Choice-free Petri Nets: A Model for Deterministic Concurrent Systems with Bulk Services and Arrivals

Enrique Teruel, José Manuel Colom, and Manuel Silva

Abstract— **Among discrete event systems, those exhibiting concurrency are especially challenging, requiring the use of formal methods to deal with them. Petri nets are a well-established such formalism. Structure theory aims at overcoming the state space explosion problem, inherent to the analysis of concurrent systems, by bridging structural and behavioral properties. To date, this has been successfully achieved mainly for some subclasses of ordinary nets. Nevertheless weights are a modeling convenience in many situations. In this paper we study a formal model for a subclass of concurrent systems with bulk services and arrivals which structurally avoids conflicts. Structural results and techniques for dealing with them are introduced. These include structural conditions on properties of correct behavior and a unified framework for checking general behavioral properties by reasoning solely on the structure.**

I. Introduction and Related Work

The subtle interactions between concurrently active processes compels the use of *formal* methods to deal with parallel and distributed discrete event systems. *Petri nets* (see for instance [10], [12], [15]) are a well-established formalism for concurrent systems.

A salient feature of concurrent systems is the complexity of their behavior compared to the structure of the model. This is known as the *state space explosion problem*. More precisely, the associated transition system or reachability graph of a given Petri net model, when finite, is much bigger than the net model itself.

A fruitful approach to the problem is the so called *structure theory*. The idea is to get useful information about the behavior by way of reasoning on the structure of the model and the initial marking. When general concurrent systems are considered, current structural techniques give necessary or sufficient conditions for the studied properties to hold [7], [9], [16].

A typical trend in structure theory is to restrict the models under study [2], [3], [6], [8], [13], [17], [18], [20]. The behavior of general systems is of course richer, but proper limitations lead to useful *subclasses*, which are able to model some practical systems and to give insight on the relationships between the behavior and the structure in more general systems. Most of the classes under consideration have been ordinary (i.e., no multiple arcs), while it is often presumed that the gap between ordinary and weighted Petri nets is small. In this sense, multiple arcs are usually presented as being convenient for dealing with *bulk* consumption or production of resources which is a situation that often arises, for instance, when modeling flexible manufacturing systems or certain features of computer operating systems and databases. Even though it is clear that for a given weighted model we can build an ordinary one that behaves the same way, there is something else that must be pointed out: *the structure of the underlying graph of the two nets may not be that similar.* More precisely: (1) the ordinary model is artificially complex, and its size grows as the weights do, and (2) the resulting ordinary net no longer belongs to the classes that are amenable to the most interesting structural techniques. Therefore it can be said that *the expressive power of a given subclass of weighted systems is* greater *than that of the "corresponding" ordinary subclass.* The difference arises due to the weighted arcs from a place to a transition, which represent a sort of "synchronization": a number of tokens must assemble in front of the transition to enable it, no matter where they come from.

In this paper we study a subclass of weighted systems were *conflicts* are forbidden. The concept of conflict is behavioral. Nets without conflicts (*persistent*, or *deterministic*), are studied for instance in [6]. If a net does not have forward branched places (called *choices*), then it cannot exhibit conflicts, although the converse is not true. From the structural point of view, it is quite natural to consider the class of systems without choices in order to deal with systems without conflicts. In fact, this class — first defined in [4] — has been called *Conflict-free* in the literature, but we prefer the name *Choice-free* since a net with choices could be free of conflicts. Some properties of Choice-free systems were studied in [6], [8].

Choice-free systems are a natural *generalization* of other structurally deterministic or persistent subclasses, namely *Marked Graphs* or *T-systems* [2], [10], *weighted T-systems* [18], and *computation graphs* [5]. Either ordinary or weighted T-systems are adequate for modeling many deterministic concurrent systems, although *their expressive power is lower than that of Choice-free systems.* In particular, since attributions are now allowed, tokens can arrive at any place from different sources. If transitions play the role of agents, while places are channels or buffers that the agents use to communicate with one another, then channels with public input — receiving messages from different agents — can be modeled.

The authors are with the Departamento de Informática e Ingeniería de Sistemas, Centro Politécnico Superior, Universidad de Zaragoza, María de Luna 3, E-50015 Zaragoza, Spain.

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Our main goal is to present (a substantial part of) the structure theory for Choice-free systems and its application. This means answering some questions, like: Which properties must the net (the pure structure of the model) enjoy in order to allow "good" behavior? (We assume *boundedness* and *liveness* are two essential properties of practical reactive systems. *Reversibility*, or at least *existence of home states*, is often required too.) What might the possible problems in the net be and how can they be detected and corrected? Looking solely at the structure, how could behavioral properties be analyzed? Given a correct net, how can an "optimal" initial state leading to "good" behavior be computed?

The rest of the paper is organized as follows. In Section II we recall the basic concepts. Choice-free systems are defined in Section III and a motivating example is presented. Some basic properties are stated in Section IV. In Section V the purely structural properties of Choice-free nets are studied and the structural conditions permitting "good" behavior are obtained. In Section VI we analyze the structure of the potential reachability graph. This analysis leads to structural characterizations of behavioral properties. Finally, Section VII outlines the application of the structure theory of Choice-free systems to the design and verification of deterministic concurrent systems.

II. Basic Concepts and Notation

Introductory presentations of Petri net concepts are, for instance, [10], [12], [15]. Here we present the basic concepts to be used. For the sake of readability, whenever a net or system is defined, it "inherits" the definition of all the characteristic sets, functions, parameters... with names conveniently marked.

A P/T *net* is a bipartite weighted directed graph $\mathcal{N} = (P, T, W)$ where P and T are disjoint sets of *places* and *transitions*, and $W : (P \times T) \cup (T \times P) \to \mathbb{N}$ defines the *weighted flow relation*: if $W(u, v) > 0$, then we say that there is an *arc* from u to v, with *weight* or *multiplicity* $W(u, v)$. *Ordinary* nets are those where all weights are one. For a given node $v \in P \cup T$, its *preset* and *postset* are given by ${}^\bullet v = \{u \mid W(u,v) > 0\}$ and $v^\bullet = \{u \mid W(v,u) > 0\}$, respectively. The preset (postset) of a set of nodes is the union of presets (postsets) of its elements. A place p such that $|p^\bullet| > 1$ is called a *choice*, and it is a basic construct to model conflicts. A transition t such that $|{}^\bullet t| > 1$ is called a *join*, and it is a basic construct to model synchronizations. A place p such that $|{}^\bullet p| > 1$ is called an *attribution*, and a transition t such that $|t^\bullet| > 1$ is called a *fork*.

The weighted flow relation can be alternatively defined by: $Pre(p, t) = W(p, t)$, $Post(p, t) = W(t, p)$. These functions can be represented by matrices. Regarding vector and matrix notations, they are often indexed by the sets P and T. For instance, the submatrix of A corresponding to row $p \in P$ and columns in $\tau \subseteq T$ is denoted by $A[p, \tau]$. The usual multiplication of scalars, vectors and/or matrices A and B is denoted by $A \cdot B$. The componentwise comparison of A and B is denoted by $A \geq B$, while $A \gneqq B$ denotes $A \geq B$ but $A \neq B$. The *token-flow* or *incidence matrix* is defined as $C = Post - Pre$.

By reversing arcs or interchanging places and transitions we get the *reverse* net, $\mathcal{N}^r$, or the *dual* net, $\mathcal{N}^d$, of $\mathcal{N}$, with incidence matrices $-C$ and $-C^T$, respectively. Both transformations together lead to the *reverse-dual* net, $\mathcal{N}^{rd}$, with incidence matrix C^T. A net $\mathcal{N}'$ is *subnet* of $\mathcal{N}$ (written $\mathcal{N}' \subseteq \mathcal{N}$) iff $P' \subseteq P$, $T' \subseteq T$ and W' is the *restriction* of W to P' and T'. A subnet generated by a subset V of nodes of a single kind is assumed to be the one generated by $V \cup {}^\bullet V \cup V^\bullet$. Subnets generated by a subset of places (transitions) are called *P-(T-)subnets*.

A *marking* is a function $M : P \to \mathbb{N}$. It can be represented by a vector. A P/T *system* is a pair $(\mathcal{N}, M_0)$ where $\mathcal{N}$ is a P/T net and M_0 is the *initial* marking.

A transition t is *enabled* at M iff $M \geq Pre[P, t]$. Being enabled, t may *occur* (or *fire*) yielding a new marking $M' = M + C[P, t]$, and this is denoted by $M \xrightarrow{t} M'$. An *occurrence sequence* from M is a sequence of transitions $\sigma = t_1 \cdots t_k \cdots$ such that $M \xrightarrow{t_1} M_1 \cdots M_{k-1} \xrightarrow{t_k} \cdots$. The *number of occurrences* of a transition t in a sequence σ is denoted by $\#(t, \sigma)$. For a set of transitions τ, $\#(\tau, \sigma)$ denotes the sum of all $\#(t, \sigma)$ for $t \in \tau$. The *length* of a sequence σ is $\#(T, \sigma)$. The *firing count vector* of a sequence σ is defined as $\vec{\sigma}[t] = \#(t, \sigma)$. We will say that a sequence σ or firing count vector $\vec{\sigma}$ is *greater* than σ' or $\vec{\sigma}'$ when $\vec{\sigma} \gneqq \vec{\sigma}'$. A sequence σ' is a *permutation* of σ if $\vec{\sigma}' = \vec{\sigma}$.

The set of all the occurrence sequences from M_0, i.e., the *language*, is denoted by $\mathrm{L}(\mathcal{N}, M_0)$, and the set of all the markings reachable from M_0, the *reachability set*, is denoted by $\mathrm{R}(\mathcal{N}, M_0)$. The *reachability graph* is a labeled directed graph $\mathrm{RG}(\mathcal{N}, M_0) = (\mathrm{R}(\mathcal{N}, M_0), E, l)$ given by:

$$((M, M') \in E \wedge l(M, M') = t) \Leftrightarrow M \xrightarrow{t} M' \ .$$

The *state equation* is an algebraic equation that gives a *necessary* condition for a marking to be reachable: if $\vec{\sigma}$ exists so that $M = M_0 + C \cdot \vec{\sigma}$ then M is said to be *potentially reachable*, denoted by $M \in \mathrm{PR}(\mathcal{N}, M_0) \supseteq \mathrm{R}(\mathcal{N}, M_0)$. The *potential reachability graph* is defined in the same way as the reachability graph, but the set of nodes is now $\mathrm{PR}(\mathcal{N}, M_0)$ instead of $\mathrm{R}(\mathcal{N}, M_0)$. Observe that $\mathrm{RG}(\mathcal{N}, M_0)$ is a subgraph of $\mathrm{PRG}(\mathcal{N}, M_0)$.

Let $(\mathcal{N}, M_0)$ be a P/T system with incidence matrix C. *Flows* (*semiflows*) are non-null integer (natural) annullers of C. Right and left annullers are called T- and P-(semi)flows respectively. A semiflow is called *minimal* when its support (i.e., the set $\|v\|$ of the non-zero components of vector v) is not a proper superset of the support of any other, and the g.c.d. of its elements is one. Flows are important because they induce certain (linear) invariant relations which are useful for reasoning on the behavior. Several structural (str.) properties are defined in terms of the existence of certain annullers, or similar vectors:

- $\mathcal{N}$ is *consistent (str. repetitive)* iff $X > 0$ exists such that $C \cdot X = \ (\geq)\ 0$. X is said to be a *consistent (repetitive) vector.*
- $\mathcal{N}$ is *conservative (str. bounded)* iff $Y > 0$ exists such that $Y \cdot C = \ (\leq)\ 0$.

A P/T system is *bounded* when every reachable marking is finite, it is *live* when every transition can ultimately occur from every reachable marking, and it is *deadlock-free* when at least one transition is enabled at every reachable marking. M is a *home state* in $(\mathcal{N}, M_0)$ iff it is reachable from every reachable marking, and $(\mathcal{N}, M_0)$ is *reversible* iff M_0 is a home state. Boundedness is necessary whenever the system is to be implemented, while liveness is often required, especially in reactive systems. A net $\mathcal{N}$ is *str. bounded* when $(\mathcal{N}, M_0)$ is bounded for *every* M_0 and it is *str. live* when *there exists* M_0 such that $(\mathcal{N}, M_0)$ is live. Consequently, if a net $\mathcal{N}$ is str. bounded and str. live some marking M_0 exists such that $(\mathcal{N}, M_0)$ is bounded and live (B&L). In such a case, non B&L is exclusively imputable to the marking and we say that the net is *well-formed*. A well-known polynomial necessary condition for well-formedness simply based on purely structural properties is conservativeness and consistency (Cv&Ct). Some relations between these concepts are summarized in the following statement:

Theorem 1 ([9], [14]) Let $(\mathcal{N}, M_0)$ be a P/T system.
1. $(\mathcal{N}, M_0)$ B&L $\Rightarrow \mathcal{N}$ strongly connected and consistent.
2. $\mathcal{N}$ well-formed $\Rightarrow \mathcal{N}$ Cv&Ct.
3. $\mathcal{N}$ Cv&Ct $\Rightarrow \mathcal{N}$ strongly connected.

III. Definition of Choice-free and Related Systems

Definition 2 ([17]) Let $\mathcal{N}$ be a P/T net.
1. $\mathcal{N}$ is a *weighted T-graph* iff $\forall p \in P : |p^\bullet| = |{}^\bullet p| = 1$.
2. $\mathcal{N}$ is *Choice-free* (CF) iff $\forall p \in P : |p^\bullet| \leq 1$.
3. $\mathcal{N}$ is *Join-free* (JF) iff $\forall t \in T : |{}^\bullet t| \leq 1$.
4. $\mathcal{N}$ is *Fork-Attribution* (FA) iff it is CF and JF.
5. $\mathcal{N}$ is *Equal Conflict* (EC) iff ${}^\bullet t \cap {}^\bullet t' \neq \emptyset \Rightarrow Pre[P,t] = Pre[P,t']$.

$(\mathcal{N}, M_0)$ is, in turn, a weighted T-system, CF system, etc.

The first subclass (see [18]) generalizes Marked Graphs by allowing weights. The second is a further generalization allowing attributions and is the class we are mainly interested in here. JF nets do not have proper synchronizations. They are defined reverse-dually to CF nets, so their purely structural properties can be automatically obtained by applying the reverse-dual transformation to the corresponding properties of CF nets that will be presented. We will show how strongly connected FA nets generalize ordinary circuit nets to some extent, in the sense that they play the same role in CF nets as circuits do in Marked Graphs. Finally, EC systems (see [20]) are a generalization of (Extended) Free Choice systems [3].

Let us illustrate the modeling capabilities of Choice-free net systems by means of a simple example which will be used throughout the paper. Assume we are modeling a small manufacturing system which produces tables. It consists of: (1) Two different machines to make table-legs, a new one which produces two legs at a time, and the old one, which makes legs one by one; (2) A machine to produce the table-boards; (3) A machine to assemble four legs and a board; (4) A big painting line which paints two tables at once. The painting line has greater capacity than the other machines, so more unpainted tables are brought in from another factory. So far we have enumerated the six agents of our system — the other factory can be regarded as an agent that produces unpainted tables — together with some of their input and output "messages". We are not very interested in their details, so we assume all of the agents are simple: they stay in an idle state until there is work to be done (there are enough messages in the input buffers), then they start working. They put their products in an output buffer and then become idle again. The communication structure consists of the various buffers. Some of them contain parts: leg-storage, where the two leg-makers leave their products, and from which the assembler collects four legs to assemble a table, board-storage for the boards, and unpainted-table-storage for the tables produced either by the assembler or by the other factory which are waiting to be painted. The others contain *Kanban*-like work orders. Whenever the painting line finishes a couple of tables, it distributes work orders to the leg-makers, the board-maker, and the other factory. Because of timing and cost considerations, we want to assemble 50% of the tables and bring the remaining 50% from the other factory. We also want 75% of the legs to be produced by the new leg-maker and 25% by the old one. A net model of the system is represented in Figure 1. (This system could be embedded in a bigger one, appearing as a "complex" agent consuming raw materials and producing painted tables. . .) Observe that T-systems are not suitable for modeling this case because places/buffers receiving tokens from different transitions/actions are forbidden. Some questions that we may ask are: What should the weights $wi1$, $wo1$,. . . be in order for the system to be live and bounded? Compute a "minimal" initial marking (e.g., a number of work orders present in places p_1,. . . p_4) that makes the system live and bounded. Given an initial marking, is the system live and bounded (and reversible)? What is the required capacity of the leg-storage?

IV. First Properties of Choice-free Systems

A first salient property of CF systems is inherited from the more general, although non syntactical, class of persistent systems [6]. This is *confluence*, which informs us about the structure of the reachability graph. More precisely, it states that whenever we can fire two distinct sequences from a given marking and we reach two different markings, then either of the two sequences can be completed firing transitions that are not common to both, thus reaching the same marking. We state this result as in [8], where a proof can be found.

Theorem 3 (Confluence [8]) Let $(\mathcal{N}, M_0)$ be a CF system. If $M_0 \xrightarrow{\sigma} M_1$ and $M_0 \xrightarrow{\tau} M_2$, then a marking M_3 must exist such that $M_1 \xrightarrow{\alpha} M_3$ and $M_2 \xrightarrow{\beta} M_3$, for some α and β with $\vec{\alpha}[t] = \max\{\vec{\sigma}[t], \vec{\tau}[t]\} - \vec{\sigma}[t]$ and $\vec{\beta}[t] = \max\{\vec{\sigma}[t], \vec{\tau}[t]\} - \vec{\tau}[t]$ for every $t \in T$.

A crucial consequence of confluence is the close relationship between liveness and repetitiveness. In particular, it suffices to find a repetitive sequence, i.e., a sequence that

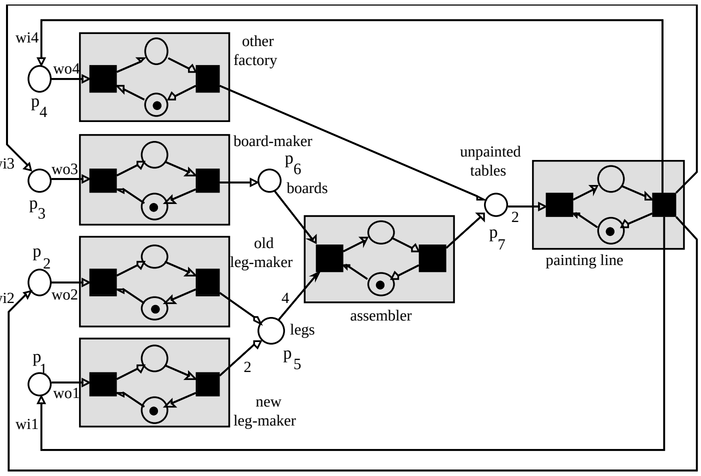


Fig. 1. First model of the table factory.

fires every transition and that can be fired over and over, to prove liveness.

Corollary 4: Let $(\mathcal{N}, M_0)$ be a CF system. $(\mathcal{N}, M_0)$ is live iff a reachable marking $M \in \mathrm{R}(\mathcal{N}, M_0)$ and a firing sequence $\sigma \in \mathrm{L}(\mathcal{N}, M)$ such that $\vec{\sigma} > 0$ and $C \cdot \vec{\sigma} \geq 0$ exist.

Proof: Let $M_x \in \mathrm{R}(\mathcal{N}, M_0)$ be an arbitrary reachable marking, and let $\sigma_x \in \mathrm{L}(\mathcal{N}, M_0)$ be the sequence such that $M_0 \xrightarrow{\sigma_x} M_x$. Let σ_0 be a sequence leading from M_0 to M. Since σ is a repetitive sequence fireable at M, then there exists some sequence $\sigma_k \in \mathrm{L}(\mathcal{N}, M)$ such that $\vec{\sigma}_k = k \cdot \vec{\sigma}$ and $\vec{\sigma}_0 + \vec{\sigma}_k \geq \vec{\sigma}_x$ for some k. Let $M_0 \xrightarrow{\sigma_0\sigma_k} M_k$. Clearly, σ is enabled at M_k, so every transition can be ultimately enabled from M_k. On the other hand, according to Theorem 3, M_k is reachable from M_x, so every transition can be ultimately enabled from M_x, hence $(\mathcal{N}, M_0)$ is live.

The "only if" part holds for arbitrary Petri nets. ■

A first consequence of this is *liveness monotonicity* wrt. the initial marking:

Corollary 5: If $(\mathcal{N}, M_0)$ is a live CF system and $M_0{}' \geq M_0$, then the system $(\mathcal{N}, M_0{}')$ is also live.

Proof: By liveness a sequence σ such that $\vec{\sigma} > 0$ and $C \cdot \vec{\sigma} \geq 0$ is fireable at some $M \in \mathrm{R}(\mathcal{N}, M_0)$. Clearly $M' = M + M_0{}' - M_0 \in \mathrm{R}(\mathcal{N}, M_0{}')$ and σ is fireable at M'. Therefore, by Corollary 4, $(\mathcal{N}, M_0{}')$ is live. ■

To end this section we recall two important properties of EC systems, which strictly include CF systems. The first one shows that liveness is equivalent to the weaker property of deadlock-freeness in bounded strongly connected EC systems. This indicates that it should be easier to analyze in these nets than in general ones. The second is related, though not comparable, to confluence: the existence of common successors in live systems is stated, but, differently from confluence, the location of the common successor is not given — which makes it weaker — and it also applies to potentially reachable markings — which makes it stronger. Both properties together lead to a method for analyzing liveness in terms of absence of solutions to a system of linear inequalities over the integers [19]. Naturally, this method could also be applied to CF systems, although we derive specific analysis methods which typically perform far better.

Theorem 6 ([19], [20]) A bounded strongly connected EC system is live iff it is deadlock-free.

Theorem 7 ([19], [20]) Let $(\mathcal{N}, M_0)$ be a live EC system. Then $\mathrm{PRG}(\mathcal{N}, M_0)$ is directed:

$$\forall M_1, M_2 \in \mathrm{PR}(\mathcal{N}, M_0) : \mathrm{R}(\mathcal{N}, M_1) \cap \mathrm{R}(\mathcal{N}, M_2) \neq \emptyset \ .$$

V. Well-formedness, Consistency, and Other Structural Properties

In general nets Cv&Ct, which implies strong connectedness, is necessary for well-formedness which is not strictly necessary for B&L. If we restrict to the CF subclass much more can be said. First of all consistency is the relevant property while conservativeness *follows* from consistency and strong connectedness. Secondly Cv&Ct is also *sufficient* for well-formedness and finally well-formedness is *necessary* for B&L. That is, a CF net cannot be said to be correct — in the sense that it permits "good" behavior — unless it is consistent and strongly connected. Moreover we will identify the problems in the net when it is not correct in terms of its P-subnets, that are strongly connected FA nets and whose role is similar to that of circuits in Marked Graphs and weighted T-systems.

Let us start studying the purely structural properties of CF nets by considering T-semiflows. Roughly speaking, systems without non-determinism present a unique possible course of events. Structurally this fact is reflected by the existence of a unique minimal T-semiflow.

Theorem 8: Let $\mathcal{N}$ be a strongly connected CF net.

1. $\mathcal{N}$ is consistent iff it has a T-semiflow.
2. $\mathcal{N}$ has at most one minimal T-semiflow.

Proof: We claim that the support of every T-semiflow of a strongly connected CF net is the whole T. Let X be a T-semiflow of $\mathcal{N}$. If $t \in \|X\|$, then $t^{\bullet\bullet} \subseteq \|X\|$, because it is always the case that for every $p \in t^\bullet$ a $t' \in p^\bullet \cap \|X\|$ exists and now $|p^\bullet| = 1$. If $\mathcal{N}$ is strongly connected, this leads to $\|X\| = T$. For Part 1, only "$\Leftarrow$" needs to be proven, and it follows immediately from the above claim. For Part 2, assume $\mathcal{N}$ has two minimal T-semiflows. Since both cover

T, they are either proportional (i.e., the same one) or we can linearly combine them to obtain another with smaller support, against minimality. ■

Notice that, unlike in weighted T-nets, the strong connectedness assumption is necessary in this case. The leftmost net in Figure 2 has a T-semiflow, but it is not consistent. On the other hand, the rightmost one is an example

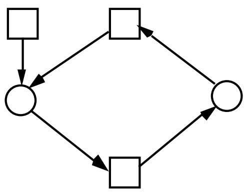
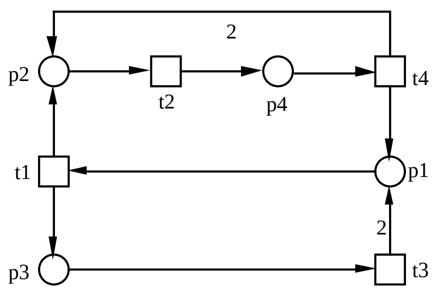


Fig. 2. Two non strongly connected CF nets.

of a consistent CF net with two minimal T-semiflows.

It easily follows from Theorem 8 that the rank of the incidence matrix of a consistent strongly connected CF net equals $|T|-1$. But, perhaps surprisingly, the converse is not true, as the net in Figure 3 shows (it is not even true for FA nets!). Fortunately, it holds for the case of *conservative*



$$\begin{bmatrix} -1 & 0 & 2 & 1 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} . \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 0$$

Fig. 3. A not consistent FA net whose $\mathrm{rank}(C) = |T| - 1$.

CF nets, as stated by the following result which has proven its importance for larger classes ([13], [20]).

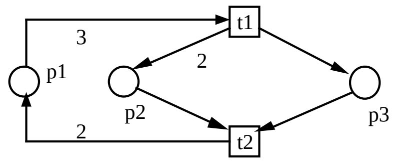
Theorem 9: Let $\mathcal{N}$ be a conservative CF net.

1. For every T-flow, X, of $\mathcal{N}$ some T-semiflow, X', exists such that $\|X'\| \subseteq \|X\|$.
2. Assuming $\mathcal{N}$ is also strongly connected, then it is consistent iff $\mathrm{rank}(C) = |T| - 1$.

Proof: For Part 1, without loss of generality, flow X can be broken up as follows: $X = X' - X''$, where $X' \gneq 0$, $X'' \geq 0$, and $\|X'\| \cap \|X''\| = \emptyset$. We claim that X' is indeed a T-semiflow. Since $C \cdot X = 0$, it follows that $C \cdot X' = C \cdot X''$. We prove first that $C \cdot X' \geq 0$. Assume the contrary. Let $C[p,T] \cdot X' = C[p,T] \cdot X'' < 0$. Since all entries in X' and X'' are non-negative and, by the CF assumption, only one entry in row $C[p,T]$, is negative, namely $C[p, p^\bullet]$, it follows that the only element in $p^\bullet$ is in $\|X'\| \cap \|X''\|$, which was assumed to be empty, hence there is a contradiction. Therefore $C \cdot X' \geq 0$. If $C \cdot X' \gneq 0$, then, by the Farkas' Lemma [11], $Y > 0$ would not exist such that $Y \cdot C \leq 0$, so $\mathcal{N}$ would not be conservative which is against the hypothesis. Then, necessarily $C \cdot X' = 0$.

For Part 2, the "$\Rightarrow$" is due to the unicity of the minimal T-semiflow in a consistent strongly connected CF net (Theorem 8). For the converse, observe that $\mathrm{rank}(C) < |T|$ implies the existence of a T-flow. Then, there is a T-semiflow (by Part 1), hence $\mathcal{N}$ is consistent (Theorem 8.1). ■

Next, we prove that consistent strongly connected CF nets are also conservative. (Observe that a conservative strongly connected CF net need not be consistent, as shown by Figure 4.) Moreover, it will be shown that the P-



$$[1 \;\; 1 \;\; 1] . \begin{bmatrix} -3 & 2 \\ 2 & -1 \\ 1 & -1 \end{bmatrix} = [0 \;\; 0]$$

Fig. 4. A strongly connected and conservative CF net that is not consistent.

subnets that are strongly connected FA nets play the role of (weighted) circuits in (weighted) T-nets. To achieve this, we first prove that a strongly connected CF net is covered by P-subnets of this kind, and that in these nets consistency and conservativeness are equivalent.

Lemma 10: Let $\mathcal{N}$ be a strongly connected CF net. Every place of $\mathcal{N}$ belongs to some strongly connected FA P-subnet, $\mathcal{N}'$.

Proof: We prove existence in a constructive manner by applying the following algorithm.

Input the place
Let $\mathcal{N}'$ be any elementary circuit containing the place
while $(\exists p \in P' : {}^\bullet p \not\subseteq T')$ **do**
  Let $\mathcal{N}_\pi$ be a subnet such that
    $P_\pi \cap P' = \{p\}$
    $|T_\pi \cap T'| = 1$ (let $\{t\} = T_\pi \cap T'$)
    $\mathcal{N}_\pi$ is an elementary path from t to p
  Let $\mathcal{N}'$ be the former $\mathcal{N}'$ merged with $\mathcal{N}_\pi$
od

The existence of the seed is guaranteed by strong connectedness, while the existence of the successive $\mathcal{N}_\pi$ is due to strong connectedness and the absence of choice places. The algorithm must terminate by finiteness of the net. The condition "$\mathcal{N}'$ is a strongly connected FA net" is invariant during the loop. The algorithm terminates when $\mathcal{N}'$ is generated by its set of places, so when $\mathcal{N}'$ is a P-subnet. Therefore, the specification is met. ■

We apply the algorithm to the net in Figure 5 to illustrate it. Let p_1 be the place to be covered. Let the seed circuit be the one defined by $\{p_1, t_1, p_3, t_3\}$. Since not all the input transitions of p_3 are in this circuit, it is not a P-subnet and we have to incorporate more nodes. By merging $\mathcal{N}_\pi$ defined by $\{t_3, p_2, t_2, p_3\}$, the strongly connected FA P-subnet is found.

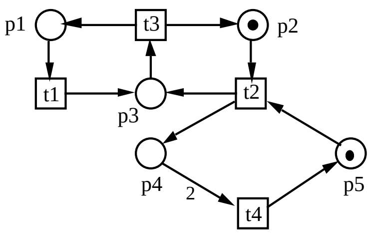


Fig. 5. A strongly connected CF net that is neither str. bounded nor str. repetitive.

Lemma 11: A strongly connected FA net is conservative iff it is consistent.

Proof: If an FA net is strongly connected and conservative, then it has a unique minimal P-semiflow ("reverse-dual" of Theorem 8), so $\text{rank}(C) = |P| - 1 = |T| - 1$ (because strongly connected FA nets have as many places as transitions), hence it is consistent (Theorem 9). For the converse, observe that the class is closed under the reverse-dual transformation. ∎

We are ready to show that consistency implies conservativeness in strongly connected CF nets. Moreover it is shown that the minimal P-semiflows correspond to P-subnets that are strongly connected FA nets. This generalizes a property of T-systems, namely that their minimal P-semiflows correspond to their circuits.

Theorem 12: Let $\mathcal{N}$ be a consistent CF net. Then, $\mathcal{N}$ is conservative iff it is strongly connected.

Proof: The "⇒" is Theorem 1.3. For the "⇐", let $\mathcal{N}' \subseteq \mathcal{N}$ be a strongly connected FA P-subnet. Since $\mathcal{N}$ is consistent, clearly so is $\mathcal{N}'$. Consequently $\mathcal{N}'$ is conservative (Lemma 11) and it has a unique minimal P-semiflow ("reverse-dual" of Theorem 8). Such P-semiflow — enlarged with zeros — is also a minimal P-semiflow of $\mathcal{N}$. In case $\mathcal{N}$ is strongly connected, it is covered by strongly connected FA P-subnets (Lemma 10), and the result follows. ∎

It follows that a Cv&Ct CF net is formed by Cv&Ct FA P-subnets. Strongly connected FA nets can be proven [17] to be Cv&Ct, or not str. repetitive, or not str. bounded in the same way as weighted circuits [8], [18]. So they can be classified accordingly as *neutral* (the Cv&Ct ones), *absorbing* (the non str. repetitive), and *generating* (the non str. bounded). In case a CF net were not Cv&Ct, then the problem would be caused by the existence of some absorbing or generating FA P-subnets, which would be identified in polynomial time by solving the system $\{Y \cdot C \lneq 0; Y \geq 0\}$ or $\{Y \cdot C \gneq 0; Y \geq 0\}$, respectively. In the example net of Figure 5, the P-subnet generated by $\{p_1, p_2, p_3\}$ is not str. bounded while the one generated by $\{p_4, p_5\}$ is not str. repetitive.

We can now prove the main result in this section. It states a polynomial time characterization of well-formedness, and that this is necessary for B&L: a system that is not well-formed (easy to check now) can never be B&L.

Theorem 13: Let $(\mathcal{N}, M_0)$ be a CF system.

1. If $(\mathcal{N}, M_0)$ is B&L, then $\mathcal{N}$ is well-formed.
2. $\mathcal{N}$ is well-formed iff it is Cv&Ct (equivalent to strongly connected and consistent in this case).

Proof: For Part 1, if $(\mathcal{N}, M_0)$ is B&L, then $\mathcal{N}$ is strongly connected and consistent (Theorem 1.1), hence conservative (Theorem 12).

For Part 2, the "⇒" is Theorem 1.2. For the converse, consistency means that $\vec{\sigma} > 0$ exists such that $C \cdot \vec{\sigma} = 0$. It is always possible to find a large enough marking to enable this vector, e.g., for all $p \in P$, let $M_0[p] = W(p, p^\bullet) \cdot \vec{\sigma}[p^\bullet]$. So, according to Corollary 4, it is always possible to mark the net lively, i.e., it is str. live. (The equivalence to strong connectedness and consistency is Theorem 12.) ∎

VI. B&L and Other Behavioral Properties

Theorem 13 states that well-formedness is necessary for B&L, although it is clearly not sufficient. There is a sufficient condition for B&L in terms of the liveness of the strongly connected FA P-subnets, similarly to the T-systems liveness characterization in terms of the circuits:

Theorem 14: Let $(\mathcal{N}, M_0)$ be a CF system without source places. $(\mathcal{N}, M_0)$ is live iff the system $(\mathcal{N}', M_0[P'])$ is live for every $\mathcal{N}'$ strongly connected FA P-subnet of $\mathcal{N}$.

Proof: The "⇒" is trivial, because if a strongly connected FA P-subsystem of $\mathcal{N}$ is not live, then the rest of the system cannot help it to run.

For the "⇐" the kernel of the proof is considering strongly connected CF nets, similarly to what is done in (weighted) T-systems [2], [18]. Take a dead transition. At least one of its input places has fewer tokens than needed, so take it and all its input transitions which must be dead too. Proceed backwards, and finally the required P-subnet $\mathcal{N}'$ will be found. ∎

According to Corollary 4, if $\mathcal{N}$ is well-formed and a repetitive (actually cyclic, i.e., a T-semiflow) sequence is fireable from M_0, then $(\mathcal{N}, M_0)$ is B&L. But this is not necessary, as manifested by the B&L CF system in Figure 6, which is not reversible. We claim that well-formedness and the

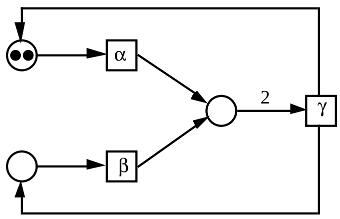


Fig. 6. A non reversible B&L CF system.

fireability of a cyclic sequence characterize B&L *and reversibility*. Below we prove this claim. Let us start with a technical lemma regarding the structure of the potential reachability graph.

Lemma 15: Every circuit in the potential reachability graph of a strongly connected CF system, $(\mathcal{N}, M_0)$, belongs to its unique terminal strongly connected component.

Proof: We prove first that there is a unique terminal strongly connected component. Let $M \in \text{PR}(\mathcal{N}, M_0)$, and let $M \xrightarrow{\sigma} M$ be the sequence corresponding to the circuit. Clearly $\vec{\sigma}$ is a T-semiflow, since the marking is not changed. According to Theorem 8, $\mathcal{N}$ is consistent and $\|\vec{\sigma}\| = T$, hence $(\mathcal{N}, M)$ is live (Corollary 4). Therefore $\mathrm{PRG}(\mathcal{N}, M)$, which is obviously equal to $\mathrm{PRG}(\mathcal{N}, M_0)$, is directed (Theorem 7). According to Theorem 12, $\mathcal{N}$ is conservative. Then $(\mathcal{N}, M_0)$ is bounded and so $\mathrm{PRG}(\mathcal{N}, M_0)$, being directed, must have a unique terminal strongly connected component.

Let U be the set of nodes of the strongly connected component of $\mathrm{PRG}(\mathcal{N}, M_0)$ containing M. To prove that it is terminal assume, on the contrary, that some arc from a node in U to a node not in U exists. Denote by t_0 the transition whose firing at $M_1 \in U$ yields $M_1^0 \notin U$. By strong connectedness some transition other than t_0, say t_1, must be fireable at M_1 yielding a marking $M_2 \in U$. By confluence (Theorem 3), t_1 is fireable at M_1^0 and t_0 is fireable at M_2 yielding, in both cases, the marking M_2^0 that cannot belong to U since M_1^0 did not. Applying the same argument repeatedly, we conclude that every node in U has an arc labeled t_0 that departs from U, so no arc in the strongly connected component is labeled t_0, against $\|\vec{\sigma}\| = T$. ■

A consequence of the above result is that every circuit in the potential reachability graph is in the *home space*. That is, if we find a reachable marking at which a cyclic sequence is fireable, then the system is B&L and that marking is a home state. Some questions still remain open, though:

- Cyclic sequences correspond to circuits in the reachability graph, and their firing count vector is proportional to the unique minimal T-semiflow. Could it be the case that minimal circuits in the reachability graph were sequences with firing count vector k times the minimal T-semiflow, with k greater than one? That is, is the minimal T-semiflow fireable, or *realizable* at every home state?
- Is there any way of knowing whether the home space has been reached? We know that every circuit of the reachability graph belongs to the home space, but nothing has been said yet about the "straight" paths leading to the home space from a non reversible initial marking.

In the case of weighted T-systems, B&L implies reversibility [18], so the second question does not apply. Regarding the first one, it is proven in [18] that the minimal T-semiflow X is indeed fireable, which is used to derive an algorithm to prove B&L of a well-formed system with polynomial complexity with respect to $\#(T, X)$. Unfortunately the proof heavily uses the fact that places have just one input transition, so it cannot be extended to CF systems. For our purposes, we will prove first a result on the reordering of sequences. It says that if a sequence σ with $\vec{\sigma} \geq X$ is fireable, X being a T-semiflow, then a sequence σ_0 with $\vec{\sigma}_0 \ngeq X$ is also fireable, so a sequence *greater than* X can be permutated leaving X till the end.

Lemma 16: Let $(\mathcal{N}, M)$ be a CF system, and let X be such that $C \cdot X = 0$. If $\sigma \in \mathrm{L}(\mathcal{N}, M)$ exists such that $\vec{\sigma} \geq X$, then also $\sigma_0(\sigma_X)^k \in \mathrm{L}(\mathcal{N}, M)$ exists such that $\vec{\sigma}_0 \ngeq X$, $\vec{\sigma} - \vec{\sigma}_0 = k \cdot X$, and $\vec{\sigma}_X = X$.

Proof: We shall prove that

$$\exists \sigma \in \mathrm{L}(\mathcal{N}, M) : \vec{\sigma} \geq X \Rightarrow \exists \sigma_0 \in \mathrm{L}(\mathcal{N}, M) : \vec{\sigma}_0 = \vec{\sigma} - X \ ,$$

because the result follows from repeated application of this.

We prove it by induction on the length of σ, $\#(T, \sigma)$.

The base of the induction, i.e., $\#(T, \sigma) = \#(T, X)$, is trivial, because in such case $\vec{\sigma} = X$, so $\vec{\sigma}_0 = 0$, and the empty sequence is "fireable" at every marking.

For the induction step, sequence σ can be written as

$$M \xrightarrow{\sigma_1} M_1 \xrightarrow{t} M_2 \xrightarrow{\sigma_2} \ ,$$

where $\vec{\sigma}_1 \leq X$ and $\vec{\sigma}_1[t] = X[t]$, i.e., t is the first transition occurring "in excess" with respect to X. Since $C \cdot X = 0$, obviously $C[{}^\bullet t, T] \cdot X = 0$. Since $\vec{\sigma}_1[t] = X[t]$, $t \notin \|X - \vec{\sigma}_1\|$. Since $\mathcal{N}$ is CF, t is the only transition that can remove tokens from places in ${}^\bullet t$, so $C[{}^\bullet t, T] \cdot (X - \vec{\sigma}_1) \geq 0$. Therefore,

$$C[{}^\bullet t, T] \cdot \vec{\sigma}_1 = C[{}^\bullet t, T] \cdot X - C[{}^\bullet t, T] \cdot (X - \vec{\sigma}_1) \leq 0 \ ,$$

hence $M_1[{}^\bullet t] = M[{}^\bullet t] + C[{}^\bullet t, T] \cdot \vec{\sigma}_1 \leq M[{}^\bullet t]$, so t enabled at M_1 implies that it is also enabled at M. It follows, applying Theorem 3, that sequence σ can be reordered firing t first:

$$M \xrightarrow{t} M_1' \xrightarrow{\sigma_1'} M_2 \xrightarrow{\sigma_2} \ ,$$

where $\vec{\sigma}_1' = \vec{\sigma}_1$. Let $\sigma' = \sigma_1' \sigma_2$. Since $\vec{\sigma}' \geq X$ and $\#(T, \sigma') < \#(T, \sigma)$, by the induction hypothesis $\sigma_0' \in \mathrm{L}(\mathcal{N}, M_1')$ exists such that $\vec{\sigma}_0' = \vec{\sigma}' - X$. Then, the sequence $\sigma_0 = t\sigma_0'$ is fireable at M, and $\vec{\sigma}_0 = \vec{\sigma} - X$. ■

As a consequence of this reordering lemma the minimal T-semiflow is indeed realizable which allows us to derive the characterization of B&L and reversibility that we have been looking for:

Theorem 17: Let $(\mathcal{N}, M_0)$ be a well-formed CF system, with minimal T-semiflow X. It is B&L iff $\sigma \in \mathrm{L}(\mathcal{N}, M_0)$ exists such that $\vec{\sigma} \geq X$ and $\vec{\sigma} - X \ngeq X$. Moreover, $M_0 + C \cdot (\vec{\sigma} - X)$ is a home state.

Proof: For the "$\Rightarrow$", by liveness at least one sequence σ' with $\vec{\sigma}' \geq X$ is fireable. Applying Lemma 16 we can put occurrences of X at the end of it. The remainder and the first occurrence of X in such a permutation is the σ we were looking for.

For the "$\Leftarrow$", applying Lemma 16 we can fire a σ_X, with $\vec{\sigma}_X = X$, at the end. Let $M_0 \xrightarrow{\sigma_0} M \xrightarrow{\sigma_X} M$ with $\vec{\sigma}_0 = \vec{\sigma} - X$. Liveness derives from Corollary 4 and by Lemma 15 M is a home state. ■

Therefore, sequences leading to the home space from transient initial markings are not greater than the minimal T-semiflow. Notice that we said "not greater" instead of "smaller". In fact, this sequence may *not* be smaller than the minimal T-semiflow. In the system of Figure 7, the shortest sequence leading to the home space is of length $\sum_{i=0}^{n-1}(2^{n-i} - 1) = 2^{n+1} - 2 - n$, while $X = \mathbb{1}$, so $\#(T, X) = 2n + 1$. Therefore, the length of the sequence σ_0 is exponential with respect to the length of the minimal T-semiflow!

From the above, it follows that reachability of a marking in a B&L and reversible CF system can be decided in polynomial time with respect to the length of the minimal T-semiflow.

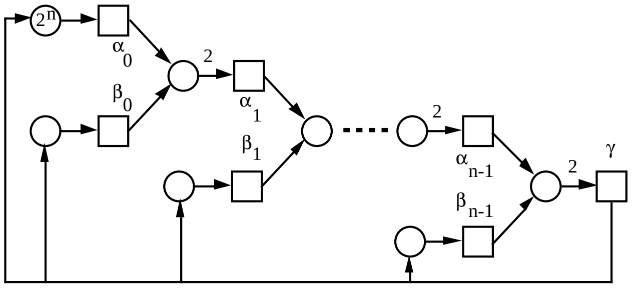


Fig. 7. A non reversible B&L CF system where the length of the shortest sequence leading to the home space is exponential with respect to the length of the minimal T-semiflow.

Theorem 18: Let $(\mathcal{N}, M_0)$ be a B&L and reversible CF system, with minimal T-semiflow X. $M \in \mathrm{R}(\mathcal{N}, M_0)$ iff $M \in \mathrm{PR}(\mathcal{N}, M_0)$ and X fireable from M. Therefore, deciding reachability is polynomial wrt. $\#(T, X)$.

Proof: X is fireable from every reachable (home) state, according to Theorem 17, and only from there, according to Lemma 15. Deciding whether $M \in \mathrm{PR}(\mathcal{N}, M_0)$ consists on checking solvability of the state equation — which has $|T|$ variables — on the natural domain. Thanks to consistency, existence of an integer solution implies existence of a natural one (add X as needed). Therefore deciding whether $M \in \mathrm{PR}(\mathcal{N}, M_0)$ is a problem of solvability of the state equation on the integer domain which is polynomial wrt. the size of the system [11]. Fireability of X is polynomial wrt. $\#(T, X)$, because the order of firings does not matter according to Theorem 3. ∎

(Note that in the case of weighted T-systems it suffices to check potential reachability [18], so the problem is polynomial wrt. $|T|$.) A theoretically appealing consequence of this — later we shall discuss its practical limitations — is that the state space of a B&L and reversible CF system can be described by a set of linear inequalities in the integer domain.

Theorem 19: Let $(\mathcal{N}, M_0)$ be a B&L and reversible CF system, with minimal T-semiflow X. Then, $\mathrm{R}(\mathcal{N}, M_0)$ is the set of all $M \in \mathbb{N}^{|P|}$ that are (part of) natural solutions to the following system of linear inequalities:

$$\begin{array}{rcl} M & = & M_0 + C \cdot \vec{\sigma} \\ M & \geq & Pre \cdot \vec{\sigma}_1 \\ M + C \cdot \sum_{i=1}^{k} \vec{\sigma}_i & \geq & Pre \cdot \vec{\sigma}_{k+1} \ , \\ & & \forall\, 1 \leq k \leq \#(T, X) - 1 \\ \sum_{i=1}^{\#(T,X)} \vec{\sigma}_i & = & X \ . \end{array} \tag{1}$$

Proof: Let us first consider the following system of linear inequalities:

$$\begin{array}{rcl} M & = & M_0 + C \cdot \vec{\sigma} \\ M & \geq & Pre \cdot \vec{\sigma}_1;\ M_1 = M + C \cdot \vec{\sigma}_1 \\ M_1 & \geq & Pre \cdot \vec{\sigma}_2;\ M_2 = M_1 + C \cdot \vec{\sigma}_2 \\ & \ldots & \\ M_{\#(T,X)-1} & \geq & Pre \cdot \vec{\sigma}_{\#(T,X)} \\ \sum_{i=1}^{\#(T,X)} \vec{\sigma}_i & = & X \ . \end{array} \tag{2}$$

As required by Theorem 18, the first (vector) equation states that M is potentially reachable, while the remaining constraints impose the fireability of the minimal T-semiflow from M. More precisely, such constraints impose the existence of a set of intermediate markings, M_i, with $1 \leq i < \#(T, X)$. Each of such intermediate markings, as well as M, enables a step (i.e., a multiset of transitions which can be fired simultaneously, represented in vector form) $\vec{\sigma}_j$, with $1 \leq j \leq \#(T, X)$, and the concatenation of all such steps leads to the sequence with firing count vector equal to X. It is not difficult to see that all the intermediate marking variables in (2) are unnecessary, so they can be eliminated leading to (1). ∎

This result provides a unified framework for checking general properties of B&L systems using *integer linear algebra* and *mathematical programming* [11], as will be briefly discussed and illustrated further on. An adaptation of the above arguments leads to a method for computing *optimal initial markings* making the system B&L, and reversible, as well as satisfying other special purpose (linear) constraints. This technique will also be illustrated in the next section.

Theorem 20: Let $(\mathcal{N}, M_0)$ be a well-formed CF system, with minimal T-semiflow X. The set of all the possible initial markings such that $(\mathcal{N}, M_0)$ is B&L and reversible is the set of all $M_0 \in \mathbb{N}^{|P|}$ that are (part of) natural solutions to the following system of linear inequalities:

$$\begin{array}{rcl} M_0 & \geq & Pre \cdot \vec{\sigma}_1 \\ M_0 + C \cdot \sum_{i=1}^{k} \vec{\sigma}_i & \geq & Pre \cdot \vec{\sigma}_{k+1} \ , \\ & & \forall\, 1 \leq k \leq \#(T, X) - 1 \\ \sum_{i=1}^{\#(T,X)} \vec{\sigma}_i & = & X \ . \end{array} \tag{3}$$

Proof: These constraints impose the fireability of the minimal T-semiflow from M_0, as in Theorem 19. Considering Theorem 17, the result follows. ∎

From the practical point of view, the main drawback of this approach is the complexity of integer programming, especially taking into account the size of the problem mainly due to the constraints imposing fireability of the minimal T-semiflow (note that practical systems are expected to have $\#(T, X)$ of reasonable size with respect to $|T|$, although this is not guaranteed in general). Nevertheless, this provides a solution to problems that were open (e.g., there is no other general technique to compute optimal initial markings even for the class of weighted circuits), while the high theoretical complexity does not mean that these problems are never solvable in practice. To make the method more applicable, several technical possibilities are envisaged, such us *reducing the net system* preserving the properties of interest [1], eliminating *redundant constraints*, taking advantage of the extreme *sparsity and regularity* of the constraints, etc.

VII. Application

Let us briefly discuss the application of the presented results to systems validation and design, while illustrating them by means of the example from Section III.

We know that a CF net is well-formed iff it is strongly connected and consistent (Theorem 13), which are properties that can be checked in polynomial time with respect to the size of the model. Well-formed nets have a unique

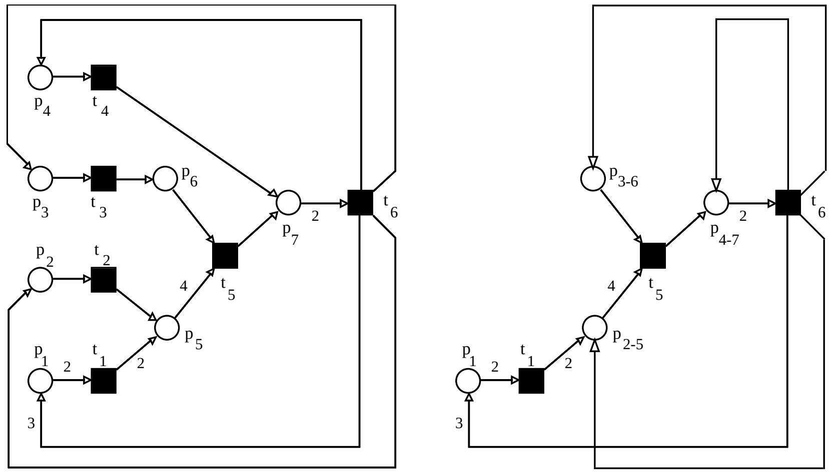


Fig. 8. Reduced models of the table factory: (a) After machine abstraction. (b) After transition post-fusion reduction (T-semiflow: $X = [3\,2\,2]$.

minimal T-semiflow (Theorem 8). In our example, the relative rate of occurrence of transitions is given by their entries in the unique minimal T-semiflow, since every infinite sequence must be proportional to it. Translating the different rates of occurrence into a desired minimal T-semiflow is straightforward, leading to $X[t] = 3$ for every transition in the new leg-maker and $X[t] = 2$ for any other transition. Then, we have to compute the appropriate weights that make the net consistent, solving $C \cdot X = 0$ for the unknowns, which are the weights indicated by $wi1, wo1, \ldots$ in Figure 1. The resulting model — reducing the agents to transitions — is depicted in Figure 8 (a). After some *transition post-fusion reductions* [1], the model in Figure 8 (b) is obtained.

Once $\mathcal{N}$ is well-formed and the minimal T-semiflow, X, is known, we concentrate on the properties of $(\mathcal{N}, M_0)$. We know that if it is possible to fire a sequence greater than or equal to X (in any order, even concurrently), then the system is live, and the reached marking is a home state (Theorem 17). Otherwise the unique deadlock will be found. In the example, $\#(T, X) = 7$ in the reduced form, so checking whether a marking makes the system live and reversible takes *at most* 7 firings. For instance, checking that a marking with 75 tokens in p_1 and 50 tokens in p_2, p_3, and p_4 makes the system live and reversible can be done by simply firing one step: the given marking corresponds in the reduced net to 75 tokens in p_1 and 50 tokens in p_{2-5}, p_{3-6}, and p_{4-7}, from which the minimal T-semiflow, $X = [3\,2\,2]$, can be fired at once.

Once we know $(\mathcal{N}, M_0)$ is B&L and reversible, more sophisticated properties of the system can be analyzed. For instance, proving reachability of a given marking can be accomplished in polynomial time by (1) checking its potential reachability and (2) in case of success, checking whether the minimal T-semiflow is fireable (Theorem 18). More generally, since the state space can be described by a set of linear constraints (Theorem 19), problems like submarking reachability, pairwise concurrency of transitions, mutual exclusion, and others can be solved by checking feasibility of a system of linear inequalities over the integers. For instance, we wonder whether it is possible that both p_4 and p_7 are empty at any reachable marking. The answer is obviously not, because whenever we remove two tokens from p_7 by firing t_6 we put a token in p_4 that can only go back to p_7 again. To verify it, we check — in the reduced model — that $M[p_{4-7}] = 0 \wedge (1)$ has no solution. Observe that there are potentially reachable markings with $M[p_{4-7}] = 0$, like $M = [0\,325\,100\,0]$, but they do not enable the minimal T-semiflow, so they are proven to be non-reachable. Similarly, we could check that no reachable marking (actually no potentially reachable marking) puts all the tokens in p_{4-7}, by checking absence of solutions to $M[p_1] + M[p_{2-5}] + M[p_{3-6}] = 0 \wedge (1)$.

Even *optimization* problems, with constraints including (1), can be solved using conventional mathematical programming techniques [11]. Some such problems are the computation of marking and enabling bounds, synchronic measures [16], etc. In the case of our example it is interesting to compute the bounds of buffer places, which give us the necessary size of the corresponding stores. For instance, the required capacity of leg-storage is: $\max\{M[p_{2-5}] \mid (1)\} = 321$, corresponding to marking $M = [0\,321\,99\,1]$ which is reachable by a sequence with firing count vector $\vec{\sigma} = [111\,0\,49]$. The number of constraints and variables in (1) would be 532 and 357 with the original model of Figure 1, but it is only 38 and 32 using the reduced net of Figure 1 (b). A solution using an "off-the-shelf" branch-and-bound solver — the public domain program *lp_solve* by M.R.C.M. Berkelaar, from Eindhoven University of Technology — took about one second in a shared SPARC station 20. If the constraints imposing fireability of the minimal T-semiflow are removed, leading to 4 constraints and 7 variables, the result is simply an upper bound. In the example it gives 325, corresponding to the potentially reachable marking $M = [0\,325\,100\,0]$, which is not actually reachable as we have already remarked.

The problem of synthesizing an optimal initial marking which makes the system B&L, reversible, and satisfies any

<table>
<tr><th>T-systems (Marked Graphs)</th><th>Weighted T-systems</th><th>Choice-free systems</th></tr>
<tr><td>Ordinary</td><td colspan="2">Weighted</td></tr>
<tr><td colspan="2">$|{}^\bullet p| = |p^\bullet| = 1$</td><td>$|p^\bullet| \leq 1$</td></tr>
<tr><td>Always Ct</td><td colspan="2">Not always Ct</td></tr>
<tr><td colspan="2">At most one minimal T-semiflow</td><td>… when str. conn.</td></tr>
<tr><td>$\mathrm{rank}(C) = |T| - 1$</td><td>… iff Ct</td><td>… iff Ct, when Cv and str. conn.</td></tr>
<tr><td>Well-formed iff str. conn.</td><td colspan="2">… and Ct</td></tr>
<tr><td colspan="3">Well-formedness is necessary for B&L</td></tr>
<tr><td colspan="3">Liveness is monotonic wrt. the marking</td></tr>
<tr><td>Live iff every circuit marked</td><td>… every circuit live</td><td>… every str. conn. FA P-subnet live</td></tr>
<tr><td colspan="3">Assume $\mathcal{N}$ well-formed and let X be the minimal T-semiflow:</td></tr>
<tr><td colspan="2">Live and reversible iff X fireable</td><td>Live iff $\vec{\sigma} \geq X$ fireable, $\vec{\sigma} - X \ngeq X$
$M_0 + C \cdot (\vec{\sigma} - X)$ is a home state</td></tr>
<tr><td colspan="3">Assume $(\mathcal{N}, M_0)$ B&L and reversible, and let X be the minimal T-semiflow:</td></tr>
<tr><td colspan="2">Reachability is polynomial wrt. $|T|$:
M reachable iff potentially reachable</td><td>… wrt. $\#(T, X)$:
… and X fireable</td></tr>
</table>

TABLE I

Properties of Marked Graphs, weighted T-systems, and Choice-free systems.

other special purpose constraints, can also be solved using a similar approach:

$$\text{minimize}\{\text{cost}(M_0) \mid \text{special purpose constraints} \cup (3)\} \ ,$$

where any sensible cost function should be marking invariant. In our example the minimal P-semiflows are $Y_1 = [1\,1\,0\,4]$ and $Y_2 = [0\,0\,1\,1]$, so a suitable cost function is $[1\,1\,1\,5] \cdot M$. If we minimize $[1\,1\,1\,5] \cdot M_0$ subject to (3), we obtain one of the "cheapest" B&L and reversible initial markings, $M_0 = [1\,4\,1\,1]$, costing 11, from which the minimal T-semiflow is indeed fireable in six steps: $\{t_5\}\{t_6\}\{2 * t_1\}\{t_5\}\{t_6\}\{t_1\}$. The solution of the integer linear programming problem, having 25 variables and 31 constraints, once again took about one second.

In large and difficult to reduce models, finding optimal initial markings may take far longer. If a suboptimal solution were sufficient, we could use a *hill-climbing* based algorithm, consisting in trying to fire the T-semiflow from a *seed marking*. Whenever we get stuck before completing the firing of the T-semiflow, we add tokens to enable one of the transitions that needs to be fired, selecting it so as to produce the lowest cost increment. As seed we take the most expensive marking that enables just one transition, for this is surely needed to fire the T-semiflow. Let us illustrate it with the (reduced) example net:

- *Selection of the seed marking*: To enable t_1 we need 2 tokens in p_1, costing 2; To enable t_5 we need 4 tokens in p_5 and 1 in p_6, costing 5; To enable t_6 we need 2 tokens in p_7 costing 10, so we select $M_{10} = [0\,0\,0\,2]$ as seed.
- *Hill-climbing*: t_6 and then t_1 can be fired from M_{10} leading to $[1\,3\,1\,1]$, which is a deadlock. We add the "cheapest" amount of tokens that enables some transitions. Adding 1 token to p_1, costing 1, enables t_1. From $M_{11} = [2\,3\,1\,1]$, costing 11, the sequence $t_1\,t_5\,t_6\,t_1\,t_5$ can be fired. This completes the T-semiflow, so M_{11} is a live initial marking.

Of course, it will not always be the case that we find an optimal marking using this algorithm. In any case, the cost of the obtained solution can be given as an upper bound for the minimum cost which may speed up the calculations when using a branch-and-bound solver.

VIII. Conclusion

Choice-free are a class of Place/Transition net systems that structurally forbids the existence of conflicts. That is, they are persistent for every initial marking. They are proposed as a rather general formal model for deterministic concurrent systems with bulk services and arrivals, supported by a reasonably complete structure theory developed in this paper. We want to stress that this theory is not only relevant for Choice-free systems alone, but it is essential for studying more complex subclasses which allow conflicts in a restricted fashion. To date, this has been found to be the case of Equal Conflict systems [20] (a weighted generalization of Free Choice ones) and Deterministic Systems of Sequential Processes [13] (State Machines cooperating via restricted message passing).

Assuming that the central fact leading to the attractive properties of Marked Graphs is their structurally imposed absence of conflicts, we wondered whether we could remove any other constraints while maintaining the major properties. In [18] it was shown that allowing weights is not detrimental to several major properties, although they do change somehow. Now we have weakened the topological constraint. It has been shown that many central properties apply to this more general setting, although what was earned in expressive power was paid for by the loss, or weakening, of some of the results as compared to Marked Graphs or weighted T-systems. The main properties of Choice-free systems are summarized and compared to those of Marked Graphs and weighted T-systems in Table I.

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