# On deadlock-freeness analysis of autonomous and timed continuous mono-T-semiflow nets

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**Abstract**

Liveness analysis of autonomous and timed continuous systems is a difficult problem. The subclass of mono-T-semiflow systems offers in practice an interesting modeling power. For this subclass, the equivalence between liveness and deadlock-freeness allows a more satisfactory treatment. This paper focuses on the interleaving of autonomous and timed properties of continuous systems. This allows a partial characterization of structural liveness for autonomous nets based on the analysis of timed systems, generalizing the well-known rank Theorem.

## 1 Introduction

Deadlock-freeness (DF) is a very important safety property, asking for non-existence of blocking states, i.e. states without successor. This paper concentrates on the analysis of DF for a subclass of *Petri nets* (PN) called *mono-T-semiflow* (MTS) nets [2]. The essential properties of MTS nets are purely structural: consistency with a single T-semiflow (i.e. all transitions are covered by the unique minimal T-semiflow) and conservativeness (i.e. all places are covered by P-semiflows). Therefore the membership problem for MTS can be decided in polynomial time. From a modeling expressive power point of view, in particular, MTS generalizes *choice-free* nets [10] by allowing *generalized mutual exclusion constraints* (monitors). A subclass of choice free nets are weighted-T-systems, a weighted generalization of the well-known subclass of marked graphs.

Continuization of discrete models is, in general, a classical relaxation aiming at computationally more efficient analysis techniques, at the price of losing some precision. Nevertheless, it should be pointed out that not all MTS systems allow continuization, if such a basic qualitative property as DF should be preserved. Figure 1 shows examples of autonomous MTS systems in which DF of the discrete model is neither necessary nor sufficient for DF of the relaxed continuous approximation [8]. This fact, that may be surprising at a first glance, can be easily accepted if we think, for example, on the existence of non-linearizable differential equations models (for example, due to the existence of a chaotic behavior).

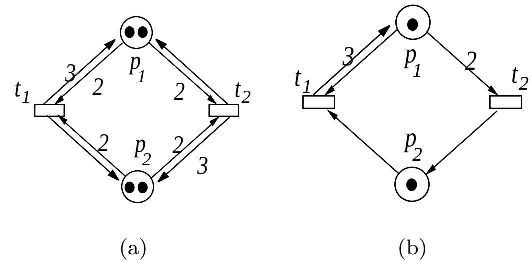


**Figure 1:** Two MTS systems which behave in very different ways if seen as discrete or as continuous: (a) is non-live as discrete, but never gets completely blocked as continuous unless an infinitely long sequence is considered. (b) is live as discrete, but non-live as continuous.

As an example of these "mismatches" among properties of the four cases, discrete *vs.* continuous, and autonomous *vs.* timed, it can be pointed out that the addition of an *infinite servers semantics* time interpretation [6] (*variable speed* in [1]) may allow the timed continuous model to have infinite behavior (DF), while a "similar" timing in the discrete system leads to a deadlock. Under classical markovian time interpretation, for the stochastic system in Figure 2(a) seen as discrete, the probability of arriving into a deadlock state is "1" (this is a particular case of the classical "gambler's ruin problem"). If $\lambda_1 = \lambda_2$ the mean time for deadlock is quadratic w.r.t. $k$, while it is "almost" linear in other case. However, if $\lambda_1 = \lambda_2$ the continuous system is live, what may be interpreted as a "very large" transient to deadlock (see [9] for more details). Liveness of a transition can also be affected when considering a system as discrete or as continuous. Figure 2(b) shows a non-MTS system (it is consistent and conservative, but has two T-semiflows), that considered as discrete is live (thus DF), but for which a deterministic timing of transitions with $t_4$ faster than $t_2$ (i.e. $\theta_4 < \theta_2$) makes $t_3$ non-live (in fact, to starve). Nevertheless, considering the model as continuous, it is live both for the autonomous and the timed interpretations.

One important property of discrete (and continuous)

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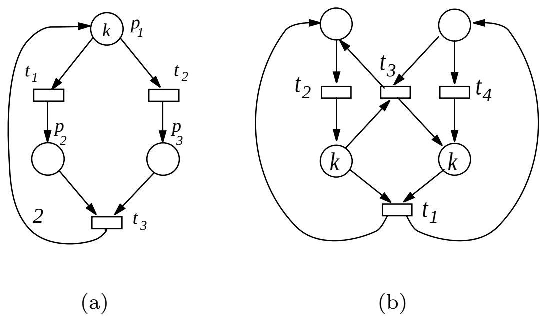


**Figure 2:** (a) A continuous MTS system that deadlocks as autonomous, and that with a timing that assigns the same firing speed to $t_1$ and $t_2$ has an infinite behavior. (b) A non-MTS system that, seen as discrete, is live as autonomous, and for which a deterministic timing with $t_4$ faster than $t_2$ prevents firing $t_3$.

MTS systems is that DF is equivalent to liveness [2], because all the infinite behaviors are "essentially conformed" by infinite repetition of sequences having the T-semiflow as the firing count vector. Even more, for MTS systems, DF of the autonomous model leads, for any arbitrary transition-time semantics (deterministic, exponential, coxian...), to non null throughput. Thus there exists a one-way bridge from logical or qualitative properties to performance properties.

Section 2 is devoted to basic concepts and notations, while Sections 3 and 4 deal with DF analysis of autonomous and infinite servers timed interpretation of continuous MTS systems. Section 5 extracts results for autonomous systems making use of some properties of timed systems. Section 6 summarizes the main results.

## 2 Basic concepts and preliminary results

The reader is assumed to be familiar with Petri nets (PNs) (see [5, 3]). The usual PN system, $\langle \mathcal{N}, \mathbf{m_0} \rangle$ ($\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post} \rangle$), will be said to be *discrete* so as to distinguish it from a *continuous* relaxation. The main difference between continuous and discrete PNs is in the marking, which in a discrete PN is restricted to be in the naturals, while in continuous PNs is released into the non-negative real numbers. This is a consequence of the firing, which is modified in the same way: A transition $t$ is *enabled* at $\mathbf{m}$ iff for every $p \in {}^{\bullet}t$, $\mathbf{m}[p] > 0$. In other words, the enabling condition of continuous systems and that of discrete ordinary systems can be expressed in an "analogous" way: every input place is marked. As in discrete systems, the *enabling degree* at $\mathbf{m}$ of a transition measures the maximal amount in which the transition can be fired in one go, i.e. $\text{enab}(t, \mathbf{m}) = \min_{p \in {}^{\bullet}t} \{\mathbf{m}[p]/\mathbf{Pre}[p,t]\}$. The firing of $t$ in a certain amount $\alpha \leq \text{enab}(t, \mathbf{m})$ leads to a new marking $\mathbf{m'} = \mathbf{m} + \alpha \cdot \mathbf{C}[P,t]$, where $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$ is the token flow matrix. Hence, as in discrete systems, the state (or fundamental) equation ($\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}$) summarizes the way the marking evolves along time. Notice that for a continuous transition being enabled or not does not depend on the arc weights, although they are important to compute the enabling degree and to obtain the new marking.

All the concepts based on the representation of the net as a graph can be directly applied to continuous nets, in particular, the conflict relationships. Two transitions, $t$ and $t'$, are said to be in *structural conflict relation* if ${}^{\bullet}t \cap {}^{\bullet}t' \neq \emptyset$. The *coupled conflict* relation is defined as the transitive closure of the structural conflict relation. Each equivalence class is called a *coupled conflict set* denoted, for a given $t$, CCS($t$). The set of all the equivalence classes is denoted by SCCS. When $\mathbf{Pre}[P,t] = \mathbf{Pre}[P,t'] \neq \mathbf{0}$, $t$ and $t'$ are in *equal conflict* relation.

Right and left natural annullers are called T- and P-semiflows, respectively. We call a semiflow $\mathbf{v}$ *minimal* when its support, $\|\mathbf{v}\|$, is not a proper superset of the support of any other, and the g.c.d. of its elements is one. When $\mathbf{y} \cdot \mathbf{C} = \mathbf{0}$, $\mathbf{y} > \mathbf{0}$ the net is said to be *conservative*, and when $\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ the net is said to be *consistent*.

We will not consider here the most technical aspects, but it must be remarked that an immediate extension of the liveness concept may lead to weird situations. For example, a net that as discrete is non-live with any marking, as continuous can allow the firing of arbitrarily long firing sequences. (If a time base is added, the problem is similar to the discharge of an R-C electric circuit.) To avoid falling down into nonsense, in [8] the idea of a marking being reachable at the limit (lim-*reachability*) was introduced.

**Definition 1** *Let* $\langle \mathcal{N}, \mathbf{m_0} \rangle$ *be a continuous system. We say that a marking* $\mathbf{m} \in (\mathbb{R}^+ \cup \{0\})^{|P|}$ *is* lim-reachable*, iff a sequence of reachable markings* $\{\mathbf{m}_i\}_{i \geq 1}$ *exists verifying*

$$\mathbf{m_0} \xrightarrow{\sigma_1} \mathbf{m}_1 \xrightarrow{\sigma_2} \mathbf{m}_2 \cdots \mathbf{m}_{i-1} \xrightarrow{\sigma_i} \mathbf{m}_i \cdots$$

*and* $\lim_{i \to \infty} \mathbf{m}_i = \mathbf{m}$. *The set of* lim-*reachable markings is denoted* lim-RS($\mathcal{N}, \mathbf{m_0}$).

It can be proved that contrary to what happens in the discrete case, in most practical cases there are no spurious solutions of the state equation.

**Property 2 ([8])** *If* $\mathcal{N}$ *is consistent and every transition can be fired,* $\exists\, \boldsymbol{\sigma} > \mathbf{0}, \mathbf{m_0} \xrightarrow{\sigma}$*, then:*

$$\text{lim-RS}(\mathcal{N}, \mathbf{m_0}) = \{\mathbf{m} \mid \mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma} \geq \mathbf{0}, \boldsymbol{\sigma} \geq \mathbf{0}\}$$

Liveness and deadlock-freeness properties can immediately be extended.

**Definition 3** *Let* $\langle \mathcal{N}, \mathbf{m_0} \rangle$ *be a continuous PN system.*

- $\langle \mathcal{N}, \mathbf{m_0} \rangle$ lim-deadlocks *iff a marking* $\mathbf{m} \in$ lim-RS($\mathcal{N}, \mathbf{m_0}$) *exists such that* $\text{enab}(t, \mathbf{m}) = 0$ *for every transition* $t$

- $\langle \mathcal{N}, \mathbf{m_0} \rangle$ *is* lim-live *iff for every transition t and for any marking* $\mathbf{m} \in$ lim-RS$(\mathcal{N}, \mathbf{m_0})$ *a successor* $\mathbf{m}'$ *exists such that* $\text{enab}(t, \mathbf{m}') > 0$.
- $\mathcal{N}$ *is* structurally lim-live *iff* $\exists\ \mathbf{m_0}$ *such that* $\langle \mathcal{N}, \mathbf{m_0} \rangle$ *is lim-live.*

For the sake of notation, in the following we will shorten lim-deadlock, lim-liveness or structurally lim-liveness to just deadlock, liveness or str. liveness. Clearly, str. liveness is a necessary condition for liveness.

For the timing interpretation of *continuous* PNs we will use a first order (or deterministic) approximation of the markovian discrete case [6], assuming that the delays associated to the firing of transitions can be approximated by their mean values. Then, the state equation has an explicit dependence on time $\mathbf{m}(\tau) = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}(\tau)$. Deriving with respect to time, $\dot{\mathbf{m}}(\tau) = \mathbf{C} \cdot \dot{\boldsymbol{\sigma}}(\tau)$ is obtained. We will denote $\mathbf{f} = \dot{\boldsymbol{\sigma}}$, since it represents the *flow* through the transitions.

Different semantics have been defined for continuous PNs, the most important being *infinite servers* (or *variable speed*) and *finite servers* (or *constant speed*) [1, 6]. Infinite servers semantics will be considered here. In this paper we will concentrate on the class of nets for which a unique minimal T-semiflow exists [2].

**Definition 4** *A PN is a* mono -T-semiflow *(MTS) net iff it is conservative and has a unique minimal T-semiflow whose support contains all the transitions.*

## 3 Liveness analysis of autonomous continuous mono-T-semiflow nets

In MTS systems any subset of transitions, $T' \subset T$, can be disabled just by firing (indefinitely) every transition in $T'$.

**Lemma 5** *[4] Let* $\mathcal{N}$ *be a MTS net. For every* $\mathbf{m_0}$ *and every* $T' \subsetneq T$ *a marking* $\mathbf{m} \in$ lim-RS$(\mathcal{N}, \mathbf{m_0})$ *exists such that for all* $t \in T'$ $enab(t, \mathbf{m}) = 0$. *Moreover, this marking can be reached firing only transitions in* $T'$.

Notice that disabling a subset of transitions is not equivalent to killing them, since they could be enabled if other transitions not contained in the subset of disabled transitions are fired.

Lemma 5 leads to the equivalence between deadlock-freeness and liveness for continuous systems, something well-known for the discrete case [2].

**Property 6** *[4] A continuous MTS system is live iff it is deadlock-free.*

Suppose that in a given system, $\langle \mathcal{N}, \mathbf{m_0} \rangle$, there is a transition, $t$, such that for any reachable marking $t$ is never the only enabled transition. This means that if the rest of transitions, $T - \{t\}$, are disabled at a given marking $\mathbf{m}$, all the transitions are disabled at $\mathbf{m}$. Since every transition of the set $T - \{t\}$ can be disabled in the limit (Lemma 5), we can infer that $\langle \mathcal{N}, \mathbf{m_0} \rangle$ is not live.

**Theorem 7** *Let* $\langle \mathcal{N}, \mathbf{m_0} \rangle$ *be a MTS system. If* $\langle \mathcal{N}, \mathbf{m_0} \rangle$ *is live then, for every transition t,* $\exists\ \mathbf{m} \in$ lim-RS$(\mathcal{N}, \mathbf{m_0})$ *such that t is the only enabled transition at* $\mathbf{m}$.

Theorem 7 establishes a necessary liveness condition that is illustrated in Figure 3. In that system, for every reachable marking in which $t_2$ is enabled either $t_3$ or $t_4$ is enabled. Hence, $t_2$ is never unavoidably "forced" to fire. Firing several times $t_3$ and $t_4$ we reach a deadlock.

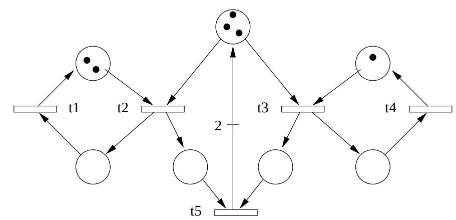


**Figure 3:** A non-live system according to Theorem 7

Although the condition of Theorem 7 is in general not easy to check, a simple structural condition (i.e. applicable independently of the initial marking) can be extracted.

**Corollary 8** *Let* $\mathcal{N}$ *be a MTS net. If* $\mathcal{N}$ *is str. live then for every* $t \neq t'$, $^\bullet t \not\subseteq {}^\bullet t'$

**Proof:** If there exist $t \neq t'$ such that $^\bullet t \subseteq {}^\bullet t'$, for every marking in which $t'$ is enabled, $t$ is also enabled. Thus, Theorem 7 can be directly applied and non-liveness for an arbitrary initial marking is deduced. ■

Hence, topological conflicts in which the set of input places of one transition are contained in the set of input places of other transition must be forbidden if the net is wanted to be live. For example in the system in Figure 4, for any reachable marking if $t_2$ is enabled then $t_1$ is also enabled. Firing $t_3$ and then $t_1$ with their maximal enabling degree, a deadlock is reached for any initial marking. Remarkably this system is live if seen as discrete!

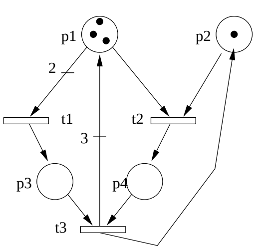


**Figure 4:** A system for which Corollary 8 detects non-liveness

In other words, Corollary 8 detects a kind of "structural contradiction" in the MTS net: on the one hand all transitions are included in the only repetitive sequence (the T-semiflow), and on the other hand there exists $t \neq t'$ such that $^\bullet t \subseteq {}^\bullet t'$, thus, the net gives the possibility of never firing transition $t'$. The result of this contradiction entails a deadlock.

## 4 Analysis of timed continuous mono-T-semiflow nets

### 4.1 Steady state and timed-liveness

Under infinite servers semantics, the flow through a transition $t$ is the product of the firing speed, $\lambda[t] > 0$, and the enabling degree of the transition, i.e., $\mathbf{f}[t] = \lambda[t] \cdot \text{enab}(t, \mathbf{m}) = \lambda[t] \cdot \min_{p \in {}^\bullet t}\{\mathbf{m}[p]/\mathbf{Pre}[p,t]\}$, leading to non-linear ordinary differential and deterministic systems. A continuous timed system will be represented as $\langle \mathcal{N}, \lambda, \mathbf{m_0} \rangle$.

Although there is not a general formal proof, the research community widely accepts that *under the infinite servers semantics any bounded system always reaches a steady state in which the flow through transitions and the marking remain constant.* This is obviously true for Join Free (JF) nets, for which the timed evolution is ruled by a single linear system. Notice that a system under other firing semantics may never reach a steady state. For example, when the flow through a transition is defined as the product of the markings of the input places, the system may describe orbits and even chaotic behaviors [9].

In the sequel we will assume that any timed MTS system evolves through a transient and eventually reaches a steady state.

A performance measure that is often used in discrete PN systems is the throughput of a transition in the steady state, i.e., the number of firings per time unit. In the continuous approximation, this corresponds to the flow of the transition. Observe that in the steady state $\dot{\mathbf{m}}(\tau) = 0$, and so, from the state equation, $\mathbf{C} \cdot \mathbf{f}_{ss} = \mathbf{0}$ where $\mathbf{f}_{ss}$, or more explicitly $\mathbf{f}_{ss}(\mathcal{N}, \lambda, \mathbf{m_0})$, is the flow vector of the timed system in the steady state, $\mathbf{f}_{ss} = \lim_{\tau\to\infty} \mathbf{f}(\tau)$. Since $\mathbf{f}_{ss} \geq \mathbf{0}$, the flow in the steady state is proportional to the T-semiflow. Let us denote as $\mathbf{m_{ss}}$ the marking at the steady state.

A classical concept in queueing network theory is the *visit ratio.* The visit ratio of transition $t_j$ with respect to $t_i$, $\mathbf{v}_j^{(i)}$, is the average number of times $t_j$ is visited (fired) per visit to (firing of) the reference transition $t_i$. Observe that $\mathbf{v}^{(i)}$ is a "normalization" of the flow vector in the steady state, i.e., $\mathbf{v}_j^{(i)} = \lim_{\tau\to\infty}(\mathbf{f}[t_j](\tau)/\mathbf{f}[t_i](\tau))$.

Liveness definitions of autonomous systems can be extended to timed systems. We will say that $\langle \mathcal{N}, \lambda, \mathbf{m_0} \rangle$ is *timed-live* iff $\mathbf{f}_{ss}(\mathcal{N}, \lambda, \mathbf{m_0}) > \mathbf{0}$. The addition of firing speeds, $\lambda$, to a net $\mathcal{N}$ produces a timed-net, $\langle \mathcal{N}, \lambda \rangle$. We will say that $\langle \mathcal{N}, \lambda \rangle$ is *str. timed-live* iff there exists an initial marking $\mathbf{m_0}$ such that $\mathbf{f}_{ss}(\mathcal{N}, \lambda, \mathbf{m_0}) > \mathbf{0}$. As in autonomous nets str. timed-liveness is a necessary condition for timed-liveness.

If a MTS timed system deadlocks, we can conclude that, seen as autonomous, the system is non-live since the evolution of the timed system just gives a particular trajectory, i.e. a firing sequence, that can be fired in the autonomous system reaching the same deadlock marking. Therefore liveness is a sufficient condition for timed-liveness. The reverse is not true (Figure 2(a) for $\lambda_1 = \lambda_2$). Analogously, str. liveness is a sufficient condition for str. timed-liveness. Relationships among liveness definitions are depicted in Figure 5.

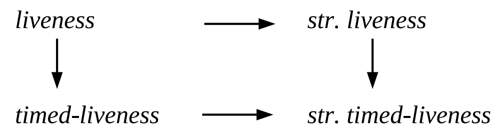


**Figure 5:** Relationships among liveness definitions for MTS models

### 4.2 $\lambda$'s influence on the existence of a non-dead steady state

The $\lambda$ vector plays a crucial role in the evolution to the steady state. Even str. non-live systems can be saved from deadlocking by choosing an adequate $\lambda$. One possible idea is to chose a $\lambda$ that avoids any transient state, thus making the initial marking equal to the marking in the steady state and therefore avoiding a deadlock, thus a positive initial marking is required for that.

**Proposition 9** *[4] Given a MTS net, $\mathcal{N}$, for every initial marking $\mathbf{m_0} > \mathbf{0}$, there exists $\lambda$ such that $\langle \mathcal{N}, \lambda, \mathbf{m_0} \rangle$ is timed-live.*

For example, the continuous system in Figure 6 is non-live as autonomous. However, defining $\lambda = (1\ 1\ \lambda_3)$ the timed system never deadlocks.

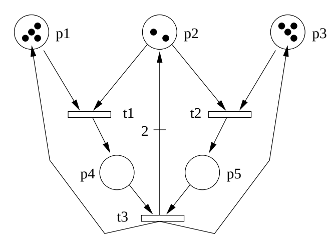


**Figure 6:** A non-live autonomous system that never deadlocks as timed with $\lambda = (1\ 1\ 1)$. Every transition owns a CF place but the timed system deadlocks with $\lambda = (2\ 1\ 1)$

Another interesting problem consists in determining which timed-nets are str. timed-live (i.e. given $\langle \mathcal{N}, \lambda \rangle$, $\exists\ \mathbf{m_0}$ such that $\mathbf{f}_{ss}(\mathcal{N}, \lambda, \mathbf{m_0}) > \mathbf{0}$?)

**Proposition 10** *[4] $\langle \mathcal{N}, \lambda \rangle$ is str. timed-live iff $\mathbf{m}$ defined as*

$$\mathbf{m}[p] = \max_{t \in p^\bullet} \left\{ \frac{\mathbf{Pre}[p,t] \cdot \mathbf{v}_t^{(1)}}{\lambda[t]} \right\}$$

*is a steady state marking for $\langle \mathcal{N}, \lambda \rangle$.*

Remark that $\mathbf{m}$, defined according to Proposition 10, is a steady state marking with $\mathbf{f}_{ss} = \mathbf{v}^{(1)}$ iff the following condition holds:

$$\forall\ t\ \ \exists\ p \in {}^\bullet t\ \ \mathbf{m}[p] = \frac{\mathbf{Pre}[p,t] \cdot \mathbf{v}_t^{(1)}}{\lambda[t]}$$

From Proposition 10 we obtain that the net in Figure 4 is not str. timed-live with $\lambda = (4\ 1\ 1)$. In the steady state marking ($\mathbf{m_{ss}}$) the flow of transitions $t_1$ and $t_2$ has to be the same since the T-semiflow of the net is $(1\ 1\ 1)$. Since $\lambda[t_1]$ is four times greater than $\lambda[t_2]$, then $\text{enab}(t_2, \mathbf{m_{ss}}) = 4 \cdot \text{enab}(t_1, \mathbf{m_{ss}})$ is required. This is not possible since for every marking $\mathbf{m}$, $\text{enab}(t_2, \mathbf{m}) \leq 2 \cdot \text{enab}(t_1, \mathbf{m})$. Therefore, no strictly positive marking verifies the steady state conditions, and so for this $\lambda$ the system will always deadlock.

### 4.3 Characterization of the $\mathbf{\Lambda}_{\mathcal{N}}$ set

Given $\mathcal{N}$, an interesting problem lies in determining the set of $\lambda$ vectors for which $\langle \mathcal{N}, \lambda \rangle$ is str. timed-live. In other words, we are interested in computing a set defined as follows:

**Definition 11** $\mathbf{\Lambda}_{\mathcal{N}} \equiv \{\lambda \mid \langle \mathcal{N}, \lambda \rangle$ *is str. timed-live* $\}$.

We have seen that if $\mathcal{N}$ is str. live then for any $\lambda$, $\langle \mathcal{N}, \lambda \rangle$ is str. timed-live. Hence for str. live nets $\mathbf{\Lambda}_{\mathcal{N}}$ will be equal to all positive real vectors ($\mathbf{\Lambda}_{\mathcal{N}} \equiv (\mathbb{R}^+)^{|T|}$).

We will show that the computation of $\mathbf{\Lambda}_{\mathcal{N}}$ can be simplified by considering separately each coupled conflict set. An easy to state formula can be used to express all the vectors contained in it.

Let us suppose that $\mathcal{N}$ has $q$ coupled conflict sets, $CCS_1 \ldots CCS_q$ with $|CCS_1| = n_1 \ldots |CCS_q| = n_q$, $|{}^\bullet CCS_1| = s_1 \ldots |{}^\bullet CCS_q| = s_q$, and that transitions and places are sorted according to the coupled conflict they belong to: $t_{1,1} \ldots t_{1,n_1}, \ldots, t_{q,1} \ldots t_{q,n_q}$ and $p_{1,1} \ldots p_{1,s_1}, \ldots, p_{q,1} \ldots p_{q,s_q}$. We define $\mathbf{\Lambda_{CCS}}$ as the a set of vectors associated to each coupled conflict set as follows:

**Definition 12** $\mathbf{\Lambda}_{CCS_j} \equiv \{\lambda_j \mid \lambda_j \in (\mathbb{R}^+)^{n_j}$ *and* $\exists\ \mathbf{m} \in (\mathbb{R}^+)^{s_j}$ *such that* $\forall\ t \in CCS_j \quad \mathbf{v}_t^{(1)} = \lambda_j[t] \cdot \text{enab}(t, \mathbf{m})\}$

$\mathbf{\Lambda}_{\mathcal{N}}$ can be expressed as the cartesian product of all the $\mathbf{\Lambda_{CCS}}$ of the net.

**Theorem 13** *[4]*

$$\mathbf{\Lambda}_{\mathcal{N}} \equiv \{(\lambda_{1,1}, \ldots, \lambda_{q,n_q}) \mid (\lambda_{i,1}, \ldots, \lambda_{i,n_i}) \in \mathbf{\Lambda}_{CCS_i}\}$$

### 4.4 Critical timed-liveness

It has been seen that those $\lambda$ vectors not included in $\mathbf{\Lambda}_{\mathcal{N}}$ do not allow a MTS system to reach a steady state with throughput greater than zero. Although $\mathbf{\Lambda}_{\mathcal{N}}$ is never an empty set (for every positive initial marking there exists $\lambda \in \mathbf{\Lambda}_{\mathcal{N}}$), its "size" can be much smaller than desired. For example it is not desirable to use a vector of $\mathbf{\Lambda}_{\mathcal{N}}$ such that a minimum change in one of its components puts the vector out of $\mathbf{\Lambda}_{\mathcal{N}}$. It would mean that a small variation in the firing speed of one transition can kill the system. Hence, a new concept is needed to define wether a system can be robust enough to bear the irregularities and variations of real world.

**Definition 14** $\langle \mathcal{N}, \lambda \rangle$ *is critically str. timed-live iff* $\lambda$ *is a border point of* $\mathbf{\Lambda}_{\mathcal{N}}$

In some cases the net structure can reduce dramatically the dimension of $\mathbf{\Lambda}_{\mathcal{N}}$. For every coupled conflict with $n$ transitions $\mathbf{\Lambda_{CCS}}$ is contained in $(\mathbb{R}^+)^{|n|}$. Therefore the maximum dimension that a given region of the $\mathbf{\Lambda_{CCS}}$ set can have is $n$. Apart from this constraint, the effective dimension of $\mathbf{\Lambda_{CCS}}$ is also limited by the number of input places of the coupled conflict set, since $\mathbf{\Lambda_{CCS}}$ is generated by as many independent variables as input places So the maximum dimension of any region of $\mathbf{\Lambda_{CCS}}$ is bounded by the number of places and transitions in the coupled conflict set.

If there is no region in $\mathbf{\Lambda}_{\mathcal{N}}$ whose dimension is equal to the number of transitions, all the points in $\mathbf{\Lambda}_{\mathcal{N}}$ are in fact border points. For example if $\mathcal{N}$ has a CCS with less input places than transitions (as $\{t_1, t_2\}$ in Figure 2(a)), all the the points in $\mathbf{\Lambda}_{\mathcal{N}}$ are border points. In that case for a given $\lambda$, if $\lambda \notin \mathbf{\Lambda}_{\mathcal{N}}$ then for every initial marking the system will finally deadlock, and if $\lambda \in \mathbf{\Lambda}_{\mathcal{N}}$ then $\langle \mathcal{N}, \lambda \rangle$ is critically str. timed-live.

The above means that in practice, all the coupled conflicts sets should have at least as many input places as transitions, otherwise the system will die or will remain in a critical timed-liveness state.

In comparison with critical str. timed-liveness, one could ask which features should be required to a system in order to be safe, i.e. arbitrary variations in $\lambda > 0$ do not cause $\lambda$ to be out of $\mathbf{\Lambda}_{\mathcal{N}}$. In other words, we are looking for those net structures that for any $\lambda$ they allow a non-dead steady state.

A place $p$ is said to be *choice-free (CF)* iff $|p^\bullet| = 1$, i.e. $p$ has a single output transition. We will say that a transition *owns* its (input) CF places.

**Theorem 15** *[4]* $\forall\ \lambda > 0$ $\langle \mathcal{N}, \lambda \rangle$ *is str. timed-live iff every transition owns at least one CF place.*

From a different point of view, Theorem 15 states that the transitions can be enabled independently iff every transition owns a CF place. Remark that this condition does not guarantee that the system will always reach a non-dead steady state for every initial marking. For example in Figure 6, every transition owns a CF place (and it is not a CF net). However, choosing $\lambda = (2\ 1\ 1)$ the system cannot reach a live steady state in which the flow of $t_1$ and $t_2$ is the same. This happens because the enabling degree of $t_1$ and $t_2$ is always the same, since $p_1$ and $p_3$ are implicit places [7], so they can be removed without changing the possible behaviors (trajectories for the timed case) of the system. From Theorem 15 it can be inferred that for those nets that have a transition without CF places there exists a $\lambda$ for which the timed system deadlocks independently of the initial marking. Transitions $\{t_1, t_2, t_3\}$ in Figure 7 do not own CF places. For $\lambda = (1\ 1\ 2\ 1\ 1\ 1)$, $\lambda \notin \mathbf{\Lambda}_{\mathcal{N}}$, the system will evolve to a deadlock.

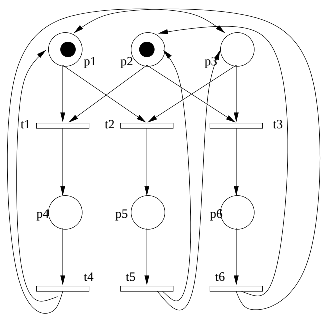


**Figure 7:** Transitions $t_1, t_2$ and $t_3$ do not own CF places. There exists a $\lambda = (1\ 1\ 2\ 1\ 1\ 1)$ for which no steady state with positive throughput is possible for the timed system.

## 5 Coming back to structural liveness

According to Theorem 15, if $\mathcal{N}$ has a transition without CF places, a $\lambda$ exists such that $\langle \mathcal{N}, \lambda \rangle$ is not str. timed-live. Therefore $\mathcal{N}$ is not str. live, since str. timed-liveness is a necessary condition for str. liveness.

**Theorem 16** *Let $\mathcal{N}$ be a MTS net. If $\mathcal{N}$ is str. live then every transition owns at least one CF place.*

The system shown in Figure 7 is non-live according to Theorem 16, since there are three transitions, $t_1, t_2$ and $t_3$ that do not own a CF place. In this case, the firing of a sequence that corresponds to vector $\sigma = (0\ 1\ 0\ 0\ 1\ 0)$ leads to marking $\mathbf{m} = (0\ 2\ 0\ 0\ 0\ 0)$, where the system deadlocks. Notice that in consistent continuous systems in which every transition can be fired at least once, there do not exist spurious solutions of the state equation (Proposition 2), hence a sequence can be fired that reaches marking $\mathbf{m}$.

Transition $t$ does not own a CF place iff all the input places of $t$ are contained in the set of input places of the rest of transitions. Thus, Theorem 16 can be rewritten as: If $\mathcal{N}$ is str. live then for every $t$, $^\bullet t \not\subseteq {}^\bullet(T \setminus \{t\})$. Notice the similarity of this statement to that of Corollary 8. In fact, Theorem 16 and Corollary 8 express exactly the same condition if all the coupled conflict sets of the net have at most two transitions, for the rest of nets Theorem 16 imposes a stronger condition.

## 6 Conclusions

In MTS systems the firing of all the transitions is required to make the system live. If the firing of a transition can be avoided from any reachable marking, the autonomous system is not live. Corollary 8 translates this fact to a necessary str. liveness condition.

The addition of time to the basic formalism entails new liveness definitions. Timing restricts considerably the behavior of systems, to the point that an adequate timing can "save" a non str. live system from deadlocking. Str. timed-live nets have been fully characterized as those timed- nets that allow a non-dead steady state. The existence of a relationship between autonomous and timed-liveness definitions allows to obtain a new liveness condition: If a net is str. live then every transition, $t$, has a input place whose only output transition is $t$.

In [7] the rank Theorem states a necessary str. liveness condition for general nets. The application of the rank theorem to continuous MTS systems yields that if a net has two transitions in equal conflict relation then the net is not str. live. Theorem 16 improves this necessary condition by disregarding arc weights at conflicts and paying attention exclusively to topological features.

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