* $S \in \mathrm{AC}_0$ *such that* $\dim_{\mathrm{FS}}(S) = r$.

The main result in this section is that we can characterize the finite-state dimensions of individual sequences in terms of finite-state compressibility. We first recall the definition of an information-lossless finite-state compressor. (This idea is due to Huffman [Huf59]. Further exposition may be found in [Koh78] or [Kur74].)

Definition 10. A *finite-state compressor* (*FSC*) is a 4-tuple

$$C = (Q, \delta, \nu, q_0),$$

where

- Q is a nonempty, finite set of *states*,
- $\delta : Q \times \{0,1\} \to Q$ is the *transition function*,
- $\nu : Q \times \{0,1\} \to \{0,1\}^*$ is the *output function*, and
- $q_0 \in Q$ is the *initial state*.

For $q \in Q$ and $w \in \{0,1\}^*$, we define the *output* from state q on input w to be the string $\nu(q,w)$ defined by the recursion

$$\begin{aligned} &\nu(q,\lambda) = \lambda \\ &\nu(q,wb) = \nu(q,w)\nu(\delta(q,w),b) \end{aligned}$$

for all $w \in \{0,1\}^*$ and $b \in \{0,1\}$. We then define the *output* of C on input $w \in \{0,1\}^*$ to be the string

$$C(w) = \nu(q_0, w).$$

Definition 11. An FSC $C = (Q, \delta, \nu, q_0)$ is *information-lossless* (*IL*) if the function

$$\begin{aligned} \{0,1\}^* &\to \{0,1\}^* \times Q \\ w &\mapsto (C(w), \delta(w)) \end{aligned}$$

is one-to-one. An *information-lossless finite-state compressor* (*ILFSC*) is an FSC that is IL.

That is, an ILFSC is an FSC whose input can be reconstructed from the output and final state reached on that input.

Intuitively, an FSC C compresses a string w if $|C(w)|$ is significantly less than $|w|$. Of course, if C is IL, then not all strings can be compressed. Our interest here is in the degree (if any) to which the prefixes of a given sequence $S \in \mathbf{C}$ can be compressed by an ILFSC.

Definition 12. 1. If C is an FSC and $S \in \mathbf{C}$, then the *compression ratio* of C on S is

$$\rho_C(S) = \liminf_{n\to\infty} \frac{|C(S[0..n-1])|}{n}.$$

2. The *finite-state compression ratio* of a sequence $S \in \mathbf{C}$ is

$$\rho_{\mathrm{FS}}(S) = \inf\{\rho_C(S) | C \text{ is an ILFSC}\}.$$

The following theorem says that finite-state dimension and finite-state compressibility are one and the same for individual sequences.

Theorem 14. *For all $S \in \mathbf{C}$,*

$$\dim_{\mathrm{FS}}(S) = \rho_{\mathrm{FS}}(S).$$

Theorem 14 is a new instance of the relation of dimension and information.

Finite-state dimension is a real-time effectivization of a powerful tool of fractal geometry. As such it should prove to be a useful tool for improving our understanding of real-time information processing.

6 Dimension in Complexity Classes

We now use resource-bounded gales to develop dimension in complexity classes.

Let p be the class of polynomial time computable functions from $\{0,1\}^*$ to $\mathbb{R}$. Let pspace be the class of polynomial space computable functions from $\{0,1\}^*$ to $\mathbb{R}$. Using these two classes we define p-dimension and pspace-dimension in the natural way.

Definition 13. [Lut00a] Let Δ be p or pspace. Let $X \subseteq \mathbf{C}$.

1. $\mathcal{G}_\Delta(X)$ is the set of all $s \in [0, \infty)$ such that there is an s-gale $d \in \Delta$ for which $X \subseteq S^\infty[d]$.
2. The Δ-dimension of a set $X \subseteq \mathbf{C}$ is $\dim_\Delta(X) = \inf \mathcal{G}_\Delta(X)$.

Let $\mathrm{E} = \mathrm{DTIME}(2^{\mathrm{linear}})$ be the class of languages that can be recognized in linear exponential time. Let $\mathrm{ESPACE} = \mathrm{DSPACE}(2^{\mathrm{linear}})$ be that class of languages that can be recognized in linear exponential space.

The following result implies that p and pspace are the *right* dimension bounds for the classes E and ESPACE, respectively.

Theorem 15. *[Lut00a]*

1. $\dim_{\mathrm{p}}(\mathrm{E}) = 1$.
2. $\dim_{\mathrm{pspace}}(\mathrm{ESPACE}) = 1$.

Therefore we can define dimension in the classes E and ESPACE as follows.

Definition 14. [Lut00a] For each $X \subseteq \mathbf{C}$,

1. $\dim(X|\mathrm{E}) = \dim_{\mathrm{p}}(X \cap \mathrm{E})$.
2. $\dim(X|\mathrm{ESPACE}) = \dim_{\mathrm{pspace}}(X \cap \mathrm{ESPACE})$.

These definitions endow the classes E and ESPACE with internal dimension structure. Several interesting results have already been proven ([Lut00a], [Hit01], [ASMRS01]), indicating that dimension can give a new insight to open questions in Computational Complexity, in particular since many sets of interest in Computational Complexity have "fractal-like" structures.

We mention two of these results.

Theorem 16. *[Lut00a] For all $\alpha \in \mathbb{Q}$,*

$$\dim(\mathrm{FREQ}(\alpha) \mid E) = \dim_{\mathrm{p}}(\mathrm{FREQ}(\alpha)) = \mathcal{H}(\alpha).$$

Theorem 16 is proven in [Lut00a] for α a "p-computable" real number. Notice that we have proven Eggleston theorem in each of our formulations (for appropriate restrictions of α). This indicates that, no matter how we limit our power to compute dimension, a bounded asymptotic frequency of the elements implies a restriction on the dimension of the set.

We finish with a result on nonuniform classes that generalizes the known fact ([Lut92]) that $\mathrm{SIZE}(\frac{2^n}{n})$ has measure 0 in ESPACE.

Theorem 17. *[Lut00a] For every real $\alpha \in [0,1]$,*

$$\dim\left(\mathrm{SIZE}\left(\alpha\frac{2^n}{n}\right) \,\Big|\, ESPACE\right) = \alpha.$$

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