Dimension is Compression

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Abstract. Effective fractal dimension was defined by Lutz (2003) in order to quantitatively analyze the structure of complexity classes. Interesting connections of effective dimension with information theory were also found, in fact the cases of polynomial-space and constructive dimension can be precisely characterized in terms of Kolmogorov complexity, while analogous results for polynomial-time dimension haven't been found.
In this paper we remedy the situation by using the natural concept of reversible time-bounded compression for finite strings. We completely characterize polynomial-time dimension in terms of polynomial-time compressors.

1 Introduction

Effective fractal dimension was defined in [13] in order to quantitatively analyze the structure of complexity classes. See [12, 16] for a summary of the main results.

In parallel, the connections of this effective dimension with algorithmic information started being patent. The cases of constructive, recursive and polynomial-space dimension were characterized precisely as the best case asymptotic compression rate when using plain, recursive, and polynomial-space-bounded Kolmogorov complexity, respectively [15, 14, 6].

But the case of polynomial-time bounds remained elusive [8]. This is not strange since computing even approximately the value of time-bounded Kolmogorov complexity seems to require an exponential search. The main difference with space-bounded Kolmogorov complexity is reversibility, in this later case the encoding phase can be performed within similar space-bounds.

In this paper we look at the usual notion of compression algorithm for finite strings. A polynomial-time compression scheme is just a pair of encoder and decoder algorithms, both working in polynomial-time. We consider encoders that do not completely start from scratch when working on an extension of a previous input. This last condition is formalized in Sect. 3 with a conditional-entropy like inequality.

We exactly characterize polynomial-time (or p) -dimension as the best case asymptotic (that is, i.o.) compression ratio attained by these polynomial-time

* This research was supported in part by Spanish Government MEC project TIC 2002-04019-C03-03

compression schemes. Dually, strong polynomial-time-dimension [2] corresponds to the worst case asymptotic compression ratio.

Several results on the polynomial-time dimension of complexity classes can be now interpreted as compressibility results. For example, the (characteristic sequences of) languages in a class of p-dimension 1 cannot be i.o. compressed by more that a sublinear amount. Here we obtain results on the compressibility of complete and autoreducible languages.

Buhrman and Longprè have given a characterization of p-measure in terms of compressibility in [4], but in that case the compressors are restricted to extenders and the encoder is required to give several alternatives, one of them being the correct output. In the light of our present results we can view effective dimension as an information content measure for infinite strings, whereas resource-bounded measure can only distinguish the extreme case of non measure 0 classes that are the most incompressible ones.

2 Preliminaries

The Cantor space $\mathbf{C}$ is the set of all infinite binary sequences. If $w \in \{0,1\}^*$ and $x \in \{0,1\}^* \cup \mathbf{C}$, $w \sqsubseteq x$ means that w is a prefix of x. For $0 \leq i \leq j$, we write $x[i \ldots j]$ for the string consisting of the i-th through the j-th bits of x. We use λ for the empty string.

Let p be the set of polynomial-time computable functions. Let E= DTIME($2^{O(n)}$).

Definition 1. *Let $s \in [0, \infty)$.*

1. *An s-gale is a function $d : \{0,1\}^* \to [0,\infty)$ satisfying*

$$d(w) = 2^{-s}[d(w0) + d(w1)]$$

   for all $w \in \{0,1\}^*$.
2. *A martingale is a 1-gale, that is, a function $d : \{0,1\}^* \to [0,\infty)$ satisfying*

$$d(w) = \frac{d(w0) + d(w1)}{2}$$

   for all $w \in \{0,1\}^*$.

Definition 2. *Let $s \in [0, \infty)$ and d be an s-gale.*

1. *We say that d succeeds on a sequence $S \in \mathbf{C}$ if*

$$\limsup_{n\to\infty} d(S[0 \ldots n]) = \infty$$

   The success set of d is $S^\infty[d] = \{S \in \mathbf{C} \mid d \text{ succeeds on } S\}$
2. *We say that d succeeds strongly on a sequence $S \in \mathbf{C}$ if*

$$\liminf_{n\to\infty} d(S[0 \ldots n]) = \infty$$

   The strong success set of d is $S^\infty_{\mathrm{str}}[d] = \{S \in \mathbf{C} \mid d \text{ succeeds strongly on } S\}$

Definition 3. *Let* $X \subseteq \mathbf{C}$*,*

1. *The* p*-dimension of* X *is*

$$\dim_{\mathrm{p}}(X) = \inf \left\{ s \in [0,\infty) \;\middle|\; \begin{array}{l} \text{there is a p-computable } s\text{-gale } d \text{ s.t.} \\ X \subseteq S^{\infty}[d] \end{array} \right\}$$

2. *The strong* p*-dimension of* X *is*

$$\mathrm{Dim}_{\mathrm{p}}(X) = \inf \left\{ s \in [0,\infty) \;\middle|\; \begin{array}{l} \text{there is a p-computable } s\text{-gale } d \text{ s.t.} \\ X \subseteq S^{\infty}_{\mathrm{str}}[d] \end{array} \right\}$$

For a complete introduction and motivation of effective dimension and effective strong dimension see [12].

3 Compressors that do not start from scratch

In this section we develop the notion of compressors that "do not start from scratch" in the sense that when encoding successively longer extensions of an input, the outputs are restricted in the way we make precise below. The extreme case of this behavior is when the compressor is a mere extender, that is, $C(w)$ is always a prefix of $C(wu)$. We consider here a much weaker restriction than extension.

Definition 4. *A pair of functions* (C, D) *(*C *the encoder,* D *the decoder)* $C, D : \{0,1\}^* \to \{0,1\}^*$ *is a polynomial-time compressor if:*

(i) C *and* D *can be computed in polynomial-time on their corresponding input length.*
(ii) For all $w \in \{0,1\}^*$*,* $D(C(w), |w|) = w$*.*

In this paper, we could make all codes prefix-free, that is, $C(\{0,1\}^n)$ is a prefix set for each n. For the asymptotic compression rates the difference is not significant.

Notice that in the previous definition there is no restriction whatsoever on the behavior of C, the encoder, when working on two inputs that are one an extension of the other. For instance, we can have $|C(wu)| \ll |C(w)|$ and $C(wu)$ can have no common prefix with $C(w)$. In definition 5 we introduce a restriction on the compressor that has an effect on the variety of $C(wu)$ for different u, that will be controlled by $|C(w)|$.

Definition 5. *A polynomial-time compressor* (C, D) *does not start from scratch if* $\forall \epsilon > 0$ *and for almost every* $w \in \{0,1\}^*$ *there exists* $k = O(\log(|w|))$*,* $k > 0$*, such that*

$$\sum_{|u| \le k} 2^{-|C(wu)|} \le 2^{\epsilon k} 2^{-|C(w)|}. \tag{1}$$

We will consider only compressors that do not start from scratch.

Notice that when there is a constant k such that $\sum_{|u|\leq k} 2^{-|C(wu)|} \leq 2^{-|C(w)|}$, condition (1) is trivial, while in general $\sum_{|u|\leq k} 2^{-|C(wu)|}$ can be as large as 1, so condition (1) is a proper restriction on compressors.

We first remark that if $C(w)$ and $C(wu)$ have a long common prefix then C fulfills condition (1).

Lemma 1. *Polynomial-time compressors for which $C(w)$ and $C(wu)$ have a common prefix of length at least $|C(w)| - O(\log(|w|))$ ($\forall\, w, u \in \{0,1\}^*$) don't start from scratch.*

We next present two easy examples of compressors not starting from scratch, including Lempel-Ziv algorithms.

Example 1. For the following polynomial-time compressor condition (1) holds

1. An extender, that is, $\forall\, w, w' \in \{0,1\}^*$

$$w \sqsubseteq w' \Rightarrow C(w) \sqsubseteq C(w').$$

2. Lempel-Ziv data compression algorithm for its three most common variants (notice that it is not an extender. See [10, 11] for details).
   In fact, Lempel-Ziv compression algorithm verifies the common-prefix condition lemma 1. Let $w \in \{0,1\}^*$. If $w = w_1 w_2 \dots w_n v$ where $w_1, w_2, \dots, w_n$ are the phrases obtained by the Lempel-Ziv parsing, then $LZ(w)$ and $LZ(wu)$ have a common prefix of length at least $|LZ(w)| - \log n \geq |LZ(w)| - \log(|w|)$.

We leave for the complete version of this paper an analysis of the case of Grammar-based compressors, that generalize Lempel-Ziv methods [9].

Polynomial-time compressors (C, D) that are length increasing in the encoder C and for which we can control, for all w and all $i \geq 0$, the number of strings u satisfying $|C(wu)| = |C(w)| + i$, don't start from scratch. More formally,

Lemma 2. *Polynomial-time compressors (C, D) that satisfy both of the following conditions don't start from scratch.*

i) For all $w, u \in \{0,1\}^*$, $|C(wu)| \geq |C(w)|$

ii) For all $\epsilon > 0$ and for almost every $w \in \{0,1\}^*$ there exists $k = O(\log(|w|))$ such that $\forall i \geq 0$

$$N_i = N_i(w,k) = \#\left\{u \in \{0,1\}^{\leq k} \;\middle|\; |C(wu)| = |C(w)| + i\right\} \leq 2^{i+\epsilon k - \log k}$$

4 Main Theorem

In the main theorem we obtain an exact characterization of polynomial-time dimension in terms of polynomial-time compressors that don't start from scratch.

Our characterization holds both for the best and worst asymptotic compression ratio, corresponding to p-dimension and strong p-dimension.

We formalize the notion of a.e. (almost everywhere) and i.o. (infinitely often) compressibility for sets of infinite sequences as the asymptotic best (respectively worse) compression ratio.

Definition 6. *For $\alpha \in [0,1]$ and $X \subseteq \mathbf{C}$,*

1. *X is* α-i.o. polynomial-time compressible *if there is a polynomial-time compressor (C,D) that does not start from scratch and such that for every $A \in X$*

$$\liminf_{n} \frac{|C(A[0 \ldots n-1])|}{n} \leq \alpha$$

2. *X is* α-a.e. polynomial-time compressible *if there is a polynomial-time compressor (C,D) that does not start from scratch and such that for every $A \in X$*

$$\limsup_{n} \frac{|C(A[0 \ldots n-1])|}{n} \leq \alpha$$

Definition 7. *Let $X \subseteq \mathbf{C}$,*

1. *X is* i.o. polynomial-time incompressible *if for every (C,D) polynomial-time compressor that does not start from scratch, there exist $A \in X$ such that*

$$\liminf_{n} \frac{|C(A[0 \ldots n-1])|}{n} = 1$$

2. *X is* a.e. polynomial-time incompressible *if for every (C,D) polynomial-time compressor that does not start from scratch, there exist $A \in X$ such that*

$$\limsup_{n} \frac{|C(A[0 \ldots n-1])|}{n} = 1$$

We next state our main theorem.

Theorem 1. *Let $X \subseteq \mathbf{C}$,*

$$\dim_{\mathrm{p}}(X) = \inf\{\, \alpha \mid X \textit{ is } \alpha\textit{-i.o. polynomial-time compressible}\}$$
$$\mathrm{Dim}_{\mathrm{p}}(X) = \inf\{\, \alpha \mid X \textit{ is } \alpha\textit{-a.e. polynomial-time compressible}\}$$

The proof of theorem 1 will be split between sections 5 and 6. In section 5 we transform each gale into a compressor that requires only a time increase of a linear factor. In section 6 we show that compression is an upper bound on dimension.

Hitchcock showed in [7] that p-dimension can be characterized in terms of on-line prediction algorithms, using the well-studied log-loss prediction ratio. Our result can thus be interpreted as a joining bridge between the performance of polynomial-time prediction and compression algorithms, both in the best and the worse case.

5 Compression is at most dimension

Proposition 1. *Let $X \in \mathbf{C}$,*

$$\dim_{\mathrm{p}}(X) \geq \inf\{\, \alpha \,|\, X \textit{ is } \alpha\textit{-i.o. polynomial-time compressible}\}$$

We first transform each gale into a simple version that requires little accuracy. Then we apply a generalization of arithmetic coding to this new gale.

For the first part we will need the following lemma stating that very simple gales characterize p-dimension.

Lemma 3. *Let $X \subseteq \mathbf{C}$. If $\dim_{\mathrm{p}}(X) = \alpha$ then $\forall s > \alpha$ there exists an s-gale d with $X \subseteq S^{\infty}[d]$ such that for all $w \in \{0,1\}^*$, there exists $m_w, n_w \in \mathbb{N}$ with $n_w \leq |w| + 1$ and*

$$d(w)2^{-|w|s} = m_w 2^{-(n_w+|w|)}$$

That is, if $\dim_p(X) < s$, then there exists a p-computable s-gale d as in the previous lemma. We define a polynomial-time compressor that doesn't start from scratch using this s-gale. Roughly speaking, the idea for the encoder C is associate to each $w \in \{0,1\}^*$ an interval of size proportionally related to $d(w)$. By the properties of d, the extreme points of such interval are dyadic rational numbers. By using the following lemma, we codify each interval with a string z. We will take $C(w) = z$.

Lemma 4. *Let a, b be dyadic numbers. and let $I = [a, b)$ be an interval of length $r \in [0, 1)$, then there exists a string z of length $-\lfloor \log(r) \rfloor + 1$ such that $a < 0.z < b$ and z can be computed in time polynomial in $|z|$.*

Proof sketch of Proposition 1.

Let s be such that $\dim_{\mathrm{p}}(X) < s$, then there exists a p-computable s-gale d as in Lemma 3 with $d(\lambda) = 1$, $X \subseteq S^{\infty}[d]$.

Let $h : \{0,1\}^* \to \mathbb{R}$ be defined as follows.

$$h(w) := \sum_{|y|=|w|, y<w} d(y)2^{(1-s)|y|-|w|}$$

where $y < w$ means that y precedes x in lexicographic order. Denote by $succ(w)$ the successor of w in lexicographic order. Notice that $h(w)$ is a dyadic number

$m2^{-n}$ with $n \leq 2|w|+1$, therefore there is a $z \in \{0,1\}^*$ such that $|z| \leq 2|w|+2$ and $h(w) < 0.z < h(succ(w))$. Let z_w be the first shortest string such that $h(w) < 0.z < h(succ(w))$. We define the encoder as $C(w) = z_w$.

It can be shown that the encoder C and the corresponding decoder form a polynomial-time compressor that does not start from scratch.

Finally, let us see that (C, D) compresses X. Notice that for each w the interval $[h(w), h(succ(w)))$ has length exactly $d(w)2^{-s|w|}$. Then by lemma 4, there is a string z of length $-\lfloor \log(2^{-s|w|}d(w)) \rfloor + 1 \leq |w| - \lfloor \log(2^{(1-s)|w|}d(w)) \rfloor + 1$ such that $h(w) < 0.z < h(succ(w))$. So,

$$|z_w| \leq |w| - \lfloor \log(2^{(1-s)|w|}d(w)) \rfloor + 1.$$

For all $A \in X$, as $X \subseteq S^\infty[d]$ then $d(A[0 \dots n-1]) > 1$ i.o. n and

$$\begin{aligned} |C(A[0 \dots n-1])| &= |z_{A[0 \dots n-1]}| \\ &\leq n - \lfloor \log(d(A[0 \dots n-1]) \rfloor + 1 \\ &\leq n - \log(2^{(1-s)n}) + 1 \\ &= sn + 1 \end{aligned}$$

□

6 Dimension is at most compression

Next we prove that compressibility is an upper bound on dimension.

Proposition 2. *Let $X \in \mathbf{C}$,*

$$\dim_{\mathrm{p}}(X) \leq \inf\{\, \alpha \mid X \textit{ is } \alpha\textit{-i.o. polynomial-time compressible}\}$$

Proof. Let $s' > s$ and $\epsilon > 0$ such that $s' - s > \epsilon$. Let N be such that condition (1) is true for each $w \in \{0,1\}^{\geq N}$. For each of these w, let $k = k(w, \epsilon) = O(\log(|w|))$ be the smallest one such that

$$\sum_{|u| \leq k} 2^{-|C(wu)|} \leq 2^{\epsilon k} 2^{-|C(w)|}$$

Let $w = w_1 \dots w_n$ with $|w_1| = N$ and $|w_i| = k(w_1 \dots w_i - 1, \epsilon)$ for $i > 0$. We define an s' gale d as follows

$$d(wu) := d(w) \frac{2^{-|C(wu)|}}{\sum_{|v| \leq k} 2^{-|C(wv)|}} 2^{s'|u|} \qquad \text{if } |u| = k(w, \epsilon)$$

$$d(wr) := \sum_{r \sqsubseteq u, |u| = k} d(wu) 2^{s'(|r| - |u|)} \qquad \text{if } |r| < k(w, \epsilon)$$

d is computable in polynomial-time. Notice that, for induction, if $w = w_1 w_2 \dots w_n$ with $|w_1| = N$ and $|w_i| = k(w_1 \dots w_{i-1}, \epsilon)$, then

$$d(w) = d(w_1) 2^{s'(|w|-N)} \prod_{h=1}^{n-1} \frac{2^{-|C(w_1 \dots w_{h+1})|}}{\sum_{|v| \le k(w_1 \dots w_h, \epsilon)} 2^{-|C(w_1 \dots w_h v)|}}$$

By condition (1),

$$\begin{aligned} d(w) &\ge d(w_1) 2^{(\epsilon - s')N} 2^{|C(w_1)|} 2^{(s'-\epsilon)|w|} 2^{-|C(w)|} \\ &\ge a 2^{(s'-\epsilon)|w|} 2^{-|C(w)|} \end{aligned}$$

where a is the minimum of

$$d(w_1) 2^{|C(w_1)|} 2^{(\epsilon - s')N}$$

for $w_1 \in \{0,1\}^N$.

For all $A \in X$,

$$\liminf_n \frac{|C(A[0 \dots n-1])|}{n} \le s$$

so there exists $(b_n)_{n \in \mathbb{N}}$ a sequence of natural numbers such that

$$\lim_n \frac{|C(A[0 \dots b_n - 1])|}{b_n} \le s$$

Let $(a_n)_{n \in \mathbb{N}}$ be defined as $a_1 = N$, $a_i = k(A[0 \dots a_{i-1} - 1])$ for $i > 1$.

Then

$$d(A[0 \dots a_{i-1} - 1]) \ge a 2^{(s'-\epsilon)a_i} 2^{-|C(A[0 \dots a_i - 1])|}$$

For each n, let $a_m < b_n \le a_{m+1}$, by condition (1)

$$|C(A[0 \dots a_m - 1])| < |C(A[0 \dots b_n - 1])| + O(\log b_n) \le b_n(s + \epsilon/2) \le a_{m+1}(s + \epsilon/2)$$

for all but finitely n.

Then,

$$\begin{aligned} d(A[0 \dots a_m - 1]) &\ge a 2^{(s'-\epsilon)a_m} 2^{-a_{m+1}(s+\epsilon/2)} \\ &\ge a 2^{a_m \epsilon/2} 2^{-O(\log(a_m))} \end{aligned}$$

And d succeeds on X.

Notice that in the last proof we didn't need the decoder so we have that for each polynomial-time encoder satisfying the condition of not starting from scratch we automatically get a polynomial-time decoder.

Corollary 1. *Let C be a polynomial-time encoder that satisfies inequality (1). Then there exist (C', D') a polynomial-time compressor that does not start from scratch and such that for every $A \in \mathbf{C}$*

$$\liminf_n \frac{C'(A[0\dots n-1])}{n} \leq \liminf_n \frac{C(A[0\dots n-1])}{n}$$

$$\limsup_n \frac{C'(A[0\dots n-1])}{n} \leq \limsup_n \frac{C(A[0\dots n-1])}{n}$$

7 Applications of the Main Result

In this section we obtain interesting consequences of our characterization for the polynomial-time compressibility of complete and autoreducible sets from previously known p-dimension results.

Notice that in this section we identify each language A with its characteristic sequence χ_A, therefore compressibility of a class always means compressibility of the corresponding characteristic sequences.

We start by showing that no polynomial-time compressor works on all many-one complete sets.

Theorem 2. *The class of polynomial-time many-one complete sets for E is i.o. polynomial-time incompressible.*

Proof. Ambos-Spies et al. prove in [1] that the class has p-dimension 1.

Next we consider $\mathrm{deg}^{\mathrm{P}}_{\mathrm{m}}(A)$, the class of sets that are equivalent to A by $\leq^{\mathrm{P}}_{\mathrm{m}}$-reductions. The compression ratio of $\mathrm{deg}^{\mathrm{P}}_{\mathrm{m}}(A)$ and $\mathrm{deg}^{\mathrm{P}}_{\mathrm{m}}(B)$, for $A \leq^{\mathrm{P}}_{\mathrm{m}} B$, is related by the following theorem.

Theorem 3. *Let A, B be sets in E such that $A \leq^{\mathrm{P}}_{\mathrm{m}} B$, then*

1. *The i.o. p-compression ratio of $\mathrm{deg}^{\mathrm{P}}_{\mathrm{m}}(A)$ is at most the i.o. p-compression ratio of $\mathrm{deg}^{\mathrm{P}}_{\mathrm{m}}(B)$.*
2. *The a.e. p-compression ratio of $\mathrm{deg}^{\mathrm{P}}_{\mathrm{m}}(A)$ is at most the a.e. p-compression ratio of $\mathrm{deg}^{\mathrm{P}}_{\mathrm{m}}(B)$.*

Proof. Ambos-Spies et al. prove 1. in [1] for p-dimension. Athreya et al. prove in [2] the strong dimension result for 2.

We next consider the property of autoreducibility. A set A is autoreducible if A can be decided by using A as an oracle but without asking query x on input x. We obtain incompressibility results both in the case of polynomial-time many-one autoreducibility and for the *complement* of i.o. p-Turing autoreducible sets. Therefore for each polynomial-time bound there are i.o. incompressible sets that are $\leq^{\mathrm{P}}_{\mathrm{m}}$-autoreducible and other that are not even i.o. $\leq^{\mathrm{P}}_{\mathrm{T}}$-autoreducible.

Theorem 4. *The class of polynomial-time many-one autoreducible sets are i.o. polynomial-time incompressible.*

Proof. Ambos-Spies et al. prove in [1] that the class has p-dimension 1.

Theorem 5. *The class of sets that are NOT i.o. polynomial-time Turing autoreducible are i.o. polynomial-time incompressible.*

Proof. Beigel et al. prove in [3] that the class has p-dimension 1.

We next show that there exist polynomial-time many-one degrees with every possible value for both a.e. and i.o. compressibility.

Theorem 6. *Let x, y be computable reals such that $0 \leq x \leq y \leq 1$. Then there is a set A in E such that the i.o. p-compression ratio of $\deg_{\mathrm{m}}^{\mathrm{P}}(A)$ is x and the a.e. p-compression ratio of $\deg_{\mathrm{m}}^{\mathrm{P}}(A)$ is y.*

Proof. Athreya et al. prove in [2] the result for p-dimension and strong p-dimension.

This last theorem includes the extreme case for which the i.o. compression ratio is 0 whereas the a.e. ratio is 1.

Finally, the hypothesis "NP has positive p-dimension" can be interpreted in terms of incompressibility. This hypothesis has interesting consequences on the approximation algorithms for MAX3SAT.

Theorem 7. *If for some $\alpha > 0$ NP is not α-i.o-compressible in polynomial-time then any approximation algorithm $\mathcal{A}$ for MAX3SAT must satisfy at least one of the following*

1. *For some $\delta > 0$, $\mathcal{A}$ uses time at least 2^{n^δ}*
2. *For all $\epsilon > 0$, $\mathcal{A}$ has performance ratio less than $7/8 + \epsilon$ on an exponentially dense set of satisfiable instances.*

Proof. Hitchcock proves in [5] that the consequence follows from NP having positive p-dimension.

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