Autonomous Continuous P/T Systems

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Abstract. Discrete event dynamic systems may have extremely large state spaces. For their analysis, it is usual to relax the description by removing the integrality constraints. Applying this idea, continuous P/T systems are defined by allowing fractional firings of transitions, and thus the existence of non-discrete markings [4, 5, 1]. In this paper we compare the behaviors of discrete and continuous systems, and observe that they are not necessarily similar. The problems that appear lead to the definition of two extensions of reachability. Many properties shall be extended differently depending on which reachability definition is being considered. Here, we concentrate on liveness and deadlock-freeness, proposing extensions and relating them to their discrete counterparts.

1 Introduction

One of the most important tools for the analysis of P/T systems is the state equation, which is based on the relaxation of the reachability condition using a *path integration* approach. This description is sometimes further relaxed by dropping integrality constraints, following the approach that is typical in the mathematical modeling of systems with large state spaces (e.g., population models). This *fluidization* allows to use *linear* programming instead of *integer* programming in the verification of certain properties.

These principles can also be applied in the reverse order, first continuization and then path integration. By disregarding first the integrality of variables, we get *continuous P/T systems* [4, 5, 1]. In these models, "fluid tokens" are contained in "deposits" (the places), the "level" of which (the marking) captures the state of the system. Transitions are regarded as "mixing valves" whose firing (opening) consumes fluid from the input places and produces fluid onto the output places in a given proportion, defined by the arc weights. These nets are interesting in the modeling of certain continuous systems, and also as an approximation of systems with large amounts of (discrete) tokens.

Autonomous continuous P/T systems were introduced in [4]. Although some work has been done in the analysis of timed continuous P/T systems [4, 5, 1], almost nothing has been done w.r.t. the analysis of autonomous continuous P/T systems. It might be thought that they cannot be that different from discrete

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P/T systems. However we will see that the behavior of a system considered as discrete may be completely different from its behavior if it is considered as continuous. Moreover, although the extension of the "token game" to continuous P/T systems is quite immediate, it is not so clear how such a basic concept as reachability should be extended. For instance, should be considered reachable a marking that cannot be obtained firing a finite sequence, but is obtained after an infinitely long one?

The basic definitions of autonomous continuous P/T nets and systems are introduced in Section 2. Some immediate properties are also proven, as the convex nature of the reachability space, or the equivalence, in case every transition is fireable, of behavioral and synchronic relations (in particular boundedness and str. boundedness). In Section 3 some examples are presented which show that the properties of a system may be very different depending on whether it is considered as discrete or continuous. A new definition of reachability, *limit reachability*, in which infinitely long firing sequences are allowed, is introduced in Section 4. If every transition is fireable, the limit reachability space of consistent systems is characterized as the set of solutions of the state equation. Section 5 is devoted to the analysis of liveness. Two definitions of liveness are introduced that correspond to the two views of reachability (Subsection 5.1). In Subsection 5.2 it is observed that both kinds of liveness are preserved if the marking is scaled. The relationship that exists among all the definitions of liveness (the two continuous definitions and the discrete one) is analyzed in Subsection 5.3. In Subsection 5.4 two necessary conditions are obtained for the liveness definition that seems more convenient. Finally, we restrict to subclasses, in particular, equal conflict [13], and free choice [6] systems, for which stronger results can be proved (Subsection 5.5).

2 Definition and first results

A continuous P/T system is a pair $\langle \mathcal{N}, \mathbf{m_0} \rangle$, where $\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post} \rangle$ is a P/T net (P and T are disjoint (finite) sets of *places* and *transitions*, and $\mathbf{Pre}$ and $\mathbf{Post}$ are $|P| \times |T|$ sized, natural valued, *incidence matrices*), and $\mathbf{m_0}$ is a *continuous marking.*

The net in a continuous P/T system is the usual P/T net. In particular,the restriction on the arc weights being integer is maintained. This is particularly reasonable when the continuous P/T system is used as an approximation of a discrete system. In the case of continuous (or hybrid) P/T systems used to model continuous systems, the integrality of the arc weights is not a big restriction because rational arc weights could be multiplied by the least common multiple of their denominators.

All the concepts based on the representation of the P/T net as a graph (strong connectedness, presets, postsets, ...) can be directly applied to continuous P/T nets. In particular, the definitions based on the annullers of the token-flow matrix ($\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$) can be immediately extended. Right and left natural annullers are called T- and P-semiflows, respectively. When $\mathbf{y} \cdot \mathbf{C} = \mathbf{0}$, $\mathbf{y} > \mathbf{0}$ the net is said to be *conservative*, and when $\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ the net is said to be *consistent.* The definitions of subclasses that depend only on the structure of the net are also generalized. For instance, in choice free nets (CF) each place has at most one output transition, free choice nets (FC) are ordinary nets in which all conflicts are equal ($^\bullet t \cap {}^\bullet t' \neq \emptyset \Rightarrow \mathbf{Pre}[P,t] = \mathbf{Pre}[P,t']$), and equal conflict nets (EQ) are the weighted counterpart of FC nets.

A continuous marking is a $|P|$ sized, non-negative, *real* valued, vector. A *continuous P/T system* is a pair $\mathcal{S} = \langle \mathcal{N}, \mathbf{m_0} \rangle$, where $\mathbf{m_0}$ is the initial continuous marking. A transition t is *enabled* at $\mathbf{m}$ iff for every $p \in {}^\bullet t$, $\mathbf{m}[p] > 0$. In other words, the enabling condition of continuous systems is the same as the enabling condition of discrete ordinary systems: every input place is marked. As in discrete systems, the *enabling degree* of a transition measures the maximal amount in which the transition can be fired in one go, i.e. $\text{enab}(t, \mathbf{m}) = \min_{p \in {}^\bullet t}\{\mathbf{m}[p]/\mathbf{Pre}[p,t]\}$. The firing of t in a certain amount $\alpha \leq \text{enab}(t, \mathbf{m})$ leads to a new marking $\mathbf{m}' = \mathbf{m} + \alpha \cdot \mathbf{C}[P,t]$. This is denoted as $\mathbf{m} \xrightarrow{\alpha t} \mathbf{m}'$. Notice that a transition being enabled or not does not depend on the arc weights, although they are important to compute the enabling degree and to obtain the new marking. A certain marking $\mathbf{m}'$ is *reachable* from $\mathbf{m}$ if a (finite) fireable sequence exists leading from $\mathbf{m}$ to $\mathbf{m}'$.

Definition 1. *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a continuous system. A certain marking $\mathbf{m} \in (\mathbb{R}^+ \cup \{0\})^{|P|}$ is* reachable *iff a finite sequence $\sigma = \alpha_1 t_1 \alpha_2 t_2 \ldots \alpha_k t_k$, exists such that*

$$\mathbf{m_0} \xrightarrow{\alpha_1\, t_1} \mathbf{m}_1 \xrightarrow{\alpha_2\, t_2} \mathbf{m}_2 \cdots \xrightarrow{\alpha_k\, t_k} \mathbf{m}_k = \mathbf{m}$$

where $t_i \in T$ and $\alpha_i \in \mathbb{R}^+$.

The reachability space, $\text{RS}_\text{C}(\mathcal{N}, \mathbf{m_0})$, *is the set of all the reachable markings.*

Given σ such that $\mathbf{m} \xrightarrow{\sigma} \mathbf{m}'$, and denoting by $\boldsymbol{\sigma}$ the firing count vector of σ, then $\mathbf{m}' = \mathbf{m} + \mathbf{C} \cdot \boldsymbol{\sigma}$. This is known as the state equation *of $\mathcal{S}$.*

The set of all the markings $\mathbf{m} \in (\mathbb{R}^+ \cup \{0\})^{|P|}$ that fulfil the state equation, with $\boldsymbol{\sigma} \in (\mathbb{R}^+ \cup \{0\})^{|T|}$, is called the linearized reachability space *(w.r.t. the state equation),* $\text{LRS}_\text{C}(\mathcal{N}, \mathbf{m_0})$.

If $\mathbf{m_0} \xrightarrow{\sigma} \mathbf{m}$, then $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}$. Thus, as in discrete P/T systems, $\text{RS}_\text{C} \subseteq \text{LRS}_\text{C}$.

The possibility of firing the transitions in any amount (up to the enabling degree) leads to the fulfilment of several properties on the set of fireable sequences related to *homothecy* and *monotony*:

Proposition 1. *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a continuous P/T system, and σ a sequence fireable at $\mathbf{m_0}$.*

- *For any $\alpha \geq 0$, $\alpha\sigma$ is fireable at $\alpha\mathbf{m_0}$, where $\alpha\sigma$ represents a sequence that is equal to σ except in the amount of each firing, that is multiplied by α.*
- *If $0 \leq \alpha \leq 1$, $\alpha\sigma$ is fireable at $\mathbf{m_0}$.*
- *For any $\mathbf{m_0}' \geq \mathbf{m_0}$, σ is fireable at $\mathbf{m_0}'$.*

This endows the reachability space with a particular structure that it does not have in discrete systems: it is a *convex set*. That is, for any two markings that can be reached from $\mathbf{m_0}$, any intermediate marking that can be expressed as their linear combination is reachable too.

Theorem 1. *The reachability space of a continuous P/T system is a convex set.*

Proof. Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a continuous system and $\mathbf{m}_1, \mathbf{m}_2$ two reachable markings, i.e., $\mathbf{m_0} \xrightarrow{\sigma_1} \mathbf{m}_1$ and $\mathbf{m_0} \xrightarrow{\sigma_2} \mathbf{m}_2$. Let $\alpha \in [0,1]$. Then, $\alpha \mathbf{m_0} \xrightarrow{\alpha\sigma_1} \alpha \mathbf{m}_1$ and $(1-\alpha)\mathbf{m_0} \xrightarrow{(1-\alpha)\sigma_2} (1-\alpha)\mathbf{m}_2$. Therefore, $\alpha \mathbf{m}_1 + (1-\alpha)\mathbf{m}_2$ is reachable from $\mathbf{m_0}$ firing $\alpha\sigma_1 + (1-\alpha)\sigma_2$. $\square$

The same idea of firing just a part of what is enabled is the basis of the following algorithm, that checks whether every transition is fireable at least once.

The algorithm fires the enabled transitions, which can lead to the enabling of other transitions, but taking care not to disable any of the former. Thus, the set of enabled transitions, T^j, never decreases. If it does not increase, a point has been reached in which the firing of the enabled transitions cannot lead to the enabling of any other one, therefore not every transition can be fired. Otherwise, since the number of transitions is finite, the algorithm stops when all have been considered.

Algorithm 1

> **Input:** A continuous P/T system, $\langle \mathcal{N}, \mathbf{m_0} \rangle$
> **Output:** The set of dead transitions, T'
> **Begin**
> Let $T^0 = \emptyset$;
> Let $T^1 = \{t \mid \text{enab}(t, \mathbf{m_0}) > 0\}$
> $j := 1$
> **While** $T^j \neq T$ and $T^j \neq T^{j-1}$ **do**
> Let σ_j be a sequence obtained firing all the transitions in $T^j \setminus T^{j-1}$ with half their enabling degree, and let $\mathbf{m}_{j-1} \xrightarrow{\sigma_j} \mathbf{m}_j$
> $T^{j+1} = \{t \mid \text{enab}(t, \mathbf{m}_j) > 0\}$
> $j := j + 1$
> **od**
> $T' := T \setminus T^j$
> **End**

In other words, in continuous systems it is equivalent that every transition is fireable or that a strictly positive marking can be reached. From this, realizability of T-semiflows can be deduced, and all the three are equivalent if the net is consistent.

Proposition 2. *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a continuous system.*

1. *It is equivalent that every transition is fireable or that a strictly positive marking can be reached.*
2. *If every transition is fireable, for every $\mathbf{x} \geq \mathbf{0}$ such that $\mathbf{C} \cdot \mathbf{x} \geq \mathbf{0}$ a marking $\mathbf{m} \in \mathrm{RS_C}(\mathcal{N}, \mathbf{m_0})$ exists such that $\mathbf{m} \xrightarrow{\sigma}$ and $\boldsymbol{\sigma} = \alpha \mathbf{x}$ with $\alpha > 0$. Moreover, both properties are equivalent if the net is consistent.*

Fireability of T-semiflows, implies that behavioral and structural synchronic relations [11] coincide in continuous systems in which every transition is fireable at least once. In particular, defining boundedness and str. boundedness as in discrete systems (a system is bounded iff $\mathbf{k}$ exists such that for every reachable marking $\mathbf{m} \leq \mathbf{k}$, and it is str. bounded iff it is bounded with every initial marking) it is immediate to see that that both concepts coincide in continuous systems in which every transition is fireable. And, as in discrete systems, str. boundedness is equivalent to the existence of $\mathbf{y} > \mathbf{0}$ such that $\mathbf{y} \cdot \mathbf{C} \leq \mathbf{0}$ (Farkas Lemma [8]).

Theorem 2. *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a continuous system in which every transition is fireable at least once. It is equivalent:*

- *$\langle \mathcal{N}, \mathbf{m_0} \rangle$ is bounded.*
- *$\mathcal{N}$ is bounded with any initial marking (str. bounded).*
- *$\mathbf{y} > \mathbf{0}$ exists such that $\mathbf{y} \cdot \mathbf{C} \leq \mathbf{0}$.*

3 Discrepancies between continuous and discrete behaviors

The simple way in which the basic definitions of discrete systems are extended to continuous systems may make us naively think that their behavior cannot be very different, provided the marking is "large enough". We will see in this section that this is not completely true.

For example, look at the system in Figure 1. Each time t_1 and t_2 are fired in their maximal enabling degree, the marking of p_1 is cut by half. Thus, we can always find a marking such that for every successor the marking of p_1 is as small as required. But it never reaches zero (remember the Zenon's paradox). Therefore, in the continuous system we can always go on firing transitions t_1 and t_2, while that is clearly not true in the discrete system, no matter how big the initial marking is.

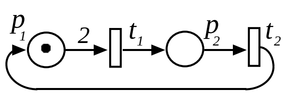


Fig. 1. A non str. live discrete system that never blocks if it is seen as continuous.

Nevertheless, the enabling degree of these continuous transitions decreases with each firing, and it could be thought that the continuous behavior is not that different from the discrete one: in the end a marking will be reached such that for every successor the enabling degree of both transitions is "almost zero". But this is a very simple system, and things may get much more entangled. Observe the system in Figure 2 (a). If we analyze this net as a discrete P/T net,

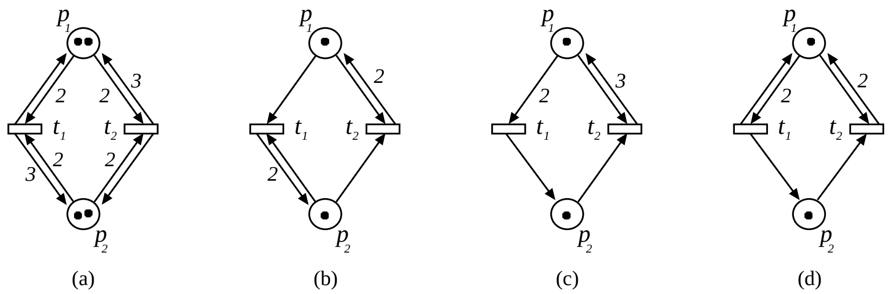


Fig. 2. Four bounded and strongly connected systems are which behave in a very different way if they are considered as discrete or as continuous:
(a) is non str. live, ϵ-live, non str. lim-live;
(b) is non str. live, non ϵ-live, non str. lim-live;
(c) is live, non str. ϵ-live, non str. lim-live;
(d) is live, ϵ-live, non str. lim-live.

it is not str. live. For any initial marking, t_1 or t_2 can be fired sufficiently many times to reach a marking in which p_1 or p_2 are marked with just one token, which is clearly a deadlock. On the other hand, the continuous system displays a completely different behavior. For instance, with the given initial marking, firing the sequence $t_1 \, \frac{1}{2}t_1 \, \frac{1}{4}t_1 \, \frac{1}{8}t_1 \ldots$ we can obtain a marking such that the enabling of t_1 and t_2 is as small as desired (observe that the marking of p_1 decreases exponentially). However, and unlike the previous example, the marking of p_1 is not unavoidably led to zero, since it can be increased again firing t_2.

The reason for the completely different behavior of the discrete and the continuous system in this case is that the continuous system considers the tokens as composed of *infinitely many* parts, and hence, we do not find the restriction that in the discrete system leads to not being able to redistribute the tokens. In other words, the "problem" is that the gap between natural numbers is discrete, and thus any decreasing sequence of natural numbers eventually reaches a lower bound, in which it may get blocked. This is not true in general for sequences of rational/real numbers.

This example shows also that systems that might be considered "equivalent" as discrete systems (their reachability graphs are isomorphic) may not be so if they are seen as continuous systems. From a discrete systems perspective, there is no difference between systems (a) and (b) in Figure 2. (In particular, the token-flow matrix of both systems is the same, although the **Pre** and **Post** matrices are different.) However, their behaviors as continuous systems are completely different. The system on the right follows the behavior of the discrete system: for any initial marking firing t_1 or t_2 in a large enough amount we reach a deadlock. On the contrary, we have seen that the system on the left never deadlocks with the firing of a finite sequence.

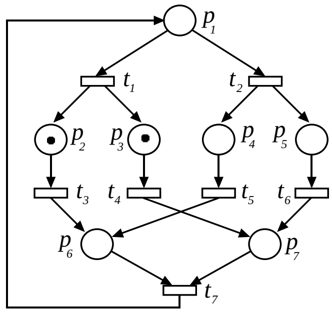
It may also happen that a system deadlocks if it is seen as continuous and does not deadlock as discrete. Look for instance at the system in Figure 2 (c). As a discrete system, it is live with the given marking. However for any initial marking, $\mathbf{m_0}$, the firing of t_1 in an amount of $\mathbf{m_0}[p_1]/2$ in the continuous system leads to $[0, \mathbf{m_0}[p_1]/2 + \mathbf{m_0}[p_2]$ and the system gets blocked.

On the other hand, the system in Figure 2 (d), which as discrete has a reachability graph isomorphic to the one of the system in Figure 2 (c), as continuous never reaches a marking with no transition enabled. Observe that although these two systems have the same behavior as discrete, there is a big difference in the underlying nets: the system in Figure 2 (c) is non live with any initial marking with an even number of tokens in p_1; while any marking greater than or equal to $[1, 1]$ makes live the system in Figure 2 (d). It is clear that a system cannot be live as continuous if its liveness as discrete relies strongly on the particular marking. That is, a system that is live as discrete with a certain marking, but not live with a multiple of it, cannot be live as continuous. Scaling liveness monotonicity, which may be desirable, but is not compulsory in discrete systems, appears a basic property if we want to study them as continuous systems.

With the different systems in Figure 2, we have seen that, even in the case of EQ nets, the behaviors of a system, if it is considered as discrete or continuous, do not necessarily coincide. We wonder whether in simpler classes, such as live and bounded FC or CF systems, discrete and continuous behaviors are analogous. Let us consider a basic property of discrete bounded systems: no infinite firing sequence exists in which the markings are all different. Two example systems, one FC and the other CF, both live, bounded and reversible (the initial marking can always be returned to), are shown in Figure 3. In both, even this simple property is violated. For the FC system on the left, all the intermediate markings when the sequence $t_3 t_4 t_7 \, \frac{1}{2}t_1 \frac{1}{2}t_2 \frac{1}{2}t_4 \frac{1}{2}t_5 \frac{1}{2}t_7 \, \frac{1}{4}t_1 \frac{1}{4}t_2 \frac{1}{4}t_4 \frac{1}{4}t_5 \frac{1}{4}t_7 \ldots \frac{1}{2^k}t_1 \frac{1}{2^k}t_2 \frac{1}{2^k}t_4 \frac{1}{2^k}t_5 \frac{1}{2^k}t_7 \ldots$ is fired, are different. The same happens to the CF system on the right if we fire the sequence $t_1 \, \frac{1}{2}t_3 \frac{1}{2}t_1 \, \frac{1}{4}t_3 \frac{1}{4}t_1 \ldots$. This is completely different from what happens in discrete systems.

4 A new concept: limit reachability

Let us go back to the system in Figure 2 (a). We have seen that a state can be reached in which the marking of p_1 is as small as desired. For some applications, it might be reasonable to consider that we can reach a marking such that this place does not contain any token. In other words, to include the marking that would be obtained in the limit as a reachable marking.



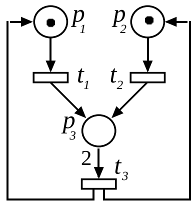


Fig. 3. Two reversible, live and bounded discrete P/T systems. When considered as continuous systems, infinitely long firing sequences exist in which the transitions are fired in their maximal enabling degree, and such that every intermediate marking appears only once.

Definition 2. *Let* $\langle \mathcal{N}, \mathbf{m_0} \rangle$ *be a continuous system. We say that a marking* $\mathbf{m} \in (\mathbb{R}^+ \cup \{0\})^{|P|}$ *is* limit reachable *iff a sequence of reachable markings* $\{\mathbf{m}_i\}_{i\geq 1}$ *exists verifying*

$$\mathbf{m_0} \xrightarrow{\sigma_1} \mathbf{m}_1 \xrightarrow{\sigma_2} \mathbf{m}_2 \cdots \mathbf{m}_{i-1} \xrightarrow{\sigma_i} \mathbf{m}_i \cdots$$

and $\lim_{i\to\infty} \mathbf{m}_i = \mathbf{m}$.

The firing sequence may be null after a finite number of firings, therefore the reachable markings are in particular limit reachable.

The limit reachability space, lim-RS$_C(\mathcal{N}, \mathbf{m_0})$, *is the set of limit reachable (and in particular reachable) markings.*

The definition of boundedness does not change with the new concept of limit reachability (if every $\mathbf{m}_i \leq \mathbf{k}$, then $\lim_{i\to\infty} \mathbf{m}_i \leq \mathbf{k}$).

There is a strong relationship between the LRS$_\mathrm{C}$ and the lim-RS$_C$ of a continuous system. In fact, they coincide in consistent systems in which every transition is fireable.

Theorem 3. *Let* $\langle \mathcal{N}, \mathbf{m_0} \rangle$ *be consistent and such that each transition can be fired at least once.*

Then lim-RS$_C(\mathcal{N}, \mathbf{m_0}) = \mathrm{LRS}_\mathrm{C}(\mathcal{N}, \mathbf{m_0})$.

Proof. It is clear that lim-RS$_C(\mathcal{N}, \mathbf{m_0}) \subseteq \mathrm{LRS}_\mathrm{C}(\mathcal{N}, \mathbf{m_0})$, since $\mathrm{LRS}_\mathrm{C}(\mathcal{N}, \mathbf{m_0})$ is a closed set that includes $\mathrm{RS}_\mathrm{C}(\mathcal{N}, \mathbf{m_0})$.

For the "$\supseteq$", let $\mathbf{m} \in \mathrm{LRS}_\mathrm{C}(\mathcal{N}, \mathbf{m_0})$, $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}$. Applying Proposition 2, we have that from $\mathbf{m_0}$ a positive marking $\mathbf{m}' = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}'$ can be reached. We will prove that $\mathbf{m} \in$ lim-RS$_C(\mathcal{N}, \mathbf{m}')$. Observe that $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma} = \mathbf{m}' + \mathbf{C} \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}')$. Being $\mathcal{N}$ consistent, a T-semiflow, $\mathbf{x}$, exists such that $\mathbf{x} + \boldsymbol{\sigma} - \boldsymbol{\sigma}' \geq \mathbf{0}$, and thus $\mathbf{m} = \mathbf{m}' + \mathbf{C} \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}' + \mathbf{x})$. Since $\mathbf{m}' > \mathbf{0}$, α and σ'' exist such that σ'' is fireable from $\mathbf{m}'$ and $\boldsymbol{\sigma}'' = \alpha(\boldsymbol{\sigma} - \boldsymbol{\sigma}' + \mathbf{x})$, i.e., a sequence proportional to the vector leading from $\mathbf{m}'$ to $\mathbf{m}$ can be fired. If $\alpha \geq 1$, it is clear that $\mathbf{m}$

can be reached from $\mathbf{m}'$. Otherwise, the firing of σ'' leads to $\mathbf{m}' + \mathbf{C} \cdot \boldsymbol{\sigma}'' = \mathbf{m}' + \alpha \mathbf{C} \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}' + \mathbf{x}) = \alpha \mathbf{m} + (1-\alpha)\mathbf{m}'$, i.e.,

$$\mathbf{m_0} \xrightarrow{\sigma'} \mathbf{m}' \xrightarrow{\sigma''} \alpha \mathbf{m} + (1-\alpha)\mathbf{m}'$$

Clearly, if σ'' was fireable from $\mathbf{m}'$, $(1-\alpha)\,\sigma''$ is fireable from $(1-\alpha)\mathbf{m}'$. Hence

$$\alpha \mathbf{m} + (1-\alpha)\mathbf{m}' \xrightarrow{(1-\alpha)\,\sigma''} \alpha \mathbf{m} + \alpha(1-\alpha)\mathbf{m} + (1-\alpha)^2 \mathbf{m}'$$

Repeating the procedure we build a sequence of markings whose limit is $\mathbf{m}$. $\square$

This does not mean that for any vector $\mathbf{x}$ such that $\mathbf{m_0} + \mathbf{C} \cdot \mathbf{x} \geq \mathbf{0}$ a sequence with this firing vector is enabled at $\mathbf{m_0}$. For instance, in the example of Figure 4, $[0,1,0,0,1] = [1,0,0,0,1] + \mathbf{C} \cdot [0,1,1,0,0]^T$ is reachable, but no sequence with firing vector $[0,1,1,0,0]$ is enabled.

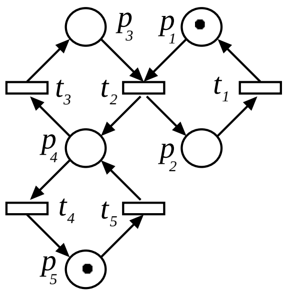


Fig. 4. In this system $[0,1,0,0,1]$ is a reachable marking, although no enabled sequence has $[0,1,1,0,0]$ as its firing vector.

The equality of the LRS_C and the lim-RS_C does not hold in general if the system is not consistent or not every transition is fireable. For instance, in the system on the left in Figure 5, which is consistent, but in which no transition can be fired, the marking $[0,1,0,0]$ belongs to the LRS_C but not to the lim-RS_C. The same happens to the marking $[0,1,0,0,1]$ in the system on the right. In this system every transition can be fired, but it is not consistent. For the moment nothing can be said about the complexity of computing the lim-RS_C in the general case, not even whether it is decidable or not. However, the setting in which the equality holds seems general enough to cover many interesting cases.

5 On liveness analysis

Two of the main properties we have been discussing about all along this work are liveness and deadlock-freeness. In this section we will present two possible extensions of the definitions of liveness to continuous P/T systems, one w.r.t.

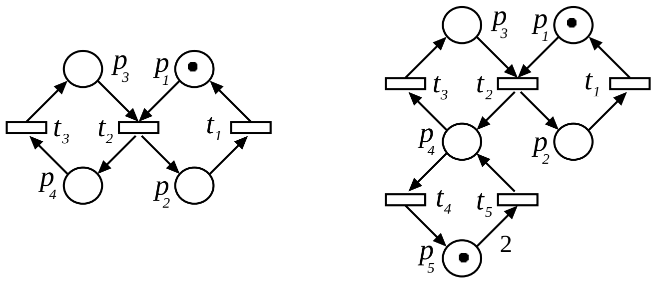


Fig. 5. Two continuous systems for which either not every transition is fireable (left) or the net is not consistent (right).

the $\mathrm{RS_C}$ (ϵ-*liveness*) and the other w.r.t. the lim-RS_C (lim-*liveness*), and the corresponding two definitions of deadlock-freeness. We will compare these two liveness definitions and also relate them to discrete liveness. This will allow to deduce that necessary conditions for discrete liveness are necessary for lim-liveness too. Then, we will concentrate on the classes of EQ and FC nets. We will see that for strongly connected and str. bounded EQ systems str. lim-liveness and (discrete) str. liveness coincide, and they are also equivalent to str. ϵ-liveness for FC nets.

5.1 Liveness definitions

Let us start defining liveness w.r.t. the $\mathrm{RS_C}$. A naive generalization of the discrete definition, following the approach used to define boundedness, leads to the following statement: a transition t of a continuous P/T system is live iff from every reachable marking, $\mathbf{m}$, another marking can be reached, $\mathbf{m}'$, at which the transition is enabled, i.e., $\mathrm{enab}(t, \mathbf{m}') > 0$.

According to this definition, transition t_1 in Figure 1 is live, since for every reachable marking a successor exists such that the marking of p_1 is greater than zero. From our discrete-biased point of view, this does not seem to be what one would desire of a live transition. Therefore, let us try to modify the definition to avoid this kind of behavior.

The problem in this example is that with a finite number of firings the marking of p_1 and p_2 can be done indefinitely small, but not zero. With the idea of not allowing this to be considered live, we introduce an improved version of the definition of liveness:

Definition 3. *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a continuous P/T system. A transition t is ϵ-*live *iff $\epsilon > 0$ exists such that for any reachable marking, $\mathbf{m}$, a successor $\mathbf{m}' \in$* $\mathrm{RS_C}(\mathcal{N}, \mathbf{m})$ *can be found such that* $\mathrm{enab}(t, \mathbf{m}') \geq \epsilon$.

So, if the enabling of a transition can be made as small as desired and never grows back, this transition is not ϵ-live. As in discrete systems, we will say that a continuous system is str. ϵ-live if a marking $\mathbf{m_0}$ exists such that in $\langle \mathcal{N}, \mathbf{m_0} \rangle$

every transition is ϵ-live. For instance, the system in Figure 2 (a) is ϵ-live with the given marking.

In an analogous way we can define deadlock-freeness.

Definition 4. *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a continuous P/T system. It* ϵ-deadlocks *iff for every $\epsilon > 0$ a reachable marking, $\mathbf{m}_\epsilon$, exists such that for every successor $\mathbf{m}'_\epsilon \in \mathrm{RS_C}(\mathcal{N}, \mathbf{m}_\epsilon)$, and every transition t, $\mathrm{enab}(t, \mathbf{m}'_\epsilon) < \epsilon$.*

Clearly, the system in Figure 1 ϵ-deadlocks.

Another possibility is to define liveness w.r.t. the lim-RS_C. With this definition of reachability, the extension of liveness is immediate:

Definition 5. *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a continuous P/T system. A transition t is* lim-live *iff for any marking $\mathbf{m} \in \text{lim-}\mathrm{RS}_C(\mathcal{N}, \mathbf{m_0})$ a successor $\mathbf{m}' \in \mathrm{RS_C}(\mathcal{N}, \mathbf{m})$ exists such that $\mathrm{enab}(t, \mathbf{m}') > 0$.*

In other words, a transition is non lim-live iff a sequence of successively reachable markings exists which converges to a marking such that none of its successors enables the transition. Observe that none of the systems in Figure 2 is lim-live. For example, in the system in 2 (a), firing once either t_1 or t_2 a deadlock is reached. For the system in 2 (b), take for instance the following sequence of markings:

$$\mathbf{m_0} \xrightarrow{t_1} \mathbf{m}_1 \xrightarrow{\frac{1}{2}t_1} \mathbf{m}_2 \xrightarrow{\frac{1}{4}t_1} \mathbf{m}_3 \xrightarrow{\frac{1}{8}t_1} \cdots$$

That is, $\mathbf{m}_k = [1/2^{(k-1)}, 4 - 1/2^{(k-1)}]$. Clearly, a limit of this sequence exists, $\mathbf{m} = [0, 4]$, and no transition is enabled there. Hence, the system is not lim-live. Even more, it can be proven that no marking makes this system lim-live. Therefore, we can say that the continuous P/T net is not str. lim-live.

The following properties of bounded lim-live systems can be immediately deduced.

Theorem 4. *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a consistent, bounded and* lim-*live continuous system. Then,*

1. $\mathrm{LRS_C}$*($\mathcal{N}$,$\mathbf{m_0}$)=*lim-RS_C*(N,$\mathbf{m_0}$), i.e. there is no spurious solution of the state equation.*
2. *$\mathcal{N}$ is str. bounded.*
3. *$\langle \mathcal{N}, \mathbf{m_0} \rangle$ is reversible w.r.t. the* lim-RS_C.

Proof. For (1), since the system is lim-live, a fireable sequence exists that contains all the transitions. Then, applying Theorem 3, the result is proved.

(2) is immediate from Theorem 2.

To prove (3), let $\mathbf{m} \in \text{lim-}\mathrm{RS}_C(\mathcal{N}, \mathbf{m_0})$. Observe that, since the net is consistent, $\mathbf{m_0} \in \mathrm{LRS_C}(\mathcal{N}, \mathbf{m})$. Then, applying (1) to $\langle \mathcal{N}, \mathbf{m} \rangle$, we obtain that $\mathbf{m_0} \in \text{lim-}\mathrm{RS}_C(\mathcal{N}, \mathbf{m})$. □

Analogously, we can define lim-deadlock:

Definition 6. *A continuous P/T system $\langle \mathcal{N}, \mathbf{m_0} \rangle$* lim-deadlocks *iff a marking $\mathbf{m} \in \text{lim-}\mathrm{RS}_C(\mathcal{N}, \mathbf{m_0})$ exists such that* $\mathrm{enab}(t, \mathbf{m}) = 0$ *for every transition t.*

For instance, all the systems in Figure 2 lim-deadlock.

5.2 Liveness monotonicity

In discrete systems it may happen that a system is live with a certain marking, and non-live with any other marking, in particular with a multiple marking. In continuous systems an ϵ-live (lim-live) system is also ϵ-live (lim-live) with any multiple/fraction of the initial marking. That is, ϵ-liveness and lim-liveness are monotonic w.r.t. scaling of the marking.

Proposition 3. *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a continuous P/T system. If it is ϵ-live (*lim-*live), then for every $\alpha > 0$ $\langle \mathcal{N}, \alpha\mathbf{m_0} \rangle$ is ϵ-live (*lim-*live).*

Proof. Assume $\langle \mathcal{N}, \mathbf{m_0} \rangle$ is ϵ-live and $\alpha > 0$ exists such that $\langle \mathcal{N}, \alpha\mathbf{m_0} \rangle$ is not ϵ-live. Since $\langle \mathcal{N}, \mathbf{m_0} \rangle$ is ϵ-live, a constant $\epsilon > 0$ exists such that for every $\mathbf{m} \in \mathrm{RS_C}(\mathcal{N}, \mathbf{m_0})$, and every transition t, a successor of $\mathbf{m}$ exists for which the enabling of t is greater than ϵ. $\langle \mathcal{N}, \alpha\mathbf{m_0} \rangle$ is not ϵ-live, therefore a transition t' and a sequence σ' exist such that $\alpha\mathbf{m_0} \xrightarrow{\sigma'} \mathbf{m}'$, and for every successor of $\mathbf{m}'$ the enabling of t' is less than $\alpha\epsilon$. Observe that $\frac{1}{\alpha}\sigma'$ can be fired at $\mathbf{m_0}$, i.e., $\mathbf{m_0} \xrightarrow{\frac{1}{\alpha}\sigma'} \frac{1}{\alpha}\mathbf{m}'$. A successor of $\frac{1}{\alpha}\mathbf{m}'$ exists such that the enabling of t' is greater than ϵ. Hence, a successor of $\mathbf{m}'$ exists for which the enabling of t' is greater than $\alpha\epsilon$, contradiction.

An analogous proof can be used in the case of lim-liveness. □

A question that naturally arises is whether discrete liveness monotonicity is a stronger result than ϵ- or lim-liveness. This is false in the case of lim-liveness. For instance, the system in Figure 2 (d) is live as a discrete system with the given marking or a larger one. But it is not str. lim-live, since for any initial marking firing $\frac{\mathbf{m_0}[p_1]}{2}t_2 \frac{\mathbf{m_0}[p_1]}{4}t_2 \frac{\mathbf{m_0}[p_1]}{8}t_2 \ldots$, in the limit a marking is reached in which no transition is enabled. However, it is true w.r.t. ϵ-liveness, that is, discrete liveness monotonicity implies ϵ-liveness.

Theorem 5. *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a consistent system which is live as discrete with any marking multiple of the initial one. Then, it is ϵ-live.*

Proof. Consider $\langle \mathcal{N}, \mathbf{m_0} \rangle$ as a continuous system. The proof will be done in two steps. First, assume a reachable marking $\mathbf{m}$ and a transition t exist such that no successor of $\mathbf{m}$ enables t. Taking a constant k big enough, a discrete sequence can be fired from $k\mathbf{m_0}$ leading to a marking which is as close as desired to $k\mathbf{m}$ (it may be exact if in the sequence leading to $\mathbf{m}$ all the transitions are fired in a rational amount). Since the discrete system is live, a successor of this marking enables t. Hence, a successor of $\mathbf{m}$ exists that enables t, contradiction.

Therefore, for every transition and every marking a successor exists in which this transition is enabled. It might happen that the system were not ϵ-live because the enabling approached zero, i.e., for every $\epsilon > 0$, t and $\mathbf{m}$ exist such that for every successor the enabling of t is less than ϵ. Since the system is consistent, reasoning as in Theorem 3, we can reach a marking that is as close as desired to any solution of the $\mathrm{LRS_C}$, in particular a making close to $\mathbf{m_0}$. $\langle N, \mathbf{m_0} \rangle$ is live as discrete, hence t can be fired again in a "big amount". □

5.3 Relations between discrete liveness, ϵ-liveness, and lim-liveness

In Section 3 we compared the properties of a system as discrete with its properties as continuous, this last one interpreted using the immediate continuous extension of reachability. The examples that appear there show in particular that a system can be str. live as discrete and not str. ϵ-live (Figure 2 (c)); and the reverse, it can be str. ϵ-live and not str. live (Figure 2 (a)).

The concept of limit reachability was then introduced, trying to bring the continuous properties nearer to what one would expect. As we have seen, lim-liveness cannot be deduced from (discrete) str. liveness (Figure 2 (c)). However, any lim-live system is str. live if it is seen as discrete. (Although not necessarily live, i.e., the structure of the net is "correct", although the marking may be "not large enough".)

Theorem 6. *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a bounded* lim*-live P/T system. Then, $\mathcal{N}$ is str. live and str. bounded as a discrete net.*

Proof. Assume $\mathcal{N}$ is not str. live as a discrete net. We will see that we can find a sequence of successively reachable markings in the continuous system, such that at the limit at least one transition is disabled, which contradicts lim-liveness.

$\langle \mathcal{N}, \mathbf{m_0} \rangle$ is not live as a discrete system, therefore a sequence σ_1 and a transition t_{j_1} exist such that $\mathbf{m_0} \xrightarrow{\sigma_1} \mathbf{m}_1$ and for every successor of $\mathbf{m}_1$ t_{j_1} is disabled.

Take now $\langle \mathcal{N}, 2\mathbf{m}_1 \rangle$. It is not live as a discrete system, therefore a sequence σ_2 and a transition t_{j_2} exist such that $2\mathbf{m}_1 \xrightarrow{\sigma_2} \mathbf{m}_2$ and for every successor of $\mathbf{m}_2$ t_{j_2} is disabled.

Analogously, $\langle \mathcal{N}, 2\mathbf{m}_2 \rangle$ is not live as a discrete system ...

Repeating this procedure, a sequence of markings of the continuous system is obtained:

$$\mathbf{m_0} \xrightarrow{\sigma_1} \mathbf{m}_1 \xrightarrow{\frac{1}{2}\sigma_2} \frac{1}{2}\mathbf{m}_2 \xrightarrow{\frac{1}{4}\sigma_3} \frac{1}{4}\mathbf{m}_3 \longrightarrow \dots \xrightarrow{\frac{1}{2^{k-1}}\sigma_k} \frac{1}{2^{k-1}}\mathbf{m}_k \longrightarrow \dots$$

For simplicity, let us denote $\mathbf{m}'_k = \dfrac{1}{2^{k-1}}\mathbf{m}_k$ and $\sigma'_k = \dfrac{1}{2^{k-1}}\sigma_k$. Then,

$$\mathbf{m_0} \xrightarrow{\sigma'_1} \mathbf{m}'_1 \xrightarrow{\sigma'_2} \mathbf{m}'_2 \longrightarrow \dots \xrightarrow{\sigma'_k} \mathbf{m}'_k \longrightarrow \dots$$

The number of transitions is finite, therefore an infinite subsequence exists such that the disabled transition is always the same. We will denote this transition as t.

Since the system is bounded, a convergent subsequence of the former subsequence exists (by Bolzano-Weierstrass Theorem, any bounded sequence contains a convergent subsequence). We will denote this latter subsequence as $\{\mathbf{m}'_{i_k}\}_{k \geq 1}$.

That is,

$$\mathbf{m_0} \xrightarrow{\sigma''_{i_1}} \mathbf{m}'_{i_1} \xrightarrow{\sigma''_{i_2}} \mathbf{m}'_{i_2} \longrightarrow \dots \xrightarrow{\sigma''_{i_k}} \mathbf{m}'_{i_k} \longrightarrow \dots$$

where $\lim_{k \to \infty} \mathbf{m}'_{i_k} = \mathbf{m}'$.

$\langle \mathcal{N}, \mathbf{m_0} \rangle$ is lim-live, therefore, $\alpha > 0$, and a firing sequence σ exist such that

$$\mathbf{m}' \xrightarrow{\sigma} \widetilde{\mathbf{m}}' \xrightarrow{\alpha t}$$

Let $\epsilon = \dfrac{1}{2} \min_{p \in P} \{\mathbf{m}'[p] \mid \mathbf{m}'[p] > 0\}$. Since $\mathbf{m}' = \lim_{k\to\infty} \mathbf{m}'_{i_k}$, a certain k_0 exists such that for every $k \geq k_0$ and every place p, $|\mathbf{m}'_{i_k}[p] - \mathbf{m}'[p]| < \epsilon$. Thus, for every $k \geq k_0$, $\mathbf{m}'_{i_k} \geq \frac{1}{2}\mathbf{m}'$ and

$$\mathbf{m}'_{i_k} \xrightarrow{\frac{1}{2}\sigma} \widetilde{\mathbf{m}}'_{i_k} \xrightarrow{\frac{\alpha}{2} t}$$

Let $K_i = 2^{i_k - 1}$. Then, $\mathbf{m}_{i_k} = K_i \mathbf{m}'_{i_k} \xrightarrow{\frac{K_i}{2}\sigma} K_i \widetilde{\mathbf{m}}'_{i_k} \xrightarrow{\frac{K_i \alpha}{2} t}$, and taking k big enough, an integer sequence that enables t can be fired from $\mathbf{m}_{i_k}$, contradiction. □

With respect to the relationship between lim-liveness and ϵ-liveness, a similar result can be proven, i.e., a bounded continuous lim-live system is ϵ-live. The reverse is not true, as the system in Figure 2 (a) shows.

Theorem 7. *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a bounded* lim-*live P/T system. Then, $\langle \mathcal{N}, \mathbf{m_0} \rangle$ is ϵ-live.*

Proof. Assume $\langle \mathcal{N}, \mathbf{m_0} \rangle$ is not ϵ-live. Then, for every $k > 0$ a transition t_{j_k} and a marking $\mathbf{m}_k \in \text{RS}_\text{C}(\mathcal{N}, \mathbf{m_0})$ exist such that for every $\widetilde{\mathbf{m}}_k \in \text{RS}_\text{C}(\mathcal{N}, \mathbf{m}_k)$, $\text{enab}(t_{j_k}, \widetilde{\mathbf{m}}_k) < 1/k$. Since the number of transitions is finite we can assume w.l.o.g. that all the t_{j_k} coincide. We will denote this transition as t.

Observe that $\mathbf{m}_1$ is reachable from $\mathbf{m_0}$, but nothing ensures that $\mathbf{m}_2$ can be reached from $\mathbf{m}_1$. By Theorem 6, this system is str. live and str. bounded, hence consistent. Thus, $\mathbf{m}_2 \in \text{LRS}_\text{C}(\mathcal{N}, \mathbf{m}_1)$. Moreover, being lim-live, every transition is fireable, and applying Theorem 3, $\mathbf{m}_2 \in \text{lim-RS}_C(\mathcal{N}, \mathbf{m}_1)$. Therefore $\mathbf{m}'_2 \in \text{RS}_\text{C}(\mathcal{N}, \mathbf{m}_1)$ exists such that $|\mathbf{m}'_2 - \mathbf{m}_2| < 1/2$. Analogously, $\mathbf{m}_3 \in \text{lim-RS}_C(\mathcal{N}, \mathbf{m}'_2)$, and repeating the reasoning $\mathbf{m}'_3 \in \text{RS}_\text{C}(\mathcal{N}, \mathbf{m}'_2)$ exists such that $|\mathbf{m}'_3 - \mathbf{m}_3| < 1/3$. In general,

$$\mathbf{m_0} \xrightarrow{\sigma_1} \mathbf{m}_1 \xrightarrow{\sigma'_2} \mathbf{m}'_2 \xrightarrow{\sigma'_3} \mathbf{m}'_3 \ldots \xrightarrow{\sigma'_k} \mathbf{m}'_k \ldots$$

and $|\mathbf{m}_k - \mathbf{m}'_k| < 1/k$. This defines a bounded sequence of markings, therefore, applying the Bolzano-Weierstrass Theorem, a convergent subsequence $\{\mathbf{m}'_{i_k}\}_{k>0}$ exists. Let $\mathbf{m}' = \lim_{k\to\infty} \mathbf{m}'_{i_k}$.

$\langle \mathcal{N}, \mathbf{m_0} \rangle$ is lim-live and $\mathbf{m}' \in \text{lim-RS}_C(\mathcal{N}, \mathbf{m_0})$, hence a (finite) sequence σ and $\alpha > 0$ exist such that

$$\mathbf{m}' \xrightarrow{\sigma} \widetilde{\mathbf{m}}' \xrightarrow{\alpha t}$$

Define $\epsilon = \frac{1}{2} \min\{\mathbf{m}'[p] \mid \mathbf{m}'[p] > 0\}$. Then, applying the limit definition and the way the sequence has been built, a certain k_0 exists such that for every $k \geq k_0$,

$|\mathbf{m}'_{i_k} - \mathbf{m}'| < \dfrac{1}{2}\epsilon$, and $|\mathbf{m}'_{i_k} - \mathbf{m}_{i_k}| < \dfrac{1}{2}\epsilon$. Thus, $|\mathbf{m}_{i_k} - \mathbf{m}'| < \epsilon$ and by definition of ϵ, $\mathbf{m}_{i_k} \geq 1/2\,\mathbf{m}'$. Therefore,

$$\mathbf{m}_{i_k} \xrightarrow{\frac{1}{2}\sigma} \widetilde{\mathbf{m}}_{i_k} \xrightarrow{\frac{\alpha}{2} t}$$

If k is big enough, $\dfrac{\alpha}{2} > \dfrac{1}{i_k}$. Contradiction, since for every successor of $\mathbf{m}_{i_k}$ the enabling of t is less than $1/i_k$. $\square$

We have seen that a system may be str. discrete live (Figure 2 (a)) or str. ϵ-live (Figure 2 (c)) and not str. lim-live. We might think that if both conditions were required, i.e., the system is str. ϵ-live as a continuous system and str. live as a discrete one, perhaps str. lim-liveness could be deduced. Actually, this is not the case, as can be observed in the system in Figure 2 (d). The problem in this example is that there are solutions of the state equation, that cannot be reached in the discrete system but are reachable at the limit, which correspond to deadlocks. For example, with the given initial marking, firing the sequence $\frac{1}{2}t_1 \frac{1}{4}t_1 \frac{1}{8}t_1 \ldots$, we reach in the limit the marking $[0, 2]$, that clearly is a deadlock.

The results in Theorem 6 and Theorem 7, and the counterexamples in Figure 2 are summarized in the diagram at Figure 6. In Subsection 5.5 we will see that these results can be improved if we restrict to selected subclasses.

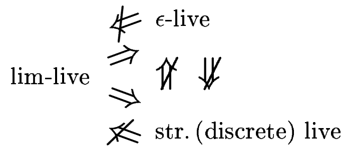


Fig. 6. Relationships among lim-liveness, ϵ-liveness and discrete liveness for general P/T nets.

5.4 Two necessary conditions for lim-liveness

From Theorem 6 it is clear that any necessary condition for a discrete system to be str. live and str. bounded, is also necessary for it to be str. lim-live and bounded. In particular the rank theorem (see [9] for a recent survey) is a necessary condition based on the existence of left and right annullers of the token-flow matrix, and the existence of an upper bound on the rank of this matrix, which is the number of *equal conflict sets*. Two transitions, t and t', are said to be in *equal conflict* (EQ) relation when $\mathbf{Pre}[P, t] = \mathbf{Pre}[P, t'] \neq \mathbf{0}$. This is an equivalence relation and the set of all the equal conflict sets is denoted by SEQS.

Theorem 8. *Let* $\langle \mathcal{N}, \mathbf{m_0} \rangle$ *be a* lim-*live and bounded continuous system. Then,* $\mathcal{N}$ *is consistent, conservative and* $\operatorname{rank}(\mathbf{C}) \leq |\mathrm{SEQS}| - 1$.

Other structural elements that are useful in the analysis of lim-liveness are siphons. A siphon is a set of places, P'', such that ${}^\bullet P'' \subseteq P''^\bullet$. Observe that an empty siphon cannot be marked. Hence, a necessary condition for lim-liveness is that no marking can be reached in which a siphon is empty. In [7] the conditions for a set of places being a siphon were stated as the solutions to a set of linear inequalities: A set $\Sigma \subseteq P$ is a siphon of $\mathcal{N}$ iff $\mathbf{y} \geq \mathbf{0}$ exists such that $\|\mathbf{y}\| = \Sigma$ and $\mathbf{y} \cdot \mathbf{C}_\Sigma \geq \mathbf{0}$, where $\mathcal{N}_\Sigma = \langle P, T, \mathbf{Pre}_\Sigma, \mathbf{Post}\rangle$ is such that $\mathbf{Pre}_\Sigma[p,t] = 0$ iff $\mathbf{Pre}[p,t] = 0$, and $\mathbf{Pre}_\Sigma[p,t] \geq \sum_{p' \in t^\bullet} \mathbf{Post}[p',t]$, otherwise. If lim-RS$_C$ = LRS$_\mathrm{C}$, the absence of a marking in which a siphon is not marked can be checked using a system of linear inequalities:

Theorem 9. *Let $\langle \mathcal{N}, \mathbf{m_0}\rangle$ be a consistent P/T continuous system. If a solution of the following system of inequalities exists, the system is not* lim-*live.*

$$\begin{aligned}\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma} &\geq \mathbf{0}\\ \mathbf{y} \cdot \mathbf{C}_\Sigma &\geq \mathbf{0}\\ \mathbf{y} \cdot \mathbf{m} &= 0\\ \mathbf{y} &\geq \mathbf{0}\\ \boldsymbol{\sigma} &\geq \mathbf{0}\end{aligned}$$

In discrete systems there exists a symmetry between traps and siphons (a trap is a set of places, P', such that $P'^\bullet \subseteq {}^\bullet P'$), in the sense that marked traps cannot be emptied and empty siphons cannot be marked. This symmetry is lost in continuous systems if lim-reachability is considered, because although an empty siphon cannot become marked, a trap can be emptied. For instance, in Figure 7, $\{p_1, p_2, p_3, p_4\}$ is a trap, that is emptied by the firing of $t_1 t_2\ \frac{1}{2}t_3 \frac{1}{2}t_4 \frac{1}{2}t_2\ \frac{1}{4}t_3 \frac{1}{4}t_4 \frac{1}{4}t_2\ \frac{1}{8}t_3 \frac{1}{8}t_4 \frac{1}{8}t_2 \ldots$ This means that traps cannot be used to improve the description of the reachability conditions given by the state equation as in discrete systems [12].

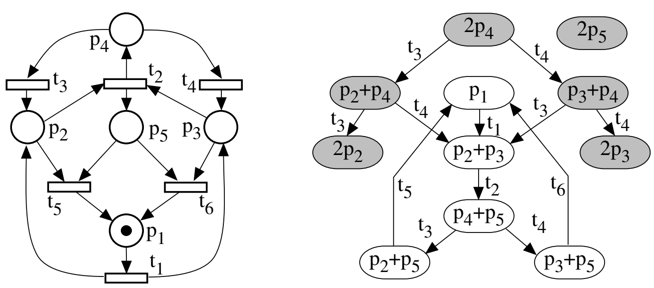


Fig. 7. A live system and its LRG.

5.5 Particular results for some subclasses

As usual, the results obtained in the general frame of continuous systems can be improved if we restrict to selected subclasses. We concentrate here on two subclasses: EQ nets and FC nets.

First, it can be seen that lim-liveness and lim-deadlock freeness coincide in bounded and strongly connected EQ systems, as happened with their discrete counterparts.

Theorem 10. *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a bounded, strongly connected, continuous EQ system. It is* lim-*live iff it is* lim-*deadlock-free.*

Proof. Assume $\langle \mathcal{N}, \mathbf{m_0} \rangle$ is not lim-live. Then, a transition t and a marking $\mathbf{m} \in$ lim-$\mathrm{RS}_C(\mathcal{N}, \mathbf{m_0})$ exist such that for every $\mathbf{m}' \in \mathrm{RS}_C(\mathcal{N}, \mathbf{m})$ the enabling of t is zero. Let $p \in {}^\bullet t$. All the transitions in $p^\bullet$ are in equal conflict relation, hence none of the output transitions of p fires again. The system is bounded, therefore a sequence of markings exists such that for their successors the enabling degree of the input transitions of p converges to zero. Applying the Bolzano-Weierstrass theorem, a convergent subsequence of markings exists. Neither t, nor any input transition of p is enabled at the marking reached in the limit, and they will never be enabled again. Repeating the reasoning, since the net is strongly connected, and the number of transitions is finite, we finally reach a marking in which no transition is enabled. □

In discrete EQ systems the rank theorem is a characterization of str. liveness and str. boundedness [13]. For continuous EQ systems this result can be improved: a characterization of lim-liveness and boundedness can be obtained analogous to the one that exists for liveness and boundedness of discrete FC systems [6]. This provides a simple, polynomial time, way to prove lim-liveness of EQ systems:

Theorem 11. *Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a continuous EQ system. The following conditions are equivalent:*

1. *The system is* lim-*live and bounded.*
2. *The system is consistent, conservative,* $\mathrm{rank}(\mathbf{C}) = |\mathrm{SEQS}| - 1$ *(or, equivalently, it is str. bounded and str. live as discrete) and the support of every P-semiflow is marked, i.e.,* $\nexists\, \mathbf{y} \geq \mathbf{0}$ *such that* $\mathbf{y} \cdot \mathbf{C} = \mathbf{0}$, $\mathbf{y} \cdot \mathbf{m_0} = 0$.

Proof. For "1⇒2", applying Theorem 6, and the characterization of str. liveness and str. boundedness for EQ nets [13], the net must be consistent, conservative and $\mathrm{rank}(\mathbf{C}) = |\mathrm{SEQS}| - 1$. Assume a P-semiflow, $\mathbf{y}$, exists that is not marked. Then, for every reachable marking, $\mathbf{m}$, $\mathbf{y} \cdot \mathbf{m} = \mathbf{y} \cdot \mathbf{m_0} = 0$, i.e., none of the places in $\|\mathbf{y}\|$ can ever be marked. Hence, their output transitions cannot be fired, contradiction.

For "2⇒1", if $\langle \mathcal{N}, \mathbf{m_0} \rangle$ is not live, it deadlocks (Theorem 10). Let $\mathbf{m}_d \in$ lim-$\mathrm{RS}_C(\mathcal{N}, \mathbf{m_0})$ be a deadlock. Then, for every transition t, a place $p \in {}^\bullet t$ exists such that $\mathbf{m}_d[p] = 0$. This set of places contains the support of a P-semiflow [13], and it is not marked, contradiction. □

The properties of EQ systems allow to extend to lim-liveness and boundedness a sufficient condition for (discrete) str. liveness and str. boundedness [9]. The idea is to transform the system into an EQ system, and apply the results of this class. More specifically, each coupled conflict set is transformed into an EQ set. The coupled conflict relation is defined as the transitive closure of the str. conflict relation, where t and t' are in str. conflict relation iff ${}^\bullet t \cap {}^\bullet t' \neq \emptyset$. The set of all the equivalence classes is denoted by SCCS. We skip the proof, since it is analogous to the one given in [9] for the discrete case.

Theorem 12. *Let $\mathcal{N}$ be consistent, conservative and* $\text{rank}(\mathbf{C}) = |\text{SCCS}| - 1$. *Then, $\mathcal{N}$ is str.* lim-*live as a continuous net. Moreover, any marking that marks every P-semiflow makes the system* lim-*live.*

In strongly connected str. bounded EQ systems str. lim-liveness is equivalent to (discrete) str. liveness, though not in general to str. ϵ-liveness (see Figure 2 (a)). For strongly connected str. bounded FC nets a stronger result holds: (discrete) str. liveness, str. ϵ-liveness, and str. lim-liveness are equivalent.

Theorem 13. *Let $\mathcal{N}$ be a strongly connected, str. bounded FC net. The following conditions are equivalent:*

1. *$\mathcal{N}$ is str.* lim-*live.*
2. *$\mathcal{N}$ is str. ϵ-live*
3. *$\mathcal{N}$ is str. (discrete) live.*

Proof. "(1)⇒(2)" is proven in Theorem 7 and "(3)⇒(1)" can be deduced from Theorem 11.

For "(2)⇒(3)", assume $\mathcal{N}$ is not str. live and let $\mathbf{m_0}$ be such that $\langle \mathcal{N}, \mathbf{m_0} \rangle$ is ϵ-live. We can assume w.l.o.g. that $\mathbf{m_0} \in \mathbb{N}^{|P|}$. The system is not live as a discrete system, hence it deadlocks, i.e., a sequence σ_d exists such that $\mathbf{m_0} \xrightarrow{\sigma_d} \mathbf{m}_d$ and no transition is enabled at $\mathbf{m}_d$. Then, for every transition t a place $p \in {}^\bullet t$ exists such that $\mathbf{m}_d[p] = 0$. The same sequence can be fired if the system is considered as continuous, and clearly it also leads to a deadlock, contradiction. □

6 Conclusions

A common practice in many fields in which systems with large state spaces appear is to relax the description by dropping integrality constraints on the state equation. Moreover, there are systems in which discrete parts are mixed with other parts, that are more naturally represented as continuous. In P/T systems this idea has led to the definition of continuous and hybrid P/T systems. Some of these models incorporate a continuous part by means of algebraic differential equations [15, 2], others allow non integer markings in some places [4, 14] (see [3] for a comparison of different approaches through the modelling of a benchmark example). This latter approach is the one that is considered here, although with one main difference, we study autonomous models, i.e. without any timed interpretation.

The basic definitions of P/T systems are extended to the continuous case in Section 2, allowing real non negative markings, and the firing of transitions in any real non negative amount. An immediate consequence of the possibility of firing transitions in non discrete amounts, is that the behavior of a continuous system does not change if the initial marking is scaled. This does not happen in discrete systems (for instance, a discrete system may be deadlock-free with a certain marking, and deadlock if the initial marking is doubled). This means that monotonicity of the properties w.r.t. scaling, which is not basic for the study of discrete systems, is a must if we want the continuous view to be coherent with the discrete one. In other words, a system should not be studied as continuous if the exact amount of tokens is so important to determine its behavior.

The relaxation of the notions of marking and firing, allowing positive real numbers, is quite intuitive. However, the extension of reachability is not so immediate. This concept is central for the eventual analysis of logic properties of the modeled systems, but it has not been properly investigated before. In this paper two possible definitions of reachability have been introduced, and their analysis explored.

With the first notion of reachability, the idea of a finite sequence of firings is preserved. Some examples are presented in Section 3 showing that with this definition of reachability the behavior of continuous and discrete systems can be completely different. The second notion, introduced in Section 4, allows infinitely long firing sequences, leading to the concept of limit reachable markings.

Another difference w.r.t. the discrete case is that in continuous systems the set of reachable markings is a convex set, independently of which reachability definition is considered. Apparently, deciding whether a certain marking is reachable or not, when the firing is not restricted to be integer, is more difficult than in the integer case. However, under the limit reachability definition, in most practical cases the set of reachable markings coincides with the solutions of the state equation.

Liveness and deadlock-freeness are studied in Section 5. Two definitions of liveness, ϵ-liveness and lim-liveness, and the corresponding two definitions of deadlock-freeness, have been introduced, each one associated to one of the definitions of reachability. These two continuous liveness definitions and the discrete liveness definition are compared. As a result it is deduced that in bounded systems lim-liveness implies ϵ-liveness and (discrete) str. liveness. For bounded strongly connected EQ systems, this result can be improved: str. lim-liveness and discrete str. liveness coincide. Moreover, they are also equivalent to ϵ-liveness in the case of FC systems. These relations among discrete and continuous liveness definitions allow to obtain two necessary conditions for lim-liveness and boundedness, and a sufficient one, which are analogous to the ones that exist in the discrete case.

From our experience after this work, we conclude that although both definitions of reachability are interesting, limit reachability seems to be specially convenient. On the one hand, the idea of treating very small quantities as zero is reasonable if continuous systems are considered as an approximation of discrete systems. On the other, the simple representation of the lim-RS_C in most practical cases (the solutions of the state equation) offers a clear advantage w.r.t. the RS_C, for which there seems to be no simple way to deduce whether a certain marking can be reached or not. However, further investigations about how to extend other properties of discrete P/T systems should be done before making a choice.

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