Finite-State Dimension

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Abstract

Classical Hausdorff dimension (sometimes called fractal dimension) was recently effectivized using gales (betting strategies that generalize martingales), thereby endowing various complexity classes with dimension structure and also defining the constructive dimensions of individual binary (infinite) sequences. In this paper we use gales computed by multi-account finite-state gamblers to develop the finite-state dimensions of sets of binary sequences and individual binary sequences. The theorem of Eggleston (1949) relating Hausdorff dimension to entropy is shown to hold for finite-state dimension, both in the space of all sequences and in the space of all rational sequences (binary expansions of rational numbers). Every rational sequence has finite-state dimension 0, but every rational number in $[0,1]$ is the finite-state dimension of a sequence in the low-level complexity class AC_0. Our main theorem shows that the finite-state dimension of a sequence is precisely the infimum of all compression ratios achievable on the sequence by information-lossless finite-state compressors.

KEYWORDS: bounded-depth circuit, finite-state compressor, finite-state dimension, finite-state gambler, gale, Hausdorff dimension, information-lossless compressor, martingale, normal sequence.

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1 Introduction

Hausdorff dimension, best known as a powerful tool of fractal geometry, has been known for over fifty years to be closely related to information. For example, Eggleston [5] proved that in the space of all infinite binary sequences, if we let $\mathrm{FREQ}(\alpha)$ be the set of sequences in which 1 appears with asymptotic frequency α $(0 \leq \alpha \leq 1)$, then the Hausdorff dimension of $\mathrm{FREQ}(\alpha)$ is precisely $\mathcal{H}(\alpha)$, the binary entropy of α. More recent investigations of Ryabko [16, 17, 18], Staiger [22, 23], and Cai and Hartmanis [3] have explored relationships between Hausdorff dimension and Kolmogorov complexity (algorithmic information).

Hausdorff dimension was originally defined topologically, using open covers by balls of diminishing radii [8, 6]. Very recently, Lutz [15] proved a new characterization of Hausdorff dimension in terms of *gales*, which are betting strategies that generalize martingales. Lutz used this characterization to effectivize Hausdorff dimension, thereby defining dimension in complexity classes [15] and the dimensions of individual sequences [14].

In this paper we extend the effectivization of dimension all the way to the level of finite-state computation. We define a *multi-account finite-state gambler* to be a finite-state gambler that maintains its capital in a portfolio of k separate accounts, so that capital in one account is shielded from losses in other accounts. (Finite-state gamblers with only one account have been investigated by Schnorr and Stimm [20] and Feder [7].) We use gales induced by multi-account finite-state gamblers to define the *finite-state dimension* $\dim_{\mathrm{FS}}(X)$ of each set $X \subseteq \mathbf{C}$, where $\mathbf{C}$ is the Cantor space, consisting of all infinite binary sequences. This definition is the natural finite-state effectivization of the gale characterization of classical Hausdorff dimension. In general, $\dim_{\mathrm{FS}}(X)$ is a real number satisfying

$$0 \leq \dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{FS}}(X) \leq 1,$$

where $\dim_{\mathrm{H}}(X)$ is the Hausdorff dimension of X. Like Hausdorff dimension, finite-state dimension has the *stability* property that

$$\dim_{\mathrm{FS}}(X \cup Y) = \max\{\dim_{\mathrm{FS}}(X), \dim_{\mathrm{FS}}(Y)\}$$

for all $X, Y \subseteq \mathbf{C}$.

We show that finite-state dimension endows $\mathbf{Q}$, the set of all binary expansions of rational numbers in $[0, 1]$, with internal dimension structure. We show that the above-mentioned theorem of Eggleston [5] (see also [1]) holds for finite-state dimension in both $\mathbf{Q}$ and $\mathbf{C}$. In particular, $\mathbf{Q}$ itself has finite-state dimension 1.

For an individual sequence $S \in \mathbf{C}$, we define the finite-state dimension of S to be $\dim_{\mathrm{FS}}(S) = \dim_{\mathrm{FS}}(\{S\})$. Each element of $\mathbf{Q}$ has finite-state dimension 0, while every sequence that is normal in the sense of Borel [2] has finite-state dimension 1. We show that every rational in $[0, 1]$ is the finite-state dimension of a sequence of very low complexity, namely, a sequence in the logspace-uniform version of the complexity class AC_0.

Our main theorem relates finite-state dimension to compressibility by information-lossless finite-state compressors, which were introduced by Huffman [9] (refining a 1948 model by Shannon [21]) and have been extensively investigated. (E.g., see [11] or [12].) Specifically, given such a compressor C and a sequence $S \in \mathbf{C}$, let $\rho_C(S)$ denote the limit infimum of all compression ratios achieved by C on prefixes of S, and let $\rho_{\mathrm{FS}}(S)$ denote the infimum of all such $\rho_C(S)$. Our main theorem says that $\dim_{\mathrm{FS}}(S)$ is precisely $\rho_{\mathrm{FS}}(S)$. Thus, with respect to finite-state computation, dimension and density of information are one and the same for individual sequences.

If $\rho_{\mathrm{LZ}}(S)$ is the limit infimum of all compression ratios achieved by (any variant of) the Lempel-Ziv compression algorithm [26] on prefixes of a sequence $S \in \mathbf{C}$, then it is well known that $\rho_{\mathrm{LZ}}(S) \leq \rho_{\mathrm{FS}}(S)$ [25]. Thus our main theorem implies that $\rho_{\mathrm{LZ}}(S) \leq \dim_{\mathrm{FS}}(S)$. However, this inequality may be proper. For example, Lathrop and Strauss [13] have shown that for every $\epsilon > 0$ there is a sequence S that is normal, whence $\dim_{\mathrm{FS}}(S) = 1$, but satisfies $\rho_{\mathrm{LZ}}(S) < \epsilon$.

We also investigate the role of multiple accounts in our model of finite-state gambling. Multiple accounts are necessary and sufficient for the associated class of gales to be closed under nonnegative, rational, linear combinations. However, we show that the restriction to single-account finite-state gamblers does not alter the finite-state dimension of any set of sequences. In our proof, the single-account gamblers have far more states than their multi-account counterparts, suggesting a possible tradeoff between accounts and states. It is an open question whether this tradeoff is real or merely a feature of our proof.

2 Preliminaries

We write $\mathbb{Z}$ for the set of all integers, $\mathbb{N}$ for the set of all nonnegative integers, $\mathbb{Z}^+$ for the set of all positive integers, and $\mathbb{Q}$ for the set of all rational numbers. We work in the set $\{0,1\}^*$ of all (finite, binary) *strings* and in the Cantor space $\mathbf{C}$ of all (infinite, binary) *sequences*. We write $|w|$ for the length of a string $w \in \{0,1\}^*$. (We also write $|X|$ for the cardinality of a finite set X, relying on context to avoid confusion.) The empty string is denoted λ. For $S \in \mathbf{C}$ and $i, j \in \mathbb{N}$, we write $S[i..j]$ for the string consisting of the i^{th} through j^{th} bits of S, with the understanding that $S[i..j] = \lambda$ if $i > j$. We write $S[i]$ for $S[i..i]$ (the i^{th} bit of S), stipulating that $S[0]$ is the leftmost bit of S. For $w \in \{0,1\}^*$ and $S \in \mathbf{C}$, we write $w \sqsubseteq S$ if w is a prefix of S, i.e., if $w = S[0..|w|-1]$.

A sequence $C \in \mathbf{C}$ is *normal* [2], and we write $S \in \mathrm{NORM}$, if for every $w \in \{0,1\}^*$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ i < n \;\middle|\; S[i..i+|w|-1] = w \right\} \right| = 2^{-|w|}.$$

That is, S is normal if every string w has asymptotic frequency $2^{-|w|}$ in S.

We use the logspace-uniform version of the bounded-depth circuit complexity class AC_0 [10]. This class consists of all sets $L \subseteq \{0,1\}^*$ for which there exist a logspace Turing machine M and a constant $d \in \mathbb{Z}^+$ such that the following conditions hold for all $n \in \mathbb{N}$.

(i) $M(0^n)$ is a standard encoding of a Boolean circuit $\nu_n : \{0,1\}^n \to \{0,1\}$ consisting of unbounded fan-in AND and OR gates and NOT gates that are applied only to inputs. All gates are allowed unbounded fan-out.

(ii) The depth of ν_n is at most d.

(iii) For all $w \in \{0,1\}^n$, $w \in L$ iff $\nu_n(w) = 1$.

Note that the logspace bound implies that there is a polynomial q such that each ν_n has at most $q(n)$ gates. Using the standard enumeration $s_0 = \lambda$, $s_1 = 0$, $s_2 = 1$, $s_3 = 00$, $\ldots$ of $\{0,1\}^*$, we say that a sequence $S \in \mathbf{C}$ is in AC_0 provided that the corresponding set $L_S = \{s_n | S[n] = 1\}$ is in AC_0.

3 Finite-State Dimension

This section develops finite-state dimension and its fundamental properties. We first review the gale characterization of classical Hausdorff dimension, which motivates our development.

Definition. [15] Let $s \in [0, \infty)$

1. An *s-gale* is a function $d : \{0,1\}^* \to [0, \infty)$ that satisfies the condition

$$d(w) = 2^{-s}\left[d(w0) + d(w1)\right] \tag{$*$}$$

   for all $w \in \{0,1\}^*$.

2. A *martingale* is a 1-gale.

Intuitively, an s-gale is a strategy for betting on the successive bits of a sequence $S \in \mathbf{C}$. For each prefix w of S, $d(w)$ is the capital (amount of money) that d has after betting on the bits w of S. When betting on the next bit b of a prefix wb of S (assuming that b is equally likely to be 0 or 1), condition $(*)$ tells us that the expected value of $d(wb)$ – the capital that d expects to have after this bet – is $(d(w0) + d(w1))/2 = 2^{s-1}d(w)$. If $s = 1$, this expected value is exactly $d(w)$ – the capital that d has before the bet – so the payoffs are "fair." If $s < 1$, this expected value is less than $d(w)$, so the payoffs are "less than fair." Similarly, of $s > 1$, the payoffs are "more than fair."

The following generalization of the Kraft inequality and its corollaries will be useful.

Lemma 3.1. [15] Let $s \in [0, \infty)$. If d is an s-gale and $B \subseteq \{0,1\}^*$ is a prefix set, then for all $w \in \{0,1\}^*$,

$$\sum_{u \in B} 2^{-s|u|} d(wu) \le d(w).$$

Corollary 3.2. [15] Let d be an s-gale, where $s \in [0, \infty)$. Then for all $w \in \{0,1\}^*, l \in \mathbb{N}$, and $0 < \alpha \in \mathbb{R}$, there are fewer than $\frac{2^l}{\alpha}$ strings $u \in \{0,1\}^l$ for which $d(wu) > \alpha 2^{(s-1)l} d(w)$.

Corollary 3.3. [15] If d is an s-gale, where $s \in [0, \infty)$, then for all $w, u \in \{0,1\}^*$,

$$d(wu) \le 2^{s|u|} d(w).$$

The objective of an s-gale is to win a lot of money.

Definition. Let d be an s-gale, where $s \in [0, \infty)$.

1. We say that d *succeeds* on a sequence $S \in \mathbf{C}$ if

$$\limsup_{n \to \infty} d(S[0..n-1]) = \infty.$$

2. The *success set* of d is

$$S^\infty[d] = \left\{ S \in \mathbf{C} \,\middle|\, d \text{ succeeds on } S \right\}.$$

Observation 3.4. Let $s, s' \in [0, \infty)$. For every s-gale d, the function $d' : \{0,1\}^* \to [0,\infty)$ defined by $d'(w) = 2^{(s'-s)|w|}d(w)$ is an s'-gale. If $s \le s'$, then $S^\infty[d] \subseteq S^\infty[d']$.

Notation. For $X \subseteq \mathbf{C}, \mathcal{G}(X)$ is the set of all $s \in [0,\infty)$ such that there is an s-gale d for which $X \subseteq S^\infty[d]$.

It was shown in [15] that the following definition is equivalent to the classical definition of Hausdorff dimension in $\mathbf{C}$.

Definition. The *Hausdorff dimension* of a set $X \subseteq \mathbf{C}$ is

$$\dim_{\mathrm{H}}(X) = \inf \mathcal{G}(X).$$

In order to define finite-state dimension, we restrict attention to s-gales that are specified by finite-state devices. These devices place bets, which we require to be rational.

Definition. A *binary bet* is a rational number r such that $0 \le r \le 1$. We let $\mathbf{B}$ denote the set of all binary bets, i.e., $\mathbf{B} = \mathbb{Q} \cap [0,1]$.

Intuitively, if a gambler whose current capital is $c \in [0,\infty)$ places a binary bet $r \in \mathbf{B}$ on a (perhaps unknown) bit $b \in \{0,1\}$, then the gambler is betting the fraction r of its capital that $b = 1$ and is betting the remainder of its capital that $b = 0$. If the payoffs are fair, then after this bet the gambler's capital will be

$$2c[(1-b)(1-r) + br] = \begin{cases} 2rc & \text{if } b = 1 \\ 2(1-r)c & \text{if } b = 0. \end{cases}$$

We now introduce the model of finite-state gambling that is used to develop finite-state dimension. Intuitively, a finite-state gambler is a finite-state device that places k separate binary bets on each of the successive bits of its input sequence. These bets correspond to k separate accounts into which the gambler's capital is divided.

Definition. If k is a positive integer, then a *k-account finite-state gambler* (*k-account FSG*) is a 5-tuple

$$G = \left(Q, \delta, \vec{\beta}, q_0, \vec{c}_0\right),$$

where

- Q is a nonempty, finite set of *states*,
- $\delta : Q \times \{0,1\} \to Q$ is the *transition function*,
- $\vec{\beta} : Q \to \mathbf{B}^k$ is the *betting function*,
- $q_0 \in Q$ is the *initial state*, and
- $\vec{c}_0 = (c_{0,1}, \ldots, c_{0,k})$, the *initial capital vector*, is a sequence of nonnegative rational numbers.

A *finite state gambler (FSG)* is a k-account FSG for some positive integer k.

Note that we require $k > 0$. No-account gamblers are not worthy of discussion.

The case $k = 1$, where there is only one account, is the model of finite-state gambling that has been considered (in essentially equivalent form) by Schnorr and Stimm [20], Feder [7], and others. In this case we do not regard $\vec{c}_0$ as a vector, but simply as a nonnegative rational number c_0, which is the initial capital of G.

If $k > 1$, it is convenient to regard the betting function $\vec{\beta} : Q \to \mathbf{B}^k$ as a vector $\vec{\beta} = (\beta_1, \ldots, \beta_k)$ of component betting functions $\beta_i : Q \to \mathbf{B}$, so that

$$\vec{\beta}(q) = (\beta_1(q), \ldots, \beta_k(q))$$

for each $q \in Q$. If $k = 1$, we write β for $\vec{\beta}$.

As usual with finite-state transition structures, we extend δ to the transition function

$$\delta^* : Q \times \{0,1\}^* \to Q$$

defined by the recursion

$$\begin{aligned} &\delta^*(q, \lambda) = q, \\ &\delta^*(q, wb) = \delta(\delta^*(q, w), b) \end{aligned}$$

for all $q \in Q$, $w \in \{0,1\}^*$, and $b \in \{0,1\}$; we write δ for δ^*; and we use the abbreviation $\delta(w) = \delta(q_0, w)$.

Intuitively, if a k-account FSG $G = (Q, \delta, \vec{\beta}, q_0, \vec{c}_0)$ is in state $q \in Q$ and its current capital vector is $\vec{c} = (c_1, \ldots, c_k) \in (\mathbb{Q} \cap [0, \infty))^k$, then for each of its accounts $i \in \{1, \ldots, k\}$, it places the binary bet $\beta_i(q) \in \mathbf{B}$. If the payoffs are fair, then after this bet G will be in state $\delta(q, b)$ and its i^{th} account will have capital

$$2c_i[(1-b)(1-\beta_i(q)) + b\beta_i(q)] = \begin{cases} 2\beta_i(q)c_i & \text{if } b = 1 \\ 2(1-\beta_i(q))c_i & \text{if } b = 0. \end{cases}$$

This suggests the following definition.

Definition. Let $G = (Q, \delta, \vec{\beta}, q_0, \vec{c}_0)$ be a k-account finite-state gambler.

1. For each $1 \leq i \leq k$, the i^{th} *martingale* of G is the function

$$d_{G,i} : \{0,1\}^* \to [0, \infty)$$

defined by the recursion

$$\begin{aligned} &d_{G,i}(\lambda) = c_{0,i}, \\ &d_{G,i}(wb) = 2d_{G,i}(w)[(1-b)(1-\beta_i(\delta(w))) + b\beta_i(\delta(w))] \end{aligned}$$

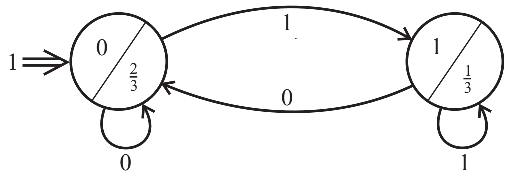
for all $w \in \{0,1\}^*$ and $b \in \{0,1\}$.

2. The *total martingale* (or simply the *martingale*) of G is the function

$$d_G = \sum_{i=1}^{k} d_{G,i}.$$

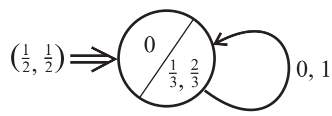
It is clear by inspection that $d_{G,1}, \ldots, d_{G,k}$, and d_G are all martingales for every k-account FSG G.

Example 3.5. The diagram



denotes the 1-account FSG $G = (Q, \delta, \beta, 0, 1)$, where $Q = \{0, 1\}$, $\delta(0,0) = \delta(1,0) = 0$, $\delta(0,1) = \delta(1,1) = 1$, $\beta(0) = \frac{2}{3}$, and $\beta(1) = \frac{1}{3}$. It is easy to verify that $d_G(\lambda) = 1$, $d_G(1) = \frac{4}{3}$, $d_G(11) = \frac{8}{9}$, and $d_G(110) = \frac{32}{27}$.

Example 3.6. The diagram



denotes the 2-account FSG $G = (Q, \delta, \vec{\beta}, 0, (\frac{1}{2}, \frac{1}{2}))$, where $Q = \{0\}$, $\delta(0,0) = \delta(0,1) = 0$, $\beta_1(0) = \frac{1}{3}$, and $\beta_2(0) = \frac{2}{3}$. Although the two components of $\vec{\beta}$ make "opposite" bets, these do not "cancel" each other. For example, note that $d_G(00) = d_G(11) = \frac{10}{9} > 1 = d_G(0) = d_G(1)$. This is because the separation of accounts causes the effect of a component bet $\beta_i(q)$ to be proportional to the current capital in the i^{th} account.

Many of the k-account FSGs that we consider have the form $G = (Q, \delta, \vec{\beta}, q_0, \vec{c})$, where $\vec{c} = (\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k})$. In this case we omit $\vec{c}$ from the notation and diagram, referring simply to the k-account FSG $G = (Q, \delta, \vec{\beta}, q_0)$. Note that such a gambler always has initial capital $d_G(\lambda) = 1$.

Lemma 3.7. If G_1 is a k_1-account FSG with n_1 states and G_2 is a k_2-account FSG with n_2 states, then there is a $(k_1 + k_2)$-account FSG G with $n_1 n_2$ states such that $d_G = d_{G_1} + d_{G_2}$.

Proof. We use a product construction. Assume the hypothesis, with

$$G_j = \left(Q_j, \delta_j, \vec{\beta}_j, q_j, \vec{c}_j\right)$$

for $j \in \{1, 2\}$, where $\vec{\beta}_j = (\beta_{j,1}, \ldots, \beta_{j,k_j})$, $\vec{c}_j = (c_{j,1}, \ldots, c_{j,k_j})$, and we assume without loss of generality that $Q_1 \cap Q_2 = \emptyset$. Define the $(k_1 + k_2)$-account FSG

$$G = (Q, \delta, \vec{\beta}, q_0, \vec{c}_0),$$

whose components are defined as follows.

(i) $Q = Q_1 \times Q_2$.

(ii) For $q' \in Q_1$, $q'' \in Q_2$, and $b \in \{0,1\}$,

$$\delta((q', q''), b) = (\delta_1(q', b), \delta_2(q'', b)).$$

(iii) For $1 \le i \le k_1 + k_2$, $q' \in Q_1$ and $q'' \in Q_2$,

$$\beta_i((q', q'')) = \begin{cases} \beta_{1,i}(q') & \text{if } i \le k_1 \\ \beta_{2,i-k_1}(q'') & \text{if } i > k_1. \end{cases}$$

(iv) $q_0 = (q_1, q_2)$.

(v) For $1 \le i \le k_1 + k_2$,

$$c_{0,i} = \begin{cases} c_{1,i} & \text{if } i \le k_1 \\ c_{2,i-k_1} & \text{if } i > k_1. \end{cases}$$

A routine induction shows that for all $1 \le i \le k_1 + k_2$ and all $w \in \{0,1\}^*$,

$$d_{G,i}(w) = \begin{cases} d_{G_1,i}(w) & \text{if } i \le k_1 \\ d_{G_2,i-k_1}(w) & \text{if } i > k_1. \end{cases}$$

It follows that for all $w \in \{0,1\}^*$,

$$\begin{aligned} d_G(w) &= \sum_{i=1}^{k_1+k_2} d_{G,i}(w) = \sum_{i=1}^{k_1} d_{G_1,i}(w) + \sum_{i=1}^{k_2} d_{G_2,i}(w) \\ &= d_{G_1}(w) + d_{G_2}(w), \end{aligned}$$

whence $d_G = d_{G_1} + d_{G_2}$. □

By Observation 3.4, an FSG G defines not only the martingale d_G, but also an s-gale for every s.

Definition. For $s \in [0, \infty)$, the *s-gale of* an FSG G is the function

$$d_G^{(s)} : \{0,1\}^* \to [0, \infty)$$

defined by

$$d_G^{(s)}(w) = 2^{(s-1)|w|} d_G(w)$$

for all $w \in \{0,1\}^*$.

In particular, note that $d_G^{(1)} = d_G$.

Definition.

1. For $s \in [0, \infty)$, a *finite-state s-gale* is an s-gale d for which there exists an FSG G such that $d_G^{(s)} = d$.
2. A *finite-state martingale* is a finite-state 1-gale.

We now use finite-state gales to define finite-state dimension.

Notation. For $X \subseteq \mathbf{C}$, $\mathcal{G}_{\mathrm{FS}}(X)$ is the set of all $s \in [0,\infty)$ such that there is a finite-state s-gale d for which $X \subseteq S^\infty[d]$.

Observation 3.8. Let $X, Y \subseteq \mathbf{C}$ and $s, s' \in [0,\infty)$.

1. If $s' \geq s \in \mathcal{G}_{\mathrm{FS}}(X)$, then $s' \in \mathcal{G}_{\mathrm{FS}}(X)$.
2. $(1,\infty) \subseteq \mathcal{G}_{\mathrm{FS}}(X) \subseteq (0,\infty)$.
3. $\mathcal{G}_{\mathrm{FS}}(X) \subseteq \mathcal{G}(X)$.
4. If $X \subseteq Y$, then $\mathcal{G}_{\mathrm{FS}}(Y) \subseteq \mathcal{G}_{\mathrm{FS}}(X)$.

Proof. Part 1 follows from Observation 3.4. Parts 3 and 4 are obvious, as is the second inclusion in part 2. For the first inclusion in part 2, let $s \in (1,\infty)$. Then the s-gale $d(w) = 2^{(s-1)|w|}$ testifies that $s \in \mathcal{G}_{\mathrm{FS}}(\mathbf{C}) \subseteq \mathcal{G}_{\mathrm{FS}}(X)$. □

Recalling that the Hausdorff dimension of a set $X \subseteq \mathbf{C}$ is $\dim_{\mathrm{H}}(X) = \inf \mathcal{G}(X)$, it is natural to define the finite-state dimension as follows.

Definition. The *finite-state dimension* of a set $X \subseteq \mathbf{C}$ is

$$\dim_{\mathrm{FS}}(X) = \inf \mathcal{G}_{\mathrm{FS}}(X).$$

Parts 1 and 2 of Observation 3.8 tell us that $\mathcal{G}_{\mathrm{FS}}(X)$ is always of the form (s^*,∞), where $0 \leq s^* \leq 1$, or of the form $[s^*,\infty)$, where $0 < s^* \leq 1$. In either case, the number s^* is the finite-state dimension of X.

Observation 3.8 has the following immediate consequences.

Observation 3.9. Let $X, Y \subseteq \mathbf{C}$.

1. $0 \leq \dim_{\mathrm{H}}(X) \leq \dim_{\mathrm{FS}}(X) \leq 1$.
2. If $X \subseteq Y$, then $\dim_{\mathrm{FS}}(X) \leq \dim_{\mathrm{FS}}(Y)$.

An important property of Hausdorff dimension is its *stability*, which is the term used by Falconer [6] for the fact that $\dim_{\mathrm{H}}(X \cup Y)$ is always the maximum of $\dim_{\mathrm{H}}(X)$ and $\dim_{\mathrm{H}}(Y)$. We now show that finite-state dimension has this property.

Theorem 3.10. For all $X, Y \subseteq \mathbf{C}$,

$$\dim_{\mathrm{FS}}(X \cup Y) = \max\left\{\dim_{\mathrm{FS}}(X), \dim_{\mathrm{FS}}(Y)\right\}.$$

Thus for all $X_1, \ldots, X_n \subseteq \mathbf{C}$,

$$\dim_{\mathrm{FS}}\left(\bigcup_{i=1}^{n} X_i\right) = \max_{1 \leq i \leq n} \dim_{\mathrm{FS}}(X_i).$$

Proof. Let $X, Y \in \mathbf{C}$. By Observation 3.9 it suffices to show that

$$\dim_{\rm FS}(X \cup Y) \leq \max\{\dim_{\rm FS}(X), \dim_{\rm FS}(Y)\}.$$

For this, let $s > \max\{\dim_{\rm FS}(X), \dim_{\rm FS}(Y)\}$; it suffices to show that $\dim_{\rm FS}(X \cup Y) \leq s$.

By our choice of s, there exist finite-state gamblers G_X and G_Y such that

$$X \subseteq S^\infty\left[d_{G_X}^{(s)}\right],\ Y \subseteq S^\infty\left[d_{G_Y}^{(s)}\right].$$

By Lemma 3.7, there is a finite-state gambler G such that $d_G = d_{G_X} + d_{G_Y}$. It follows immediately that $d_G^{(s)} = d_{G_X}^{(s)} + d_{G_Y}^{(s)}$, whence

$$X \cup Y \subseteq S^\infty\left[d_{G_X}^{(s)}\right] \cup S^\infty\left[d_{G_Y}^{(s)}\right] = S^\infty\left[d_G^{(s)}\right].$$

This implies that $\dim_{\rm FS}(X \cup Y) \leq s$. □

We conclude this section with an easy technical lemma.

Definition. A 1-account FSG $G = (Q, \delta, \beta, q_0)$ is *nonvanishing* if $0 < \beta(q) < 1$ for all $q \in Q$.

Lemma 3.11. For every 1-account FSG G and every $\epsilon > 0$, there is a nonvanishing 1-account FSG G' such that for all $w \in \{0,1\}^*$, $d_{G'}(w) \geq 2^{-\epsilon|w|} d_G(w)$.

Proof. Let $G = (Q, \delta, \beta, q_0)$ be a 1-account FSG, and let $\epsilon > 0$. For each $q \in Q$,

$$1 - 2^{-\epsilon}(1 - \beta(q)) - 2^{-\epsilon}\beta(q) = 1 - 2^{-\epsilon} > 0,$$

so we can fix a rational $\beta'(q)$ such that

$$2^{-\epsilon}\beta(q) < \beta'(q) < 1 - 2^{-\epsilon}(1 - \beta(q)).$$

Then $0 < \beta'(q) < 1$ for each $q \in Q$, so the 1-account FSG

$$G' = (Q, \delta, \beta', q_0)$$

is nonvanishing. Also, for all $q \in Q$,

$$\beta'(q) \geq 2^{-\epsilon}\beta(q)$$

and

$$1 - \beta'(q) \geq 2^{-\epsilon}(1 - \beta(q)),$$

so for all $w \in \{0,1\}^*$, $d_{G'}(w) \geq 2^{-\epsilon|w|} d_G(w)$. □

4 Accounts versus States

We have allowed our finite state gamblers to have multiple accounts. When discussing the finite-state dimensions of individual sequences, as in sections 6 and 7, the multiplicity of accounts obviously contributes nothing. On the other hand, our proof of Theorem 3.10 makes explicit use of the multi-account feature. In this section we discuss the necessity and desirability of multiple accounts.

If G is a 1-account FSG and $s \in [0, \infty)$, then we call $d_G^{(s)}$ a 1-*account finite state s-gale*, and we call d_G a 1-*account finite-state martingale.*

We begin our discussion by noting that finite-state gales (with multiple accounts) are closed under nonnegative, rational, linear combinations.

Observation 4.1. Let $s \in [0, \infty)$. If $d_1, \ldots, d_k$ are finite-state s-gales and $a_1, \ldots, a_k$ are nonnegative rationals, then $a_1 d_1 + \cdots + a_k d_k$ is a finite-state s-gale.

Proof. The case $s = 1$ follows easily from Lemma 3.7. The general case then follows by Observation 3.4. □

We next show that 1-account finite-state gales do not enjoy this closure property. Our demonstration uses the following finiteness criterion.

Observation 4.2. For every 1-account finite-state martingale d, the set

$$\left\{ \frac{d(w1)}{d(w)} \,\middle|\, d(w) > 0 \right\}$$

is finite.

Proof. Let $G = (Q, \delta, \beta, q_0, c)$ be a 1-account FSG. Then the definition of d_G tells us that for all $w \in \{0,1\}^*$,

$$d_G(w) > 0 \Rightarrow \beta(\delta(w)) = \frac{d_G(w1)}{2d_G(w)}.$$

Since the domain of β is the finite set Q, the observation follows immediately. □

Observation 4.3. If G is the 2-account FSG of Example 3.6, then d_G is not a 1-account finite-state martingale.

Proof. For all $w \in \{0,1\}^*$, we have

$$d_G(w) = \frac{1}{2}\left(\frac{2}{3}\right)^{|w|}\left[2^{\#(0,w)} + 2^{\#(1,w)}\right],$$

where $\#(b, w)$ is the number of times the bit b appears in the string w. In particular, for all $n \in \mathbb{N}$, we have

$$d_G(0^n) = \frac{1}{2}\left(\frac{2}{3}\right)^n (2^n + 1)$$

and

$$d_G(0^n 1) = \frac{1}{2}\left(\frac{2}{3}\right)^{n+1} (2^n + 2),$$

so

$$\frac{d_G(0^n 1)}{d_G(0^n)} = \frac{2}{3}\frac{2^n + 2}{2^n + 1}.$$

Since the set

$$\left\{ \frac{2}{3} \frac{2^n+2}{2^n+1} \,\middle|\, n \in \mathbb{N} \right\}$$

is infinite, it follows by Observation 4.2 that d_G is not a 1-account finite-state martingale. $\square$

Observation 4.3 tells us that multi-account FSGs cannot always be exactly simulated by 1-account FSGs. In contrast with Observation 4.1, it also gives us the following.

Observation 4.4. For all $s \in [0, \infty)$, there exist 1-account finite-state s-gales d_1 and d_2 such that $d_1 + d_2$ is not a 1-account finite-state s-gale.

Proof. Let G be the 2-account FSG of Example 3.6. For $i \in \{1, 2\}$, let $d_i = d_{G,i}^{(s)}$. Then d_1 and d_2 are 1-account finite-state s-gales, but $d_1 + d_2 = d_G^{(s)}$ is not a 1-account finite-state s-gale by Observation 4.3. $\square$

We have designed our FSG model so that the associated gales are closed under nonnegative, rational, linear combinations because this is such a useful closure property. By Observation 4.4, multiple accounts are required for this closure property to hold.

Notwithstanding the usefulness of the above closure property, the question remains whether multiple accounts are strictly necessary for a theory of finite-state dimension. That is, if we define $\mathcal{G}_{1-\text{acct}-\text{FS}}(X)$ to be the set of all $s \in [0, \infty)$ such that there is a 1-account finite-state s-gale d for which $X \subseteq S^\infty[d]$, and we define

$$\dim_{1-\text{acct}-\text{FS}}(X) = \inf \mathcal{G}_{1-\text{acct}-\text{FS}}(X),$$

is there any set $X \subseteq \mathbf{C}$ for which

$$\dim_{\text{FS}}(X) < \dim_{1-\text{acct}-\text{FS}}(X)?$$

The next result shows that multiple accounts are not strictly necessary if we are willing to accept a large blowup in the number of states.

Theorem 4.5. For each n-state, k-account FSG G and each $\epsilon \in (0, 1)$, if we let $m = \left\lceil \frac{\log k}{\epsilon} \right\rceil$ and $N = n(2^m - 1)$, then there is an N-state, 1-account FSG G' such that for all $s \in [0, 1]$,

$$S^\infty[d_G^{(s)}] \subseteq S^\infty[d_{G'}^{(s+\epsilon)}].$$

Proof. Let $G = (Q, \delta, \vec{\beta}, q_0, \vec{c}_0)$ be an n-state, k-account FSG, and let $\epsilon \in (0, 1)$. Let $c = \sum_{j=1}^k c_{0,j}$. If $k = 1$ or $c = 0$ the theorem holds trivially, so assume that $k \geq 2$ and $c > 0$. To simplify notation, let $d_j = d_{G,j}$ for each $1 \leq j \leq k$, and let $d = d_G$.

For each $q \in Q$, let $G_q = (Q, \delta, \vec{\beta}, q)$. That is, let G_q be the FSG obtained from G by changing the initial state to q and the initial capital vector to $(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k})$. For each $q \in Q$ and $1 \leq j \leq k$, let $d_{q,j} = d_{G_q,j}$, and let $d_q = d_{G_q}$. (We consistently use q as a state and j as an account index, so no confusion will arise between d_q and d_j.) Note that for all $w, u \in \{0, 1\}^*$ and $1 \leq j \leq k$, if $q = \delta(w)$, then

$$d_{q,j}(u) = \frac{1}{k} \frac{d_j(wu)}{d_j(w)},$$

where we stipulate that this fraction is 0 if $d_j(w) = 0$. This implies that for all $w, u \in \{0,1\}^*$, if $q = \delta(w)$, then

$$d_q(u) = \frac{1}{k} \sum_{j=1}^{k} \frac{d_j(wu)}{d_j(w)} \tag{4.1}$$

with the same stipulation.

Let $m = \left\lceil \frac{\log k}{\epsilon} \right\rceil$, and define the 1-account FSG $G' = (Q', \delta', \beta', q_0')$ whose components are specified as follows.

(i) $Q' = Q \times \{0,1\}^{<m}$.

(ii) For $q \in Q$, $u \in \{0,1\}^{<m}$, and $b \in \{0,1\}$,

$$\delta'((q,u),b) = \begin{cases} (q, ub) & \text{if } |u| < m-1 \\ (\delta(q,ub), \lambda) & \text{if } |u| = m-1. \end{cases}$$

(iii) For $q \in Q$ and $u \in \{0,1\}^{<m}$,

$$\beta'(q,u) = \begin{cases} \frac{d_q(u1)}{2d_q(u)} & \text{if } d_q(u) > 0 \\ \frac{1}{2} & \text{if } d_q(u) = 0. \end{cases}$$

(iv) $q_0' = (q_0, \lambda)$.

Intuitively, the states of G' are arranged in n trees, one for each state of G. The tree for q simulates the martingale d_q (which is not necessarily the martingale of any 1-account FSG) for m steps before passing control to the root of another tree. It is clear that G' is an N-state, 1-account FSG, where $N = n(2^m - 1)$. Let $d' = d_{G'}$.

We now show that if $|w|$ is a multiple of m, $|u| < m$, and $q = \delta(w)$, then

$$d'(wu) = d'(w) d_q(u). \tag{4.2}$$

We use induction on the string u. It holds trivially if $u = \lambda$. Assume that (4.2) holds for u, where $|u| < m-1$, and let $b \in \{0,1\}$. Then the definition of d' and the induction hypothesis tell us that

$$\begin{aligned} d'(wub) &= 2d'(wu)\left[(1-b)(1-\beta'(q,u)) + b\beta'(q,u)\right] \\ &= 2d'(w)d_q(u)\left[(1-b)(1-\beta'(q,u)) + b\beta'(q,u)\right]. \end{aligned}$$

If $d_q(u) = 0$, this immediately tells us that $d'(wub) = 0 = d'(w)d_q(ub)$, whence (4.2) holds for ub. If $d_q(u) > 0$, it tells us that

$$\begin{aligned} d'(wub) &= 2d'(w)d_q(u)\left[(1-b)\left(1 - \frac{d_q(u1)}{2d_q(u)}\right) + b\,\frac{d_q(u1)}{2d_q(u)}\right] \\ &= d'(w)\left[(1-b)(2d_q(u) - d_q(u1)) + bd_q(u1)\right] \\ &= d'(w)d_q(ub), \end{aligned}$$

whence (4.2) again holds for ub. This completes the verification of (4.2).

For all $w \in \{0,1\}^*$, if we write $w = w_0 w_1 \cdots w_l$, where $|w_i| = m$ for all $0 \le i < l$ and $|w_l| < m$, then (4.2) tells us inductively that

$$d'(w) = \prod_{i=0}^{l} d_{q_i}(w_i), \tag{4.3}$$

where $q_i = \delta(w_0 \cdots w_{i-1})$ for all $0 \le i \le l$. If $w_l = \lambda$ here, then (4.3) and (4.1) tell us that

$$\begin{aligned} d'(w) &= \prod_{i=0}^{l-1} d_{q_i}(w_i) \\ &= \prod_{i=0}^{l-1} \frac{1}{k} \sum_{j=1}^{k} \frac{d_j(w_0 \cdots w_i)}{d_j(w_0 \cdots w_{i-1})} \\ &\ge k^{-l} \sum_{j=1}^{k} \prod_{i=0}^{l-1} \frac{d_j(w_0 \cdots w_i)}{d_j(w_0 \cdots w_{i-1})} \\ &= k^{-l} \sum_{j=1}^{k} \frac{d_j(w)}{c_j} \\ &\ge \frac{1}{ck^l} d(w). \end{aligned}$$

This shows that for all $w \in \{0,1\}^*$, if $|w| = lm$, then

$$d'(w) \ge \frac{1}{ck^l} d(w),$$

whence our choice of m tells us that

$$\begin{aligned} d'^{(1+\epsilon)}(w) &= 2^{\epsilon l m} d'(w) \\ &\ge 2^{\epsilon l m} \frac{1}{ck^l} d(w) \\ &\ge \frac{1}{c} d(w). \end{aligned}$$

We have now shown that for all $w \in \{0,1\}^*$, if $|w|$ is a multiple of m, then

$$d'^{(1+\epsilon)}(w) \ge \frac{1}{c} d(w). \tag{4.4}$$

Now let $s \in [0,1]$. To complete the proof it suffices to show that $S^\infty[d^{(s)}] \subseteq S^\infty[d'^{(s+\epsilon)}]$. For this, let $A \in S^\infty[d^{(s)}]$. To see that $A \in S^\infty[d'^{(s+\epsilon)}]$, let r be an arbitrarily large positive integer. Since $A \in S^\infty[d^{(s)}]$, there exists $x \sqsubseteq A$ such that $d^{(s)}(x) \ge 2^m cr$. Let w be the longest prefix of x whose length is a multiple of m, and let $i = |x| - |w|$. Then $w \sqsubseteq A$, and (4.4) tells us that

$$\begin{aligned} d'^{(s+\epsilon)}(w) &= 2^{(s-1)|w|} d'^{(1+\epsilon)}(w) \\ &\geq \frac{1}{c} 2^{(s-1)|w|} d(w) \\ &\geq \frac{1}{c} 2^{(s-1)|w|-i} d(x) \\ &= \frac{1}{c} 2^{(s-1)|w|-i} 2^{(1-s)|x|} d^{(s)}(x) \\ &= \frac{1}{c} 2^{-is} d^{(s)}(x) \\ &\geq 2^{m-is} r \\ &\geq r. \end{aligned}$$

Since r is arbitrary here, this shows that $A \in S^{\infty}[d'^{(s+\epsilon)}]$. □

Corollary 4.6. For all $X \subseteq \mathbf{C}$,

$$\dim_{1-\mathrm{acct}-\mathrm{FS}}(X) = \dim_{\mathrm{FS}}(X).$$

Proof. Let $X \subseteq \mathbf{C}$ and $\epsilon \in (0,1)$. It is clear that $\dim_{1-\mathrm{acct}-\mathrm{FS}}(X) \geq \dim_{\mathrm{FS}}(X)$, so it suffices to show that $\dim_{1-\mathrm{acct}-\mathrm{FS}}(X) \leq \dim_{\mathrm{FS}}(X) + 2\epsilon$.

Let $s = \dim_{\mathrm{FS}}(X)+\epsilon$. Then there is an FSG G such that $X \subseteq S^{\infty}[d_G^{(s)}]$. By Theorem 4.5, then, there is a 1-account FSG G' such that $X \subseteq S^{\infty}[d_G^{(s)}] \subseteq S^{\infty}[d_{G'}^{(s+\epsilon)}]$. This implies that $\dim_{1-\mathrm{acct}-\mathrm{FS}}(X) \leq s + \epsilon = \dim_{\mathrm{FS}}(X) + 2\epsilon$. □

We have now shown that the finite-state dimension of a set is not affected by whether or not multiple accounts are allowed in the definition. However, if we use Theorem 4.5 to replace a k-account FSG by a 1-account FSG, then we are going from n states to roughly $n \cdot k^{\frac{1}{\epsilon}}$ states. If we are trying to approximate the dimension to within r bits of accuracy, then ϵ will be roughly 2^{-r}, so our 1-account FSG will have roughly $n \cdot k^{2^r}$ states.

At the time of this writing, we do not know whether such a large blowup in the number of states is necessary. If so, then the multi-account FSG model is quantitatively more powerful than the single-account FSG model, regardless of the qualitative identity in Corollary 4.6. If not, then we might be able to dispense with the multi-account feature. In any case, the following question appears to be significant.

Question 4.7. Given an n-state, k-account FSG G, $s \in [0,1]$, and $\epsilon \in (0,1)$, how many states are required for a 1-account FSG G' satisfying $S^{\infty}[d_G^{(s)}] \subseteq S^{\infty}[d_{G'}^{(s+\epsilon)}]$?

5 Rational Sequences

This section shows how to use finite-state dimension to define a natural notion of dimension in the set of all binary expansions of rational numbers.

Definition. Let $n \in \mathbb{Z}^+$ and $S \in \mathbf{C}$.

1. S is *eventually periodic* with *period* n, and we write $S \in \mathbf{Q}_n$, if there exist $x \in \{0,1\}^*$ and $y \in \{0,1\}^n$ such that for all $k \in \mathbb{N}$, $xy^k \sqsubseteq S$. In this case we write $S = xy^\infty$.
2. S is *eventually periodic*, and we write $S \in \mathbf{Q}$, if there exists $n \in \mathbb{Z}^+$ such that $S \in \mathbf{Q}_n$.

Note that for all $m, n \in \mathbb{Z}^+$, $\mathbf{Q}_n \subseteq \mathbf{Q}_{mn}$. Note also that $\mathbf{Q} = \cup_{n=1}^{\infty} \mathbf{Q}_n$ is precisely the set of all binary expansions of elements of $\mathbb{Q} \cap [0,1]$. For this reason, the elements of $\mathbf{Q}$ are also called *rational sequences*.

We now define dimension in the set of rational sequences.

Definition. For $X \subseteq \mathbf{C}$, the *dimension of X in $\mathbf{Q}$* is

$$\dim(X|\mathbf{Q}) = \dim_{\mathrm{FS}}(X \cap \mathbf{Q}).$$

We shall see that this definition endows $\mathbf{Q}$ with internal dimension structure. The following properties of dimension in $\mathbf{Q}$ are clear from the definition and Theorem 3.10.

Observation 5.1. Let $X, Y \subseteq \mathbf{C}$.

1. $0 \le \dim(X|\mathbf{Q}) \le \dim_{\mathrm{FS}}(X) \le 1$.
2. If $X \cap \mathbf{Q} \subseteq Y \cap \mathbf{Q}$, then $\dim(X|\mathbf{Q}) \le \dim(Y|\mathbf{Q})$.
3. $\dim(X \cup Y|\mathbf{Q}) = \max\{\dim(X|\mathbf{Q}), \dim(Y|\mathbf{Q})\}$.

We next show that, for fixed n, the set of all sequences with period at most n has dimension 0 in $\mathbf{Q}$.

Lemma 5.2. For all $n \in \mathbb{Z}^+$,

$$\dim(\mathbf{Q}_n|\mathbf{Q}) = \dim_{\mathrm{FS}}(\mathbf{Q}_n) = 0.$$

Proof. Let $n \in \mathbb{Z}^+$. For each $r \in \mathbb{N}$ and $y \in \{0,1\}^n$, let $X_{r,y}$ be the set of all $S \in \mathbf{C}$ such that there exists $x \in \{0,1\}^*$ such that $|x| \equiv r \mod n$ and for all $k \in \mathbb{N}$, $xy^k \sqsubseteq S$. Then

$$\mathbf{Q}_n = \bigcup_{r=0}^{n-1} \bigcup_{y \in \{0,1\}^n} X_{r,y},$$

and this is a finite union, so it suffices by Theorem 3.10 to show that each $\dim_{\mathrm{FS}}(X_{r,y}) = 0$. For this, fix $0 \le r < n$ and $y \in \{0,1\}^n$, and let s be such that $0 < s < 1$ and $2^{\frac{s}{2}} \in \mathbb{Q}$. It suffices to show that $\dim_{\mathrm{FS}}(X_{r,y}) \le s$.

Define the 1-account, $(r+n)$-state gambler

$$G = (Q, \delta, \beta, -r),$$

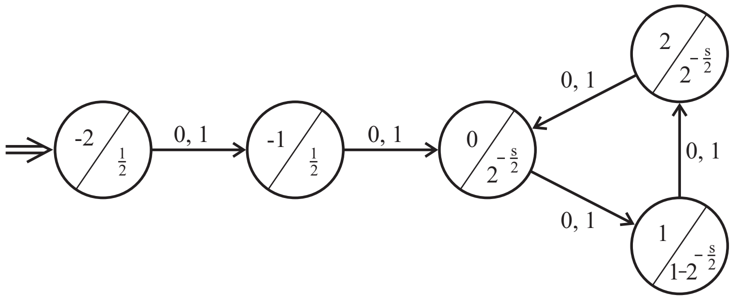
where

$$Q = \left\{ q \in \mathbb{Z} \,\middle|\, -r \leq q < n \right\},$$

$$\delta(q, b) = \begin{cases} q+1 & \text{if } q < n-1 \\ 0 & \text{if } q = n-1, \end{cases}$$

$$\beta(q) = \begin{cases} \frac{1}{2} & \text{if } q < 0 \\ 1 - 2^{-\frac{s}{2}} & \text{if } q \geq 0 \text{ and } y[q] = 0 \\ 2^{-\frac{s}{2}} & \text{if } q \geq 0 \text{ and } y[q] = 1. \end{cases}$$

For example, if $n = 3$, $r = 2$, and $y = 101$, then G has the structure



Suppose that $S \in X_{r,y}$. Then there exist $x \in \{0,1\}^*$ and $j \in \mathbb{N}$ such that $|x| = jn + r$ and for all $k \in \mathbb{N}$, $xy^k \sqsubseteq S$. Let $u, v \in \{0,1\}^*$ be such that $x = uv$ and $|u| = r$, and let $\epsilon = \frac{s}{2}n$, $a = (s-1)r + jn(s + \log(1 - 2^{-\frac{s}{2}}))$. Since G does not bet on the first r bits of S, we have

$$d_G^{(s)}(u) = 2^{(s-1)r}.$$

Since $2^{-\frac{s}{2}} > \frac{1}{2} > 1 - 2^{-\frac{s}{2}}$ and $|v| = jn$, we then have

$$\begin{aligned} d_G^{(s)}(x) &\geq d_G^{(s)}(u) \cdot 2^{s|v|}(1 - 2^{-\frac{s}{2}})^{|v|} \\ &= 2^{(s-1)r + sjn}(1 - 2^{-\frac{s}{2}})^{jn} \\ &= 2^a. \end{aligned}$$

It follows that for all $k \in \mathbb{N}$,

$$\begin{aligned} d_G^{(s)}(xy^k) &= d_G^{(s)}(x) \cdot 2^{skn}(2^{-\frac{s}{2}})^{kn} \\ &\geq 2^{a + \epsilon k}. \end{aligned}$$

Since $\epsilon > 0$, k is arbitrary, and $xy^k \sqsubseteq S$, it follows that $d_G^{(s)}$ succeeds on S. This shows that $X_{r,y} \subseteq S^\infty\left[d_G^{(s)}\right]$, whence $\dim_{\mathrm{FS}}(X_{r,y}) \leq s$. □

Notation. For $n \in \mathbb{Z}^+$, let $\mathbf{Q}_{\leq n} = \bigcup_{k=1}^{n} \mathbf{Q}_k$.

Corollary 5.3. For all $n \in \mathbb{Z}^+$,

$$\dim(\mathbf{Q}_{\leq n}|\mathbf{Q}) = \dim_{\mathrm{FS}}(\mathbf{Q}_{\leq n}) = 0.$$

Proof. This follows immediately from Lemma 5.2 and Theorem 3.10 □

Corollary 5.4. If $X \subseteq \mathbf{C}$ and $X \cap \mathbf{Q}$ is finite, then $\dim(X|\mathbf{Q}) = 0$.

In contrast with Lemma 5.2, and with the fact that every countable set of sequences has classical Hausdorff dimension 0, a set of sequences may have positive dimension in $\mathbf{Q}$. In fact, we show that the theorem of Eggleston [5] mentioned in the first paragraph of the present paper holds in $\mathbf{Q}$.

Define the *frequency* of a nonempty string $w \in \{0,1\}^*$ to be the ratio

$$\mathrm{freq}(w) = \frac{\#(1,w)}{|w|},$$

where $\#(b,w)$ denotes the number of occurrences of the bit b in w. For each $S \in \mathbf{C}$ and $n \in \mathbb{Z}^+$, let

$$\mathrm{freq}_S(n) = \mathrm{freq}(S[0..n-1]).$$

For each $\alpha \in [0,1]$, define the sets

$$\mathrm{FREQ}(\alpha) = \left\{ S \in \mathbf{C} \;\middle|\; \lim_{n\to\infty} \mathrm{freq}_S(n) = \alpha \right\},$$

$$\mathrm{FREQ}(\leq \alpha) = \left\{ S \in \mathbf{C} \;\middle|\; \limsup_{n\to\infty} \mathrm{freq}_S(n) \leq \alpha \right\}.$$

Note that if $S = xy^\infty \in \mathbf{Q}$, then $S \in \mathrm{FREQ}(\alpha)$, where $\alpha = \frac{\#(1,y)}{|y|} \in \mathbb{Q}$.

The following theorem uses the binary entropy function

$$\mathcal{H} : [0,1] \to [0,1]$$

$$\mathcal{H}(x) = x \log \frac{1}{x} + (1-x) \log \frac{1}{1-x}$$

(The values of $\mathcal{H}(0)$ and $\mathcal{H}(1)$ are both 0, so that $\mathcal{H}$ is continuous on $[0,1]$.) The proof of the theorem uses the weighted binary entropy function

$$h : (0,1)^2 \to [0,\infty)$$

$$h(x,y) = x \log \frac{1}{y} + (1-x) \log \frac{1}{1-y}.$$

This function is continuous on $(0,1)^2$. For fixed $x \in (0,1)$, $h(x,y)$ takes its minimum value $\mathcal{H}(x)$ at $y = x$ and strictly increases as y moves away from x.

Theorem 5.5. For all $\alpha \in \mathbb{Q} \cap [0,1]$,

$$\dim(\mathrm{FREQ}(\alpha) \mid \mathbf{Q}) = \dim_{\mathrm{FS}}(\mathrm{FREQ}(\alpha)) = \mathcal{H}(\alpha).$$

Proof. By Observation 5.1 it suffices to show that

$$\dim_{\mathrm{FS}}(\mathrm{FREQ}(\alpha)) \le \mathcal{H}(\alpha) \le \dim(\mathrm{FREQ}(\alpha) \mid \mathbf{Q}). \tag{5.1}$$

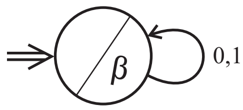
We prove that the first inequality holds for $\alpha \in \mathbb{Q} \cap [0, \frac{1}{2}]$. The proof that it holds for $\alpha \in \mathbb{Q} \cap [\frac{1}{2}, 1]$ is analogous. Let $\alpha \in \mathbb{Q} \cap [0, \frac{1}{2}]$, and let $s > \mathcal{H}(\alpha)$. To prove the first inequality in (5.1), it suffices to show that

$$\dim_{\mathrm{FS}}(\mathrm{FREQ}(\alpha)) \le s. \tag{5.2}$$

Let $\epsilon = \frac{s - \mathcal{H}(\alpha)}{2}$. If $\alpha = 0$, fix $\delta \in \mathbb{Q} \cap (0, \frac{1}{2})$ such that $\mathcal{H}(\delta) < \epsilon$. If $\alpha > 0$, fix $\delta \in \mathbb{Q} \cap (0, \frac{1}{2})$ such that $[\alpha - \delta, \alpha + \delta]^2 \subseteq (0, 1)^2$ and for all $(x, y) \in [\alpha - \delta, \alpha + \delta]^2$, $h(x, y) < \mathcal{H}(\alpha) + \epsilon$. Let

$$\beta = \begin{cases} \alpha & \text{if } \alpha > 0 \\ \delta & \text{if } \alpha = 0, \end{cases}$$

and let G be the 1-account, 1-state FSG



.

Note that for all $w \in \{0, 1\}^*$,

$$d_G(w) = 2^{|w|} \beta^{\#(1,w)} (1 - \beta)^{\#(0,w)}. \tag{5.3}$$

To prove (5.2), it suffices to show that

$$\mathrm{FREQ}(\alpha) \subseteq S^\infty[d_G^{(s)}]. \tag{5.4}$$

To see this, let $S \in \mathrm{FREQ}(\alpha)$, and let $w_n = S[0..n-1]$ for all $n \in \mathbb{N}$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $\#(1, w_n) \le (\alpha + \delta)n$. Since $1 - \beta \ge \beta$, it follows by (5.3) and our choice of δ that for all $n \ge n_0$,

$$\begin{aligned} d_G(w_n) &\ge 2^n \beta^{(\alpha+\delta)n} (1 - \beta)^{n - (\alpha+\delta)n} \\ &= \left(2^{1 - h(\alpha+\delta, \beta)}\right)^n \\ &\ge \left(2^{1 - \mathcal{H}(\alpha) - \epsilon}\right)^n, \end{aligned}$$

whence

$$d_G^{(s)}(w_n) \ge 2^{(s-1)n} \left(2^{1 - \mathcal{H}(\alpha) - \epsilon}\right)^n = 2^{\epsilon n}.$$

Thus $S \in S^\infty[d_G^{(s)}]$, confirming (5.4) and thereby completing the proof of the first inequality in (5.1).

The second inequality in (5.1) is trivial for $\alpha \in \{0, 1\}$, so let $\alpha \in \mathbb{Q} \cap (0, 1)$, let $s < \mathcal{H}(\alpha)$, and let $G = (Q, \delta, \beta, q_0)$ be a 1-account FSG. By Corollary 4.6, it suffices to show that

$$\mathrm{FREQ}(\alpha) \cap \mathbf{Q} \not\subseteq S^\infty[d_G^{(s)}]. \tag{5.5}$$

Since α is rational, we can write $\alpha = \frac{k}{n}$, where $n \geq 2$ is chosen large enough that

$$n(\mathcal{H}(\alpha) - s) \geq 2 \log n. \tag{5.6}$$

(Note that $0 < k < n$ because $\alpha \in (0,1)$.) Let

$$B = \{u \in \{0,1\}^n \mid \#(1,u) = k\}.$$

Using the well-known bound $e\left(\frac{n}{e}\right)^n < n! < en\left(\frac{n}{e}\right)^n$, it is easy to see that

$$\begin{aligned} |B| = \binom{n}{k} &> \frac{1}{ek(n-k)} 2^{n\mathcal{H}(\alpha)} \\ &\geq \frac{4}{en^2} 2^{n\mathcal{H}(\alpha)} > 2^{n\mathcal{H}(\alpha) - 2\log n}, \end{aligned}$$

whence (5.6) tells us that

$$|B| > 2^{sn}. \tag{5.7}$$

Corollary 3.2 tells us that for each $w \in \{0,1\}^*$, there are fewer than 2^{sn} strings $u \in \{0,1\}^n$ for which $d_G^{(s)}(wu) > d_G^{(s)}(w)$. It follows by (5.7) that for each $w \in \{0,1\}^*$ there exists a string $u(w) \in B$ such that $d_G^{(s)}(wu(w)) \leq d_G^{(s)}(w)$. For each $i \in \mathbb{N}$, define strings $x_i \in \{0,1\}^*$ and $u_i \in B$ by the recursion

$$x_0 = \lambda, \quad u_i = u(x_i), \quad x_{i+1} = x_i u_i.$$

Since Q is finite, there exist $i, j \in \mathbb{N}$ such that $i < j$ and $\delta(x_i) = \delta(x_j)$. Let

$$x = x_i, \quad y = u_i \cdots u_{j-1}, \quad S = xy^\infty.$$

Since $\mathrm{freq}(u) = \alpha$ for all $u \in B$, we have $\mathrm{freq}(y) = \alpha$, whence $S \in \mathrm{FREQ}(\alpha) \cap \mathbf{Q}$. On the other hand, our choice of the strings u_i and our construction of S ensure that

$$d_G^{(s)}(S[0..in-1]) \leq 1$$

for all $i \in \mathbb{N}$. It follows by Corollary 3.3 that for all $m \in \mathbb{N}$, if we write $m = qn + r$, where $q, r \in \mathbb{N}$ and $r < n$, then

$$\begin{aligned} d_G^{(s)}(S[0..m-1]) &\leq 2^{rs} d_G^{(s)}(S[0..qn-1]) \\ &\leq 2^{rs} \\ &< 2^{ns}. \end{aligned}$$

Since n is constant here, this implies that $S \notin S^\infty[d_G^{(s)}]$, confirming (5.5) and concluding the proof of the second inequality in (5.1). □

Corollary 5.6. For all $\alpha \in [0, \frac{1}{2}]$,

$$\dim(\mathrm{FREQ}(\leq \alpha) \mid \mathbf{Q}) = \dim_{\mathrm{FS}}(\mathrm{FREQ}(\leq \alpha)) = \mathcal{H}(\alpha).$$

Proof. The proof of the first inequality in (5.1) actually shows that for all $\alpha' \in \mathbb{Q} \cap [0, \frac{1}{2}]$,

$$\dim_{\mathrm{FS}}(\mathrm{FREQ}(\leq \alpha')) \leq \mathcal{H}(\alpha'). \tag{5.8}$$

Given an arbitrary $\alpha \in [0, \frac{1}{2}]$ and $\epsilon > 0$, choose $\alpha_1 \in \mathbb{Q} \cap [0, \alpha]$ and $\alpha_2 \in \mathbb{Q} \cap [\alpha, \frac{1}{2}]$ such that $\mathcal{H}(\alpha_1) \geq \mathcal{H}(\alpha) - \epsilon$ and $\mathcal{H}(\alpha_2) \leq \mathcal{H}(\alpha) + \epsilon$. Theorem 5.5 and monotonicity tell us that

$$\begin{aligned}\mathcal{H}(\alpha) - \epsilon \leq \mathcal{H}(\alpha_1) &\leq \dim(\mathrm{FREQ}(\leq \alpha_1) \,|\mathbf{Q}) \\ &\leq \dim(\mathrm{FREQ}(\leq \alpha) \mid \mathbf{Q}).\end{aligned}$$

Similarly, (5.8) and monotonicity tell us that

$$\begin{aligned}\dim_{\mathrm{FS}}(\mathrm{FREQ}(\leq \alpha)) &\leq \dim_{\mathrm{FS}}(\mathrm{FREQ}(\leq \alpha_2)) \\ &\leq \mathcal{H}(\alpha_2) \leq \mathcal{H}(\alpha) + \epsilon.\end{aligned}$$

Since ϵ may be arbitrarily small here, it follows that

$$\dim_{\mathrm{FS}}(\mathrm{FREQ}(\leq \alpha)) \leq \mathcal{H}(\alpha) \leq \dim(\mathrm{FREQ}(\leq \alpha) \mid \mathbf{Q}).$$

The corollary follows immediately from this. □

Finally, we note that the set of all rational sequences has finite-state dimension 1.

Corollary 5.7. $\dim(\mathbf{Q} \mid \mathbf{Q}) = \dim_{\mathrm{FS}}(\mathbf{Q}) = 1$.

Proof. Taking $\alpha = \frac{1}{2}$ in Theorem 5.5 and using monotonicity, we have $\dim(\mathbf{Q} \mid \mathbf{Q}) \geq \dim(\mathrm{FREQ}(\frac{1}{2}) \mid \mathbf{Q}) = \mathcal{H}(\frac{1}{2}) = 1$. The corollary follows immediately. □

6 Individual Sequences

It is natural to define the finite-state dimension of an individual sequence as follows.

Definition. The *finite-state dimension* of a sequence $S \in \mathbf{C}$ is

$$\dim_{\mathrm{FS}}(S) = \dim_{\mathrm{FS}}(\{S\}).$$

It is clear that $\dim_{\mathrm{FS}}(S) = \dim_{1-\mathrm{acct}-\mathrm{FS}}(\{S\})$, i.e., it suffices to consider single-account FSGs when working with individual sequences.

We know the finite-state dimensions of normal sequences from the next result of Schnorr and Stimm [20] on 1-account FSGs. For each martingale d, let $X(d)$ be the set of all $S \in \mathbf{C}$ such that either

(i) d is *eventually constant* on S, i.e., $d(S[0..n]) = d(S[0..n-1])$ for all sufficiently large n, or

(ii) d *decays exponentially* on S, i.e., there exists $\alpha \in (0, 1)$ such that $d(S[0..n-1]) < \alpha^n$ for all sufficiently large n.

Recall from section 2 that NORM is the set of all normal sequences.

Theorem 6.1. (Schnorr and Stimm [20]). If G is a 1-account FSG, then $\mathrm{NORM} \subseteq X(d_G)$.

By Theorem 6.1, every normal sequence has finite-state dimension 1. On the other hand, by Corollary 5.3, every rational sequence has finite-state dimension 0. The following theorem says that every rational number $r \in [0, 1]$ is the finite-state dimension of a reasonably simple sequence.

Theorem 6.2. For every $r \in \mathbb{Q} \cap [0, 1]$ there exists $S \in \mathrm{AC}_0$ such that $\dim_{\mathrm{FS}}(S) = r$.

The rest of this section is a proof of Theorem 6.2. The case $r = 1$ is given by Theorem 6.1 and the following known result.

Theorem 6.3. (Strauss [24]). There is a normal sequence in AC_0.

We use the following simple construction to obtain Theorem 6.2 from Theorem 6.3.

Construction 6.4. Given a rational number $r \in [0, 1]$, define the *r-dilution function* $g_r : \mathbf{C} \to \mathbf{C}$ as follows. Write $r = \frac{a}{b}$ in lowest terms, i.e., $a \in \mathbb{N}$, $b \in \mathbb{Z}^+$, and $\gcd(a, b) = 1$. Given $S \in \mathbf{C}$ and $i \in \mathbb{N}$, let w_i be the i^{th} block of a bits of S, i.e., $w_i = S[ai \,..\, a(i+1) - 1]$. (Note that $w_i = \lambda$ if $r = 0$.) Then

$$g_r(S) = w_0 0^{b-a} w_1 0^{b-a} \cdots .$$

Lemma 6.5. For all $r \in \mathbb{Q} \cap [0, 1]$ and $S \in \mathbf{C}$,

$$\dim_{\mathrm{FS}}(g_r(S)) = r \cdot \dim_{\mathrm{FS}}(S).$$

Proof. $g_1(S) = S$ and $g_0(S) = 0^\infty$ for all $S \in \mathbf{C}$, so these cases are obvious. Fix $r \in \mathbb{Q} \cap (0, 1)$, $S \in \mathbf{C}$. Let $a, b \in \mathbb{N}$ such that $r = \frac{a}{b}$ in lowest terms.

To see that $\dim_{\mathrm{FS}}(g_r(S)) \leq r \cdot \dim_{\mathrm{FS}}(S)$, let $s > \dim_{\mathrm{FS}}(S)$. It suffices to show that $\dim_{\mathrm{FS}}(g_r(S)) \leq r \cdot s$. By our choice of s, there is a 1-account FSG $G = (Q, \delta, \beta, q_0)$ such that $S \in S^\infty[d_G^{(s)}]$.

We define the 1-account FSG

$$G' = (Q', \delta', \beta', q_0')$$

whose components are as follows.

(i) $Q' = Q \times \{0, 1, \ldots, b - 1\}$.

(ii) If $i \leq a - 1$, $(q, i) \in Q'$, and $x \in \{0, 1\}$,

$$\delta'((q, i), x) = (\delta(q, x), i + 1).$$

(iii) If $a \leq i \leq b - 1$, $(q, i) \in Q'$, and $x \in \{0, 1\}$,

$$\delta'((q, i), x) = (q, (i + 1) \bmod b).$$

(iv) If $i \leq a - 1$, $(q, i) \in Q'$,

$$\beta'(q, i) = \beta(q).$$

(v) If $a \le i \le b-1$, $(q,i) \in Q'$,
$$\beta'(q,i) = 0.$$

(vi) $q_0' = (q_0, 0)$.

If for each i, $w_i = S[ai\,..\,a(i+1)-1]$, then

$$d_{G'}(w_0 0^{b-a} w_1 0^{b-a} \cdots w_{i-1} 0^{b-a}) = d_G(w_0 \cdots w_{i-1}) 2^{(b-a)i}.$$

For each $n \in \mathbb{N}$, let i be such that $bi \le n < b(i+1)$. Then $S[0..n-1] = w_0 \cdots w_{i-1} v$ with $|v| < b$ and $d_G^{(s)}(S[0..n-1]) < 2^{bs} \cdot d_G^{(s)}(w_0 \cdots w_{i-1})$. Therefore

$$\limsup_{i \to \infty} d_G^{(s)}(w_0 \cdots w_{i-1}) = \limsup_{n \to \infty} d_G^{(s)}(S[0..n-1]) = \infty.$$

We have that

$$d_{G'}^{(sr)}(w_0 0^{b-a} w_1 0^{b-a} \cdots w_{i-1} 0^{b-a}) = 2^{(sr-1)bi} d_G(w_0 \cdots w_{i-1}) 2^{(b-a)i} = d_G^{(s)}(w_0 \cdots w_{i-1}).$$

Therefore $g_r(S) \in S^\infty[d_{G'}^{(sr)}]$ and $\dim_{\mathrm{FS}}(g_r(S)) \le r \cdot s$.

To see that $\dim_{\mathrm{FS}}(g_r(S)) \ge r \cdot \dim_{\mathrm{FS}}(S)$, let $s' < s < \dim_{\mathrm{FS}}(S)$. It suffices to show that $\dim_{\mathrm{FS}}(g_r(S)) \ge r \cdot s'$. Let $G = (Q, \delta, \beta, q_0)$ be a 1-account FSG. It suffices to show that $g_r(S) \notin S^\infty[d_G^{(s'r)}]$.

We define the 1-account FSG
$$G' = (Q', \delta', \beta', q_0')$$
whose components are as follows.

(i) $Q' = Q \times \{0, 1, \ldots, a-1\}$.

(ii) If $i < a-1$, $(q,i) \in Q'$, and $x \in \{0,1\}$,
$$\delta'((q,i),x) = (\delta(q,x), i+1).$$

(iii) If $(q, a-1) \in Q'$, and $x \in \{0,1\}$,
$$\delta'((q,a-1),x) = (\delta(q, x0^{b-a}), 0).$$

(iv) For all $(q,i) \in Q'$,
$$\beta'(q,i) = \beta(q).$$

(vi) $q_0' = (q_0, 0)$.

If for each i, $w_i = S[ai\,..\,a(i+1)-1]$, then

$$d_{G'}(w_0 \cdots w_{i-1}) \ge d_G(w_0 0^{b-a} w_1 0^{b-a} \cdots w_{i-1} 0^{b-a}) 2^{-(b-a)i}.$$

By our choice of s, $S \notin S^\infty[d_{G'}^{(s)}]$, and since $s' < s$, $d_{G'}(w_0 \cdots w_{i-1}) < 2^{(1-s')ia}$ for almost every i, therefore

$$d_G(w_0 0^{b-a} w_1 0^{b-a} \cdots w_{i-1} 0^{b-a}) < 2^{(b-a)i} 2^{(1-s')ia} = 2^{(b-s'a)i} = 2^{(1-s'r)bi}.$$

So for each $n \in \mathbb{N}$, $d_G(g_r(S)[0..n-1]) < 2^{(1-s'r)n+b}$ and therefore $g_r(S) \notin S^\infty[d_G^{(s'r)}]$ and $\dim_{\mathrm{FS}}(g_r(S)) \geq r \cdot s'$.

□

Lemma 6.6. For all $r \in \mathbb{Q} \cap [0,1]$, $g_r(\mathrm{AC}_0) \subseteq \mathrm{AC}_0$.

Proof. $g_1(S) = S$ and $g_0(S) = 0^\infty$ for all $S \in \mathbf{C}$, so these cases are obvious. Fix $r \in \mathbb{Q} \cap (0,1)$. Let $a, b \in \mathbb{N}$ such that $r = \frac{a}{b}$ in lowest terms. Let $S \in \mathrm{AC}_0$. Let $\{C_k \mid k \in \mathbb{N}\}$ be a family of AC_0 circuits for S. We define a family of circuits that recognizes $g_r(S)$.

For each input x with $|x| = m$, a circuit will compute n such that $x = s_n$, then compute $i = \lfloor n/b \rfloor$, compute $M = b \cdot i + a$, compare n with M. If $n \geq M$ then output 0. Otherwise use a circuit from C_{m-c} to C_m to decide $y = s_{n-(b-a)i}$ (where c is a suitable constant for which $|s_{n-(b-a)i}| \geq m - c$).

Since multiplication and division by a constant, as well as comparison and addition, can all be performed in AC_0, $g_r(S) \in \mathrm{AC}_0$. □

Proof of Theorem 6.2. Let $r \in \mathbb{Q} \cap [0,1]$. By Theorem 6.3 there is a sequence $S' \in \mathrm{NORM} \cap \mathrm{AC}_0$. Then $\dim_{\mathrm{FS}}(S') = 1$ by Theorem 6.1. Let $S = g_r(S')$. Then $\dim_{\mathrm{FS}}(S) = r$ by Lemma 6.5, and $S \in \mathrm{AC}_0$ by Lemma 6.6. □

7 Dimension and Compression

In this section we characterize the finite-state dimensions of individual sequences in terms of finite-state compressibility. We first recall the definition of an information-lossless finite-state compressor. (This specific model is due to Huffman [9], refining a similar model introduced by Shannon [21] in 1948. Further exposition may be found in [11] or [12].)

Definition. A *finite-state compressor* (*FSC*) is a 4-tuple

$$C = (Q, \delta, \nu, q_0),$$

where

- Q is a nonempty, finite set of *states*,
- $\delta : Q \times \{0,1\} \to Q$ is the *transition function*,
- $\nu : Q \times \{0,1\} \to \{0,1\}^*$ is the *output function*, and
- $q_0 \in Q$ is the *initial state*.

For $q \in Q$ and $w \in \{0,1\}^*$, we define the *output* from state q on input w to be the string $\nu(q,w)$ defined by the recursion

$$\nu(q, \lambda) = \lambda,$$
$$\nu(q, wb) = \nu(q, w)\nu(\delta(q, w), b)$$

for all $w \in \{0,1\}^*$ and $b \in \{0,1\}$. We then define the *output* of C on input $w \in \{0,1\}^*$ to be the string

$$C(w) = \nu(q_0, w).$$

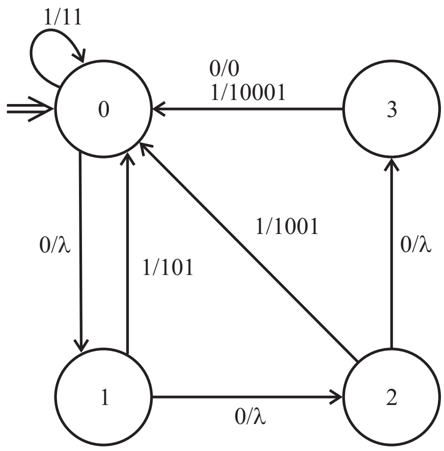
Definition. An FSC $C = (Q, \delta, \nu, q_0)$ is *information-lossless* (*IL*) if the function

$$\begin{aligned} \{0,1\}^* &\to \{0,1\}^* \times Q \\ w &\mapsto (C(w), \delta(w)) \end{aligned}$$

is one-to-one. An *information-lossless finite-state compressor* (*ILFSC*) is an FSC that is IL.

That is, an ILFSC is an FSC whose input can be reconstructed from the output and final state reached on that input.

Example 7.1. (Scheinwald [19]). The diagram



denotes the FSC $C = (Q, \delta, \nu, 0)$, where $Q = \{0,1,2,3\}$ and for all $q \in Q$ we have $\delta(q,0) = (q+1) \bmod 4$, $\delta(q,1) = 0$, $\nu(q,0) = \lambda$ if $q \le 2$, $\nu(3,0) = 0$, and $\nu(q,1) = 10^q1$. It can be seen that C is IL. For example, if $C(w) = 00101$ and $\delta(w) = 3$, it must be the case that $w = 0000000001000$.

Intuitively, an FSC C compresses a string w if $|C(w)|$ is significantly less than $|w|$. Of course, if C is IL, then not all strings can be compressed. Our interest here is in the degree (if any) to which the prefixes of a given sequence $S \in \mathbf{C}$ can be compressed by an ILFSC.

Definition. If C is an FSC and $S \in \mathbf{C}$, then the *compression ratio* of C on S is

$$\rho_C(S) = \liminf_{n\to\infty} \frac{|C(S[0\,..\,n-1])|}{n}.$$

Definition. The *finite-state compression ratio* of a sequence $S \in \mathbf{C}$ is

$$\rho_{\mathrm{FS}}(S) = \inf\left\{\rho_C(S) | C \text{ is an ILFSC}\right\}.$$

Our use of the limit infimum in defining $\rho_C(S)$ implies that our finite-state compression ratio $\rho_{\mathrm{FS}}(S)$ is sometimes less than the compression ratio $\rho(S)$ defined in [26, 19]. However, the following theorem shows that $\rho_{\mathrm{FS}}(S)$ is the compression ratio that precisely characterizes the finite-state dimensions of individual sequences.

Theorem 7.2. For all $S \in \mathbf{C}$,

$$\dim_{\mathrm{FS}}(S) = \rho_{\mathrm{FS}}(S).$$

The rest of this section is devoted to proving Theorem 7.2. We first examine a particular method of constructing finite-state gamblers from a finite-state compressor.

Construction 7.3. Given an FSC $C = (Q, \delta, \nu, q_0)$ and $k \in \mathbb{Z}^+$, we define the 1-account FSG

$$G = G(C, k) = (Q', \delta', \beta', q_0')$$

whose components are as follows.

(i) $Q' = Q \times \{0, 1, \ldots, k-1\}$.

(ii) For all $(q, i) \in Q'$ and $b \in \{0, 1\}$,

$$\delta'((q, i), b) = (\delta(q, b), (i+1) \bmod k).$$

(iii) For all $(q, i) \in Q'$,

$$\beta'(q, i) = \frac{\sigma(q, 1\{0,1\}^{k-i-1})}{\sigma(q, \{0,1\}^{k-i})},$$

where $\sigma(q, A) = \sum_{u \in A} 2^{-|\nu(q,u)|}$.

(iv) $q_0' = (q_0, 0)$.

Lemma 7.4. In Construction 7.3, if $|w|$ is a multiple of k and $u \in \{0,1\}^{\leq k}$, then

$$d_G(wu) = 2^{|u| - |\nu(\delta(w), u)|} \frac{\sigma(\delta(wu), \{0,1\}^{k-|u|})}{\sigma(\delta(w), \{0,1\}^k)} d_G(w).$$

Proof. We use induction on the string u. If $u = \lambda$ the lemma is clear. Assume that it holds for u, where $u \in \{0,1\}^{<k}$, and let $b \in \{0,1\}$. Then

$$\begin{aligned} d_G(wub) &= 2 \frac{\sigma(\delta(wu), b\{0,1\}^{k-|u|-1})}{\sigma(\delta(wu), \{0,1\}^{k-|u|})} d_G(wu) \\ &= 2^{1-|\nu(\delta(wu), b)|} \frac{\sigma(\delta(wub), \{0,1\}^{k-|u|-1})}{\sigma(\delta(wu), \{0,1\}^{k-|u|})} d_G(wu), \end{aligned}$$

so by the induction hypothesis the lemma holds for ub. □

Lemma 7.5. In Construction 7.3, if $w = w_0 w_1 \cdots w_{n-1}$, where each $w_i \in \{0,1\}^k$, then

$$d_G(w) = \frac{2^{|w|-|C(w)|}}{\prod_{i=0}^{n-1} \sigma(\delta(w_0 \cdots w_{i-1}), \{0,1\}^k)}.$$

Proof. We use induction on n. For $n = 0$, the identity is clear. Assume that it holds for n. Let $w' = w_0 w_1 \cdots w_n$ where each $w_i \in \{0,1\}^k$, and let $w = w_0 w_1 \cdots w_{n-1}$. Then Lemma 7.4 with $u = w_n$ tells us that

$$d_G(w') = \frac{2^{|w_n|-|\nu(\delta(w),w_n)|}}{\sigma(\delta(w), \{0,1\}^k)} d_G(w),$$

whence the identity holds for w' by the induction hypothesis. □

Lemma 7.6. In Construction 7.3, if C is IL and $|w|$ is a multiple of k, then

$$d_G(w) \geq 2^{|w|-|C(w)|-\frac{|w|}{k}(l+\log m+\log k+1)},$$

where $l = \lceil \log |Q| \rceil$ and $m = \max\left\{ |\nu(q,b)| \,\middle|\, q \in Q, b \in \{0,1\} \right\}$.

Proof. Following Lemma 2 in [26] we prove that for each $z \in \{0,1\}^*$, $\sigma(\delta(z), \{0,1\}^k) \leq 2^{l+\log m+\log k+1}$. To see this, fix $z \in \{0,1\}^*$ and observe that at most $|Q|$ strings $w \in \{0,1\}^k$ can have the same output from state $\delta(z)$. Therefore the number of $w \in \{0,1\}^k$ for which $|\nu(\delta(z), w)| = j$ does not exceed $|Q| \cdot 2^j$. Hence

$$\sigma(\delta(z), \{0,1\}^k) = \sum_{w \in \{0,1\}^k} 2^{-|\nu(\delta(z),w)|} \leq \sum_{j=0}^{mk} |Q| 2^j 2^{-j} = |Q|(km+1) \leq 2^{l+\log m+\log k+1}.$$

It follows by Lemma 7.5 that

$$d_G(w) \geq 2^{|w|-|C(w)|-\frac{|w|}{k}(l+\log m+\log k+1)}.$$

□

Lemma 7.7. In Construction 7.3, if C is IL, then for all $w \in \{0,1\}^*$,

$$d_G(w) \geq 2^{|w|-|C(w)|-\frac{|w|}{k}(l+\log m+\log k+1)-(km+l+\log m+\log k+1)},$$

where $l = \lceil \log |Q| \rceil$ and $m = \max\left\{ |\nu(q,b)| \,\middle|\, q \in Q, b \in \{0,1\} \right\}$.

Proof. Assume the hypothesis, let l and m be as given, and let $w \in \{0,1\}^*$. Fix $0 \leq j < k$ such that $|w| + j$ is divisible by k. By Lemma 7.6, we have

$$\begin{aligned} d_G(w) &\geq 2^{-j} d_G(w0^j) \\ &\geq 2^{-j+|w0^j|-|C(w0^j)|-\frac{|w0^j|}{k}(l+\log m+\log k+1)} \\ &= 2^{|w|-|C(w0^j)|-\frac{|w|}{k}(l+\log m+\log k+1)-\frac{j}{k}(l+\log m+\log k+1)} \\ &\geq 2^{|w|-|C(w)|-\frac{|w|}{k}(l+\log m+\log k+1)-(km+l+\log m+\log k+1)}. \end{aligned}$$

□

We next examine a method of constructing information-lossless finite-state compressors from a nonvanishing 1-account finite-state gambler.

Construction 7.8. Let $G = (Q, \delta, \beta, q_0)$ be a nonvanishing 1-account FSG, and let $k \in \mathbb{Z}^+$. For each $q \in Q$, let $G_q = (Q, \delta, \beta, q)$, and define $p_q : \{0,1\}^k \to [0,1]$ by $p_q(w) = 2^{-k} d_{G_q}(w)$. Since G is nonvanishing and each d_{G_q} is a martingale with $d_{G_q}(\lambda) = 1$, each of the functions p_q is a positive probability measure on $\{0,1\}^k$. For each $q \in Q$, let $\Theta_q : \{0,1\}^k \to \{0,1\}^*$ be the Shannon-Fano-Elias code (see, for example [4]) given by the probability measure p_q. Then

$$|\Theta_q(w)| = l_q(w),$$

where

$$l_q(w) = 1 + \left\lceil \log \frac{1}{p_q(w)} \right\rceil$$

for all $q \in Q$ and $w \in \{0,1\}^k$, and each of the sets range(Θ_q) is an instantaneous code. We define the FSC

$$C = C(G, k) = (Q', \delta', \nu', q_0')$$

whose components are as follows.

(i) $Q' = Q \times \{0,1\}^{<k}$.

(ii) For all $(q, w) \in Q'$ and $b \in \{0,1\}$,

$$\delta'((q,w),b) = \begin{cases} (q, wb) & \text{if } |w| < k-1 \\ (\delta(q, wb), \lambda) & \text{if } |w| = k-1. \end{cases}$$

(iii) For all $(q, w) \in Q'$ and $b \in \{0,1\}$,

$$\nu'((q,w),b) = \begin{cases} \lambda & \text{if } |w| < k-1 \\ \Theta_q(wb) & \text{if } |w| = k-1. \end{cases}$$

(iv) $q_0' = (q_0, \lambda)$.

Since each range(Θ_q) is an instantaneous code, it is easy to see that the FSC $C = C(G,k)$ is IL.

Lemma 7.9. In Construction 7.8, if $|w|$ is a multiple of k, then

$$|C(w)| \le \left(1 + \frac{2}{k}\right)|w| - \log d_G(w).$$

Proof. Let $w = w_0 w_1, \cdots w_{n-1}$, where each $w_i \in \{0,1\}^k$. For each $0 \le i < n$, let $q_i = \delta(w_0 \cdots w_{i-1})$.

Then

$$\begin{aligned}
|C(w)| &= \sum_{i=0}^{n-1} l_{q_i}(w_i) \\
&= \sum_{i=0}^{n-1} \left(1 + \left\lceil \log \frac{1}{\rho_{q_i}(w_i)} \right\rceil\right) \\
&\le \sum_{i=0}^{n-1} \left(2 + \log \frac{2^k}{d_{G_{q_i}}(w_i)}\right) \\
&= (k+2)n - \log \prod_{i=0}^{n-1} d_{G_{q_i}}(w_i) \\
&= (k+2)n - \log d_G(w) \\
&= \left(1 + \frac{2}{k}\right)|w| - \log d_G(w).
\end{aligned}$$

□

Lemma 7.10. In Construction 7.8, for all $w \in \{0,1\}^*$,

$$|C(w)| \le \left(1 + \frac{2}{k}\right)|w| - \log d_G(w).$$

Proof. Let $w = w'z$, where $|w|$ is a multiple of k and $|z| = j < k$. Then Lemma 7.9 tells us that

$$\begin{aligned}
|C(w)| &= |C(w')| \\
&\le \left(1 + \frac{2}{k}\right)|w'| - \log d_G(w') \\
&\le \left(1 + \frac{2}{k}\right)|w'| - \log(2^{-j} d_G(w)) \\
&= \left(1 + \frac{2}{k}\right)|w| - \log d_G(w) - \frac{2j}{k} \\
&\le \left(1 + \frac{2}{k}\right)|w| - \log d_G(w).
\end{aligned}$$

□

We now use Constructions 7.3 and 7.8 to prove the main result of this section.

Proof of Theorem 7.2. Let $S \in \mathbf{C}$. For each $n \in \mathbb{N}$, let $w_n = S[0..n-1]$.

To see that $\dim_{\mathrm{FS}}(S) \le \rho_{\mathrm{FS}}(S)$, let $s > s' > \rho_{\mathrm{FS}}(S)$. It suffices to show that $\dim_{\mathrm{FS}}(S) \le s$. By our choice of s', there is an ILFSC $C = (Q, \delta, \nu, q_0)$ for which the set

$$I = \left\{ n \in \mathbb{N} \,\middle|\, |C(w_n)| < s'n \right\}$$

is infinite. Let $l = \lceil \log |Q| \rceil$, let $m = \max\{|\nu(q,b)| \mid q \in Q, b \in \{0,1\}\}$ and fix $k \in \mathbb{Z}^+$ such that $\frac{l + \log m + \log k + 1}{k} < s - s'$. Let $G = G(C,k)$ be as in Construction 7.3. Then by Lemma 7.7, for all

$n \in I$ we have

$$\begin{aligned} d_G^{(s)}(w_n) &\geq 2^{sn-|C(w_n)|-\frac{n}{k}(l+\log m+\log k+1)-(km+l+\log m+\log k+1)} \\ &\geq 2^{(s-s'-\frac{l+\log m+\log k+1}{k})n-(km+l+\log m+\log k+1)}. \end{aligned}$$

Since $s - s' - \frac{l+\log m+\log k+1}{k} > 0$, this implies that $S \in S^\infty[d_G^{(s)}]$. Thus $\dim_{\rm FS}(S) \leq s$.

To see that $\rho_{\rm FS}(S) \leq \dim_{\rm FS}(S)$ let $s > s' > s'' > \dim_{\rm FS}(S)$. It suffices to show that $\rho_{\rm FS}(S) \leq s$. By our choice of s'', there is a 1-account FSG G such that the set

$$J = \left\{ n \in \mathbb{N} \,\middle|\, d_G^{(s'')}(w_n) \geq 1 \right\}$$

is infinite. By Lemma 3.11, there is a nonvanishing 1-account FSG G' such that $d_{G'}(w) \geq 2^{(s''-s')|w|} d_G(w)$ for all $w \in \{0,1\}^*$. Fix $k > \frac{2}{s-s'}$, and let $C = C(G', k)$ be as in Construction 7.8. Then Lemma 7.10 tells us that for all $n \in J$,

$$\begin{aligned} |C(w_n)| &\leq \left(1 + \frac{2}{k}\right) n - \log d_{G'}(w_n) \\ &\leq \left(1 + \frac{2}{k} + s' - s''\right) n - \log d_G(w_n) \\ &\leq \left(\frac{2}{k} + s'\right) n - \log d_G^{(s'')}(w_n) \\ &\leq \left(\frac{2}{k} + s'\right) n \\ &< sn. \end{aligned}$$

Thus $\rho_{\rm FS}(S) \leq s$. □

It is worthwhile to examine the number of states used in the proof of Theorem 7.2. Consider first the proof that $\dim_{\rm FS}(S) \leq \rho_{\rm FS}(S)$. If the compressor C has n states and we want the gambler G to approximate $\rho_{\rm FS}(S)$ to within r bits of accuracy, then $s - s'$ is $\Theta(2^{-r})$, so k is $\Theta(2^r \log n)$, so G has $\Theta(2^r n \log n)$ states. This increase in the number of states is modest because, roughly speaking, only $O(r + \log\log n)$ more hardware is required to implement G than to implement C.

Conversely, consider the proof that $\rho_{\rm FS}(S) \leq \dim_{\rm FS}(S)$. If the gambler G has n states and we want the compressor C to approximate $\dim_{\rm FS}(S)$ to within r bits of accuracy, then $s - s'$ is $\Theta(2^{-r})$, so k is $\Theta(2^r)$, so C has $\Theta(n \cdot 2^k) = \Theta(n \cdot 2^{2^r})$ states. This is a very large increase in the number of states. At the time of this writing, we do not know whether such a large increase is necessary or merely an artifact of the present proof. That is, the following question is open.

Question 7.11. Given an n-state, 1-account FSG G, $s \in [0,1]$, and $\epsilon \in (0,1)$, how many states are required for an ILFSC C such that $\rho_C(S) \leq s + \epsilon$ for all $S \in S^\infty[d_G^{(s)}]$?

Theorem 7.2 tells us that finite-state dimension of a sequence S can be defined using either FSGs or ILFSCs. Question 7.11 asks whether FSGs are significantly more succinct than ILFSCs for this purpose.

8 Conclusion

We have used finite-state gamblers to effectivize the gale characterization of classical Hausdorff dimension, thereby defining finite-state dimension in the Cantor space **C** and in the space **Q** of all rational binary sequences. We have shown that Eggleston's classical theorem on limiting frequencies holds for finite-state dimension in both **Q** and **C**. We have shown that the finite-state dimensions of individual sequences can be equivalently defined using either 1-account finite-state gamblers or information-lossless finite-state compressors, but our proof suggests that far more states may be required in the latter model. Similarly, we have shown that the finite-state dimensions of sets of sequences can be equivalently defined using either multi-account finite-state gamblers or 1-account finite-state gamblers, but our proof suggests that far more states may be needed in the latter model. It is to be hoped that the quantitative relationships among these three finite-state models will be clarified in the near future.

In any case, finite-state dimension is a real-time effectivization of a powerful tool of fractal geometry. As such it should prove to be a useful tool for improving our understanding of real-time information processing.

Acknowledgment

We thank Giora Slutzki for useful discussions. We are also grateful to an anonymous referee for a careful review and for pointing out an error in our original formulation of Lemma 7.6.

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