A Comparison of Weak Completeness Notions[1]

Extended Abstract

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Abstract

We compare the weak completeness notions for $\mathbf{E}$ in the sense of Lutz's resource-bounded measure theory [11] with respect to the standard polynomial time reducibilities. Our results parallel results for classical completeness by Watanabe [17] and others. We show that the weak completeness notions for 1-query reductions coincide: A set is weakly complete for $\mathbf{E}$ under 1-truth-table reducibility iff it is weakly complete for length-increasing one-one reducibility. For most of the other polynomial reducibilities, however, we obtain separations of the weak completeness notions where these reducibilities differ on $\mathbf{E}$ (Ladner et al. [10]). In fact our separations simultaneously hold for the corresponding weak completeness notions for $\mathbf{E}$ and $\mathbf{E}_2$, for the classical completeness notions, and for the weak completeness notions in the sense of the resource-bounded Baire category concepts of Ambos-Spies et al. [2] and Ambos-Spies [1].

1 Introduction

Lutz [11] introduced a resource-bounded measure theory and he showed that the polynomial-time case allows a quantitative analysis of the class $\mathbf{E} = DTIME(2^{lin})$ of the exponential time sets. He proposed an interesting extension of the fundamental concept of completeness, called weak completeness. For any polynomial-time reducibility $\leq_r^P$, a set $A \in \mathbf{E}$ is weakly p-r-complete for $\mathbf{E}$ if $\mu_p(\mathbf{P}_r(A) \cap \mathbf{E}) \neq 0$, i.e., if the class of $\leq_r^P$-predecessors of A in $\mathbf{E}$ is nonnegligible in the sense that it has nonzero measure in $\mathbf{E}$. Obviously, any p-r-complete set for $\mathbf{E}$ is also weakly p-r-complete for $\mathbf{E}$ and many interesting properties are shared by both concepts (see Juedes and Lutz [8]).

As also shown by Lutz [12], however, there are weakly p-m-complete sets for $\mathbf{E}$ which are not p-m-complete for $\mathbf{E}$. In fact, Ambos-Spies et al. [4] have shown that the class of weakly p-m-complete sets has measure 1 in $\mathbf{E}$ whereas the class of p-m-complete sets has measure 0 in $\mathbf{E}$ (Mayordomo [13]). So weakly complete sets are abundant whereas complete sets are very rare.

The previous investigations in weak completeness focussed on the case of p-m-reducibility. Here we will also consider the other standard polynomial time reducibilities and we will compare the corresponding weak completeness concepts. Starting point for these investigations is the fundamental paper of Ladner, Lynch and Selman [10], where separations for the most popular p-reducibilities in $\mathbf{E}$ are proved. In particular, there are separations of the adaptive Turing (T) and the nonadaptive truth-table (tt)-reducibility, and of special cases of the latter like the bounded query reductions (k$-$tt ($k \geq 1$), btt) and reductions with particular evaluation schemes, like the disjunctive (d) and conjunctive (c) reducibilities.

Ladner et al. [10] raised the question of whether corresponding separations can be proved for the corresponding completeness notions in $\mathbf{E}$. For reducibilities allowing at least two queries this question has been answered affirmatively by Watanabe [17]. For single-query reductions, however, the corresponding completeness notions collapse: By Berman [5], every p-many-one complete set for $\mathbf{E}$ is already p-one-one complete under length increasing reductions, and Homer et al. [7] proved the equivalence of p-m-completeness and p-1-tt-completeness. Here we obtain the corresponding results for the weak completeness notions thereby answering a question of Buhrman and Torenvliet [6]. Our collapse results for the 1-query reductions are

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based on incompressibility properties of the random sets while known biimmunity properties of the resource-bounded random sets (Mayordomo [13]) will be the basis for the separation results: To separate weak p-r-completeness from weak p-r'-completeness for some reducibilities $\leq_r^P$ and $\leq_{r'}^P$ such that $\leq_r^P$ does not imply $\leq_{r'}^P$, we construct a p-r-complete set A for $\mathbf{E}$ such that no $\leq_{r'}^P$-predecessor of A will be biimmune for a certain fixed level of the exponential hierarchy $DTIME(2^{kn})$, whence $\mu_p(\mathbf{P}_{r'}(A)) = 0$. So A is not only a witness for the separation of weak p-r-completeness and weak p-r'-completeness, but also a witness for the separation of the corresponding classical completeness notions, whence our results also imply Watanabe's separation results. Moreover, for the genericity concept of Ambos-Spies et al. [2] and the stronger genericity concepts of Ambos-Spies [1], corresponding biimmunity properties have been proved in [3]. So the set A above will also witness the separation of weak p-r-completeness and weak p-r'-completeness in the sense of the Baire categories concepts on $\mathbf{E}$ corresponding to these genericity notions (see Ambos-Spies [1] for definitions).

David Martin (personal communication) has pointed out that Watanabe's separation of $\leq_T^P$ and $\leq_{tt}^P$-completeness in $\mathbf{E}$ [17] can be used to separate $\leq_T^P$-completeness and weak $\leq_{tt}^P$-completeness.

The outline of the paper is as follows. In Section 2 we shortly review the polynomial reducibilities considered in this paper. A short account of Lutz's resource-bounded measure theory is given in Section 3. Section 4 contains the collapse results for single query reductions while Sections 5 and 6 contain the separation results for the bounded query and the unbounded query reducibilities, respectively. Finally in Section 7 we discuss extensions of our results for resource-bounded category and state some open problems.

2 Reducibilities

Here we shortly review the reducibilities discussed in this paper. For more details see e.g. Ladner et al. [10] or Watanabe [17]. Since we only consider deterministic polynomial-time bounded (p-) reducibilities, p-Turing-(p-T-) reducibility is the most general concept discussed here: A set A is *p-T-reducible* to a set B ($A \leq_T^P B$) if there is a deterministic, polynomial-time bounded oracle Turing machine M which accepts A with oracle B. (In our notation we do not distinguish between a machine and the set accepted by it and between a set and its characteristic function. So $A \leq_T^P B$ via M iff $A(x) = M^B(x)$ for all strings x.) Note that, for the set Q_x^B of queries asked by M with oracle B on input x, $|Q_x^B| \leq p(|x|)$ and $|y| \leq p(|x|)$ for all $y \in Q_x^B$, where p is a polynomial time bound of M. The query sets Q_x^B may depend on the oracle B, however, so that in general $|\bigcup \{Q_x^X : X \subseteq \Sigma^*\}|$ is only bounded by $2^{p(|x|)}$. In a nonadaptive p-T-reduction, called a *p-truth-table* (*p-tt*)*reduction* ($\leq_{tt}^P$), the queries do not depend on the oracle, i.e., for any oracle X and any string x, $Q_x^X = Q_x^\emptyset$. Special types of p-tt-reductions with particular evaluation functions are the *disjunctive* (d) and *conjunctive* (c) reducibilities: $A \leq_d^P B$ ($A \leq_c^P B$) via M if $A = M^B$ and M is a p-tt-reduction such that, for any oracle X and any input x, $x \in M^X$ iff $X \cap Q_x^X \neq \emptyset$ ($Q_x^X \subseteq X$).

Note that, for any sets A and B,

(2.1) $\quad A \leq_d^P B \ (A \leq_c^P B) \ \Rightarrow \ A \leq_{tt}^P B \ \Rightarrow \ A \leq_T^P B$

and, as shown by Ladner et al. [10], in general no other implications hold among these reducibilities.

For any p-r-reducibilities we can define bounded versions by bounding the number of queries: For any function $f: N \to N$, we say that A is *p-$f(n)$-r-reducible* to B ($A \leq_{f(n)-r}^P B$) if $A \leq_r^P B$ via a machine M such that, for any oracle X and any input x, $|Q_x^X| \leq f(|x|)$. In particular, for constant f, say $f(n) = k$, we obtain the *bounded query reducibilities* $\leq_{k-r}^P$ and we say that A is *bounded p-r* (*p-br*) *reducible* to B ($A \leq_{br}^P B$) if $A \leq_{k-r}^P B$ for some $k \geq 1$ ($r \in \{d, c, tt, T\}$). Bounded truth-table reductions can be also characterized in terms of selection functions $(f_1, \ldots, f_k)$ and an evaluation function (g): $A \leq_{k-tt}^P B$ iff there are polynomial-time computable functions $f_1, \ldots, f_k : \Sigma^* \to \Sigma^*$ and $g : \Sigma^* \times \Sigma^k \to \Sigma$ such that for any input x,

$$A(x) = g(x, B(f_1(x)), \ldots, B(f_k(x))).$$

An important special case of the btt-reductions are the single-query reductions: A set A is p-many-one (p-m) reducible to B ($A \leq_m^P B$) if there is a polynomial-time computable function f such that $A(x) = B(f(x))$ for all x. If, moreover, f is one-to-one [and length increasing] then A is *p-one-one (p-1) [p-1-li] reducible to B* ($A \leq_1^P B$ [$A \leq_{1-li}^P B$]). The relations among the 1-query-reducibilities are as follows: For any sets A, B,

$$\text{(2.2)}\qquad \begin{array}{llll} A \leq_{1-li}^P B & \Rightarrow & A \leq_1^P B & \\ & \Rightarrow & A \leq_m^P B \Leftrightarrow A \leq_{1-d}^P B \Leftrightarrow A \leq_{1-c}^P B \\ & \Rightarrow & A \leq_{1-tt}^P B \Leftrightarrow A \leq_{1-T}^P B \end{array}$$

and, in general no other implications hold (see Ladner et al. [10]). For the other bounded query reductions, the following implications hold for any sets A, B and for $k \geq 2$ and $r \in \{d, c, tt, T\}$:

$$\text{(2.3)}\qquad A \leq_{(k-1)-r}^P B \Rightarrow A \leq_{k-r}^P B \Rightarrow A \leq_{br}^P B \Rightarrow A \leq_r^P B$$

$$\text{(2.4)}\qquad A \leq_{k-d}^P B \quad (A \leq_{k-c}^P B) \Rightarrow A \leq_{k-tt}^P B \Rightarrow A \leq_{k-T}^P B$$

$$\text{(2.5)}\qquad A \leq_{k-T}^P B \Rightarrow A \leq_{(2^k-1)-tt}^P B$$

$$\text{(2.6)}\qquad A \leq_{bd}^P B \quad (A \leq_{bc}^P B) \Rightarrow A \leq_{btt}^P B \Leftrightarrow A \leq_{bT}^P B$$

Again no other implications hold as shown, explicitly or implicitly, by Ladner at al. [10]. For any of the above reducibility notions we can define corresponding completeness concepts. Let $\mathbf{P}_r(A) = \{B : B \leq_r^P A\}$. Then A is *p-r-complete* for a class $\mathbf{C}$ if $A \in \mathbf{C}$ and $\mathbf{C} \subseteq \mathbf{P}_r(A)$. Here we consider the completeness only for $\mathbf{E} = DTIME(2^{lin})$ and $\mathbf{E}_2 = DTIME(2^{poly})$, which are almost equivalent: For $A \in \mathbf{E}$, A is p-r-complete for $\mathbf{E}$ iff A is p-r-complete for $\mathbf{E}_2$. Watanabe [17] proved that for the above p-reducibilities allowing at least two queries, nonequivalent reducibilities also give rise to different completeness concepts. This is contrasted by results of Berman [5] and Homer et al. [7] which together imply that all types of 1-query reducibilities yield the same completeness concepts for $\mathbf{E}$ ($\mathbf{E}_2$), namely a set A is p-1-li-complete for $\mathbf{E}$ ($\mathbf{E}_2$) iff A is p-1-T-complete for $\mathbf{E}$ ($\mathbf{E}_2$).

Finally note that for any of the above p-r-reducibilities there is an effective enumeration $\{M_e : e \geq 0\}$ of all p-r-reductions such that, for x with $|x| \geq e$, $M_e^X(x)$ can be computed relative to X in $2^{|x|}$ steps (uniformly in e, x, X). In the following we call such an enumeration shortly an exponential time enumeration of the p-r-reductions.

3 Measure on E

In 1939 Ville [16] introduced martingales as a tool for characterizing measure 0 sets. In the early 70's Schnorr [14], [15] defined concepts of randomness based on computable martingales, and he also considered resource-bounded version of these notions. Recently, Lutz [11] extended these to a resource-bounded measure theory.

Definition 3.1 *(Ville [16], Schnorr [14],[15] and Lutz[11])*

- *A* martingale *is a function $d : \Sigma^* \to R_+$, where R_+ is the set of nonnegative reals, such that $\forall x \in \Sigma^*$ $(d(x0) + d(x1) = 2d(x))$. A martingale d* succeeds on *a set A if $limsup_n d(A \upharpoonright z_n) = \infty$. (Here z_n is the n-th string under the standard ordering and $A \upharpoonright z_n$ is the characteristic string of $A \cap \{x : x < z_n\}$.) $S^\infty[d]$ denotes the class of all sets on which the martingale d succeeds.*
- *A martingale d is* computable *if the range of d is contained in Q_+, where Q_+ is the set of nonnegative rationals, and d is recursive. A* $t(n)$-martingale *is a computable martingale $d \in DTIME(t(n))$.*
- *A set A is* $t(n)$-random *if no $t(n)$-martingale succeeds on A.*
- *A class $\mathbf{C}$ has* $t(n)$-measure 0, *($\mu_{t(n)}(\mathbf{C}) = 0$) if there is a $t(n)$-martingale d such that $\mathbf{C} \subseteq S^\infty[d]$. $\mathbf{C}$ has* $t(n)$-measure *1 ($\mu_{t(n)}(\mathbf{C}) = 1$) if $\overline{\mathbf{C}}$ has $t(n)$-measure 0.*
- *A* p-martingale *is a $q(n)$-martingale for some polynomial q.* p-randomness *and* p-measure *(μ_p) are defined correspondingly.*

As observed by Lutz [11], this polynomial-time measure allows the definition of a measure on $\mathbf{E}$.

Definition 3.2 *(Lutz [11]) A class* $\mathbf{C}$ has *measure* 0 in $\mathbf{E}$ *if* $\mu_p(\mathbf{C} \cap \mathbf{E}) = 0$, *and* $\mathbf{C}$ has *measure* 1 in $\mathbf{E}$ *if* $\overline{\mathbf{C}}$ *has measure 0 in* $\mathbf{E}$.

By Lutz [11], $\mu_p(\mathbf{E}) \neq 0$, whence this definition is consistent. Lutz's generalization of the classical completeness concept for any p-r-reducibility from Section 2 is as follows:

Definition 3.3 *(Lutz [11]) A set A is* weakly p-r-complete *for* $\mathbf{E}$ *if* $\mu_p(\mathbf{P}_r(A) \cap \mathbf{E}) \neq 0$, *i.e., if* $\mathbf{P}_r(A)$ *does not have measure 0 in* $\mathbf{E}$.

Since $\mu_p(\mathbf{E}) \neq 0$, any p-r-complete set for $\mathbf{E}$ is also weakly p-r-complete for $\mathbf{E}$.

Proposition 3.4 *(Lutz [11]) Every p-r-complete set for* $\mathbf{E}$ *is also weakly p-r-complete for* $\mathbf{E}$.

As Lutz [12] has also shown, however, there are weakly p-m-complete sets for $\mathbf{E}$ which are not p-m-complete for $\mathbf{E}$. In fact, as shown by Ambos-Spies et al. [4], the class of weakly p-m-complete sets has measure 1 in $\mathbf{E}$ whereas, by a result of Mayordomo [13], the class of the p-m-complete sets has measure 0 in $\mathbf{E}$.

Theorem 3.5 *(Lutz [11], Ambos-Spies et al. [4], Mayordomo [13]) The class of the weakly p-m-complete sets for* $\mathbf{E}$ *has measure 1 in* $\mathbf{E}$ *whereas the class of p-m-complete sets for* $\mathbf{E}$ *has measure 0 in* $\mathbf{E}$. *So, in particular, there is a weakly p-m-complete set for* $\mathbf{E}$ *which is not p-m-complete for* $\mathbf{E}$.

Note that, for any p-reducibilities $\leq_r^P$ and $\leq_{r'}^P$ such that $\leq_r^P$ implies $\leq_{r'}^P$, any weakly p-r-complete set is also weakly p-r'-complete. In the following, we will tacitly use this simple fact. Moreover, we will use the following characterization of p-measure 0 classes in terms of randomness.

Proposition 3.6 *(Ambos-Spies et al. [4]) A class* $\mathbf{C}$ *has p-measure 0 (measure 0 in* $\mathbf{E}$*) iff, for some $k \geq 2$, there is no n^k-random set in* $\mathbf{C}$ *(*$\mathbf{C} \cap \mathbf{E}$*). Hence A is weakly p-r-complete for* $\mathbf{E}$ *iff , for all $k \geq 2$, there is an n^k-random set in* $\mathbf{P}_r(A) \cap \mathbf{E}$.

In fact, Ambos-Spies et al. [4] have shown that, in Proposition 3.6, k can be fixed as $k = 2$ (but this stronger result will not be used in the following). The only nontrivial results on the p-measure required for our investigations are the observations by Mayordomo and Juedes and Lutz that the classes of certain biimmune and incompressible sets have p-measure 1. We state these results in terms of random sets and, for the sake of completeness, we sketch the proofs.

Lemma 3.7 *(Mayordomo [13]) Let A be n^{k+1}-random ($k \geq 1$). Then A is $DTIME(2^{kn})$-biimmune, i.e., there is no infinite set $B \in DTIME(2^{kn})$ such that $A \cap B \in DTIME(2^{kn})$.*

Proof. For a contradiction assume that B is infinite and $B, A \cap B \in DTIME(2^{kn})$. Define a martingale d by letting $d(\lambda) = 1$; $d((X \restriction x)0) = d((X \restriction x)1) = d(X \restriction x)$ for $x \notin B$; and $d((X \restriction x)A(x)) = 2d(X \restriction x)$ and $d((X \restriction x)1 - A(x)) = 0$ for $x \in B$. Then d is an n^{k+1}-martingale and d succeeds on A. So A is not n^{k+1}-random contrary to assumption.

Lemma 3.8 *(Juedes and Lutz [8]) Let A be n^{k+1}-random ($k \geq 1$). Then A is not compressible under $DTIME(2^{kn})$-many-one reductions, i.e., for any set B and any function $f \in DTIME(2^{kn})$ such that $A \leq_m B$ via f, f is almost one-to-one.*

Proof. For a contradiction assume that $A \leq_m B$ via f where $f \in DTIME(2^{kn})$ and

(3.1) $\exists^\infty x \, \exists \, y < x(f(x) = f(y))$.

Define a martingale d by letting $d(\lambda) = 1$; $d((X\upharpoonright x)0) = d((X\upharpoonright x)1) = d(X\upharpoonright x)$ if $f(y) \neq f(x)$ for all $y < x$; and $d((X\upharpoonright x)X(y_0)) = 2d(X\upharpoonright x)$ and $d((X\upharpoonright x)(1 - X(y_0))) = 0$ otherwise, where y_0 is the least string $y < x$ with $f(y) = f(x)$. Then d is an n^{k+1}-martingale. Moreover, since $A \leq_m B$ via f implies that $A(x) = A(y)$ for any strings x, y with $f(x) = f(y)$, it follows from (3.1) that d succeeds on A. So A is not n^{k+1}-random contrary to assumption.

We conclude with some remarks on Lutz's measure on $\mathbf{E}_2$. Lutz observed that the p_2-measure, where p_2 means the functions $2^{(logn)^k}$ for some $k \geq 0$, defines a consistent measure on $\mathbf{E}_2$. Since the development of the measure on $\mathbf{E}_2$ parallels the one of the measure on $\mathbf{E}$, we do not introduce this concept here explicitly and refer the reader to Lutz [11] and to Juedes and Lutz [9] for a discussion of the relations on the weak completeness concepts between $\mathbf{E}$ and $\mathbf{E}_2$, where the following proposition is implicitly shown.

Proposition 3.9 *Every weakly p-m-complete set for $\mathbf{E}$ is weakly p-m-complete for $\mathbf{E}_2$. If $\leq_m^P$ implies $\leq_r^P$ then every weakly p-r complete set for $\mathbf{E}$ is weakly p-r complete for $\mathbf{E}_2$.*

Here we will use the following simple criterion for the failure of weak completeness in $\mathbf{E}_2$ in the style of the Proposition 3.6.

Proposition 3.10 *Assume that for some $k \geq 1$, $\mathbf{P}_r(A)$ does not contain any n^k-random set, then A is neither weakly p-r-complete for $\mathbf{E}$ nor weakly p-r-complete for $\mathbf{E}_2$.*

4 Single Query Reductions

Theorem 4.1 *For any set A, A is weakly p-1-li-complete for $\mathbf{E}$ iff A is weakly p-m-complete for $\mathbf{E}$.*

For the proof of the theorem we need the following lemma.

Lemma 4.2 *Let $A \in DTIME(2^{kn})(k \geq 1)$ and let B be an n^c-random set $(c > k)$ such that $B \in \mathbf{P}_m(A) \cap \mathbf{E}$. There is an n^c-random set $C \in \mathbf{P}_{1-li}(A) \cap \mathbf{E}$.*

Proof. Fix f such that $B \leq_m^P A$ via f. We claim that there is a number n such that

(4.1) $\forall\, x, y \;\; (|x|, |y| \geq n \;\&\; x \neq y \;\Rightarrow\; |f(x)| > |x| \;\&\; f(x) \neq f(y))$.

To show this, fix $x_0 \in A$ and $x_1 \notin A$ and define $\widehat{f}$ by letting

$$\widehat{f}(x) = \begin{cases} f(x) & \text{if} \quad |f(x)| > |x| \\ x_0 & \text{if} \quad |f(x)| \leq |x| \;\&\; f(x) \in A \\ x_1 & \text{if} \quad |f(x)| \leq |x| \;\&\; f(x) \notin A \end{cases}$$

Then $A \leq_m B$ via $\widehat{f}$ and, by choice of f and A, $\widehat{f} \in DTIME(2^{kn})$. So, by Lemma 3.8, $\widehat{f}$ is almost $1-1$. It follows that $\widehat{f}(x) = f(x)$ (hence $|f(x)| > |x|$) for almost all x, whence f is almost $1-1$ too. Obviously this implies (4.1).

Now define C as follows. For x with $|x| \geq n$ let $C(x) = B(x)$, and for the m-th string z_m of length $< n$ let $C(z_m) = A(u_m)$ where u_m is the m-th string of length n $(0 \leq m < 2^n)$. Then C is a finite variant of B whence $C \in \mathbf{E}$ and C is n^c-random. Moreover, $C \leq_{1-li}^P A$ via g where $g(z_m) = u_m$ for $m < 2^n$ and $g(x) = f(x)$ for x with $|x| \geq n$.

Proof of Theorem 4.1. Fix $A \in \mathbf{E}$. By Lemma 4.2 there is a number k such that for all $c > k$, $\mathbf{P}_m(A) \cap \mathbf{E}$ contains an n^c-random set iff $\mathbf{P}_{1-li}(A) \cap \mathbf{E}$ contains an n^c-random set. So the claim follows from Proposition 3.6.

Theorem 4.3 *For any set A, A is weakly p-1-tt-complete for $\mathbf{E}$ iff A is weakly p-m-complete for $\mathbf{E}$.*

For the proof of the theorem we need the following two observations.

Proposition 4.4 *Let A, B be sets such that $B \in \mathbf{P}_{1-tt}(A)$. There is a set $C \in \mathbf{P}$ such that, for $\widehat{B} = (B \cap C) \cup (\overline{B} \cap \overline{C})$, $\widehat{B} \in \mathbf{P}_m(A)$.*

Proof. Fix a p-1-tt-reduction f, g from B to A, i.e., $f : \Sigma^* \to \Sigma^*$ and $g : \Sigma^* \times \Sigma \to \Sigma$ are polynomial-time computable and, for every string x, $B(x) = g(x, A(f(x)))$. Then, for $C = \{x : g(x, 0) < g(x, 1)\}$, $C \in \mathbf{P}$ and, as one can easily check, $\widehat{B} \leq_m^P A$.

Lemma 4.5 *Let B,C be sets such that B is n^k-random ($k \geq 2$) and $C \in \mathbf{P}$. Then $(B \cap C) \cup (\overline{B} \cap \overline{C})$ is n^k-random.*

Proof. Let $\widehat{B} = (B \cap C) \cup (\overline{B} \cap \overline{C})$ and, for a contradiction, assume that there is an n^k-martingale d which succeeds on $\widehat{B}$. Consider the function t: $\Sigma^* \to \Sigma^*$ which maps every string w to the string $t(w)$ with $|w| = |t(w)|$ and $t(w)(i) = w(i)$ for $z_i \in C$ and $t(w)(i) = 1 - w(i)$ for $z_i \notin C$ $(0 \leq i < |w|)$. Then t is n^2-computable and $t(B \restriction x) = \widehat{B} \restriction x$ for all x. So, for d' defined by $d'(w) = d(t(w))$, it is easy to check that d' is a martingale, d' is n^k-computable, and $d'(B \restriction x) = d(\widehat{B} \restriction x)$, whence d' succeeds on B. It follows that B is not n^k-random contrary to assumption.

Proof of Theorem 4.3. Fix $A \in \mathbf{E}$ and $k \geq 2$. Then, by Proposition 4.4 and Lemma 4.5, $\mathbf{P}_{1-tt}(A) \cap \mathbf{E}$ contains an n^k-random set iff $\mathbf{P}_m(A) \cap \mathbf{E}$ contains an n^k-random set. So the claim follows from Proposition 3.6.

Corollary 4.6 *For any set A, A is weakly p-1-li-complete for $\mathbf{E}$ iff A is weakly p-1-T-complete for $\mathbf{E}$.*

Proof. Since $\leq_{1-tt}^P$ and $\leq_{1-T}^P$ coincide, this follows from Theorems 4.1 and 4.3.

By straightforward modifications of the above arguments, we obtain the corresponding results for weak completeness for $\mathbf{E}_2$.

5 Bounded Query Reductions

Theorem 5.1 *Let $k \geq 1$. There is a set A such that*

(5.1) A is p-$(k+1)$-d-complete for $\mathbf{E}$ (hence weakly p-$(k+1)$-d-complete for $\mathbf{E}$ and $\mathbf{E}_2$) and

(5.2) A is neither weakly p-k-T-complete for $\mathbf{E}$ nor weakly p-k-T-complete for $\mathbf{E}_2$.

Note that Theorem 5.1 gives a very strong separation for $(k+1)$-query reductions and k-query-reductions. It simultaneously separates the corresponding completeness and weak completeness notions and it works for both adaptive and nonadaptive queries. Also note that the disjunctive reduction in (5.1) can be replaced by a corresponding conjunctive reduction, since the complement of a p-$(k+1)$-d-complete set is p-$(k+1)$-c-complete whereas weak p-k-T-completeness is invariant under complementation.

The proof of Theorem 5.1 is based on the following lemma.

Lemma 5.2 *Let $k \geq 1$. There is a set $A \in DTIME(2^{2n})$ such that (5.1) holds and*

(5.3) $\mathbf{P}_{k-T}(A) \cap BIIMMUNE(2^{2n}) = \emptyset$,

where $BIIMMUNE(2^{2n})$ is the class of $DTIME(2^{2n})$-biimmune sets.

Before we prove this lemma, we show how it implies the theorem.

Proof of Theorem 5.1. Fix A as in Lemma 5.2. It suffices to show that (5.2) holds. By (5.3) and by Lemma 3.7, $\mathbf{P}_{k-T}(A)$ does not contain any n^3-random set. So (5.2) follows from Proposition 3.10.

Proof of Lemma 5.2. We inductively define a set A with the desired properties. Let E be a p-m-complete set for $\mathbf{E}$ with $E \in DTIME(2^n)$. Then, to make A p-$(k+1)$-d-complete for $\mathbf{E}$, it suffices to ensure that

(5.4) $\forall x\ (x \in E \ \Leftrightarrow\ \exists i \leq k\, (0^i 1x \in A))$.

Since strings will be put into A only in order to satisfy (5.4), we will have

(5.5) $A \subseteq \{0^i 1x \,:\, i \leq k \,\&\, x \in E\}$.

The strategy for satisfying (5.3) is as follows. Let $\{M_e \,:\, e \geq 0\}$ be an exponential time enumeration of the p-k-T-reductions. Then it suffices to define sets B_e , $e \geq 0$, such that

(5.6) B_e infinite & $B_e \in \mathbf{P}$ & $B_e \cap M_e^A \in DTIME(2^{2n})$.

The basic idea for achieving this is to define B_e and A in such a way that for strings $x \in B_e$ the only relevant queries in the computation of $M_e^A(x)$ have length $\leq |x|$ (hence, by $A \in DTIME(2^{2n})$, can be answered in $2^{2|x|}$ steps). To make this compatible with (5.4), the sets B_e are chosen to be very sparse. Let δ be the iterated exponential function, i.e., $\delta(0) = 0$ and $\delta(n+1) = 2^{\delta(n)}$, and let $B_e = \{0^{\delta(\langle e,n\rangle)} \,:\, n \geq 0\}$. Then, obviously, B_e is infinite and, as one can easily check, $B_e \in \mathbf{P}$.

For the definition of A, let $A_s = \{0^i 1x \in A \,:\, i \leq k \,\&\, x < 0^{\delta(s)}\}$ for $s \geq 0$. Note that, by (5.5), $A = \bigcup_{s \geq 0} A_s$. So it suffices to inductively define the parts A_s of A. Let $A_0 = \emptyset$ and, given A_s, define A_{s+1} as follows: Fix e, n such that $s = \langle e,n\rangle$, let Q be the set of the k queries asked by $M_e^{A_s}(0^{\delta(s)})$, and, for every string x, let $i(x)$ be the least number $i \leq k$ such that $0^i 1x \notin Q$. Then

$$A_{s+1} \;=\; A_s \,\cup\, \{0^{i(x)} 1x \,:\, 0^{\delta(s)} \leq x < 0^{\delta(s+1)} \,\&\, x \in E\}.$$

As one can easily check, $A \in DTIME(2^{2n})$. It remains to show that (5.4) and (the third part of) (5.6) hold. Now (5.4) is immediate by definition of A. For a proof of (5.6) fix e, let p be a polynomial-time bound of M_e, and fix n_0 such that $p(n) < 2^n$ for $n \geq n_0$. It suffices to show that, for $n \geq n_0$, $M_e^A(0^{\delta(\langle e,n\rangle)})$ can be computed in $O(2^{2\delta(\langle e,n\rangle)})$ steps. Let $s = \langle e,n\rangle$ and let Q be the query set of $M_e^{A_s}(0^{\delta(\langle e,n\rangle)})$. Then, by choice of n, $|y| < \delta(s+1)$ for all $y \in Q$, whence, by defintion of A_{s+1}, no query $y \in Q - A_s$ will be put into A. So $M_e^A(0^{\delta(\langle e,n\rangle)}) = M_e^{A_s}(0^{\delta(\langle e,n\rangle)})$, both computations have the same query set Q, and, for any string $y \in Q \cap A$, $y = 0^i 1x$ for some $i \leq k$ and x with $|x| < \delta(\langle e,n\rangle)$. By $A \in DTIME(2^{2n})$ this implies that $M_e^A(0^{\delta(\langle e,n\rangle)})$ can be computed in $O(2^{2\delta(\langle e,n\rangle)})$ steps.

By a straightforward modification of the above argument we can separate weak p-tt-completeness and weak p-btt-completeness:

Theorem 5.3 *There is a set A such that*

(5.7) A is p-d-complete for $\mathbf{E}$ (hence weakly p-d-complete for $\mathbf{E}$ and $\mathbf{E}_2$) and

(5.8) A is neither weakly p-btt-complete for $\mathbf{E}$ nor weakly p-btt-complete for $\mathbf{E}_2$.

Remark. For the classical completeness notions, Watanabe claims the following result subsuming the claims corresponding to Theorems 5.1 and 5.3 in this setting: For any polynomials p and q such that $q(n) > p(n)$ a.e., there exists a p-$q(n)$-tt-complete set for $\mathbf{E}$ which is not p-$p(n)$-tt-complete for $\mathbf{E}$ (see [17], Theorem 3.7(2)). In general, however, this claim is false. For instance, assume that $p(n) = n$, $q(n) = n+1$ and let A be p-$q(n)$-complete for $\mathbf{E}$. For any set $B \in \mathbf{E}$, let $B_i = \{x \,:\, xi \in B\}$ $(i = 0, 1)$. Then $B_i \in \mathbf{E}$ whence $B_i \leq^P_{(n+1)-tt} A$ by assumption. Since, for $y = xi$, $y \in B$ iff $x \in B_i$, and $|x| = |y| - 1$, it follows that $B \leq^P_{n-tt} A$. Hence p-n-tt-completeness and p-$(n+1)$-tt-completeness coincide. The question, under which conditions (weak) p-$p(n)$-tt-completeness and (weak) p-$q(n)$-tt-completeness differ, will be treated in the full version of this paper.

6 Unbounded Queries

In this section we distinguish the weak completeness notions for reducibilities which do not differ in the number of the admissible queries but in the way the queries may be selected or evaluated. We first consider the case of adaptive versus nonadaptive queries.

Theorem 6.1 *There is a set A such that*

(6.1) A is p-T-complete for $\mathbf{E}$ *(hence weakly p-T-complete for* $\mathbf{E}$ *and* $\mathbf{E}_2$*) and*

(6.2) A is neither weakly p-tt-complete for $\mathbf{E}$ *nor weakly p-tt-complete for* $\mathbf{E}_2$.

By the observations made in Section 5, for a proof of Theorem 6.1 it suffices to prove the following lemma.

Lemma 6.2 *There is a set $A \in DTIME(2^{3n})$ such that (6.1) holds and*

(6.3) $\mathbf{P}_{tt}(A) \cap \ BIIMUNE(2^{4n}) = \emptyset$.

Proof. The pattern of the proof is the same as that of Lemma 5.2. Fix sets E and B_e ($e \geq 0$) as there and let $\{M_e\colon e \geq 0\}$ be an exponential time enumeration of the *p-tt*-reductions. Then, to satisfy (6.1) and (6.3), it suffices to ensure that $E \leq_T^P A$ and

(6.4) $B_e \cap M_e^A \in DTIME(2^{4n})$,

respectively. To achieve the former, we ensure that, for all strings x,

(6.5) $x \in E \iff \exists y(|y| = |x| \ \& \ \forall z(|z| \leq |x| \Rightarrow [y \sqsubseteq z \iff \langle x, z\rangle \in A]))$

(Note that, by a standard prefix search, (6.5) implies that $E \leq_T^P A$.)

For the definition of A, let $A = \cup_{s\geq 0} A_s$, where $A_s = \{\langle x, z\rangle \ : \ |z| \leq |x| < \delta(s)\} \cap A$ is inductively defined as follows: Let $A_0 = \emptyset$. Given $s = \langle e, n\rangle \ > \ 0$ and A_s, for the definition of A_{s+1}, let Q_s be the query set of $M_e^{\emptyset}(0^{\delta(s)})$ and distinguish the following two cases. If $|Q_s| \ \geq 2^{\delta(s)}$ or $|y| \geq 2^{\delta(s)}$ for some $y \in Q_s$, then let

$$A_{s+1} = A_s \cup \{\langle x, 0^m\rangle \ : \ 0^{\delta(s)} \leq x < 0^{\delta(s+1)} \ \& \ x \in E \ \& \ m \leq |x| \ \}.$$

Otherwise, let

$$A_{s+1} = A_s \cup \{\langle x, z\rangle \ : \ 0^{\delta(s)} \leq x < 0^{\delta(s+1)} \ \& \ z \sqsubsetneq y_x \ \} \cup \{\langle x, y_x\rangle \ : \ 0^{\delta(s)} \leq x < 0^{\delta(s+1)} \ \& \ x \in E \ \},$$

where y_x is the least string y with $|y| = |x|$ such that $\langle x, y\rangle \notin Q_s$.

Then, as one can easily check, (6.5) is satisfied and $A \in DTIME(2^{3n})$. Moreover, for any $\langle x, z\rangle$ and s, $\langle x, z\rangle \ \in A_s$ can be decided in $2^{3\delta(s)}$ steps. For a proof of (6.4) fix e. Then, for all sufficiently large numbers m, the second case applies in the definition of $A_{<e,m>+1}$. So it suffices to show that, in this case, $M_e^A(0^{\delta(<e,m>)})$ can be computed in $O(2^{4\delta(<e,m>)})$ steps. Let $s = \langle e, m\rangle$ and let Q_s be the query set of $M_e^{\emptyset}(0^{\delta(s)})$. Then Q_s can be computed in *poly*$(\delta(s))$ steps and, since M_e is a *p-tt*-reduction, Q_s is the query set of $M_e^A(0^{\delta(s)})$. So it suffices to decide $\langle x, z\rangle \in A$ in $O(2^{3\delta(s)})$ steps for all strings $\langle x, z\rangle \in Q_s$. Now if $x < 0^{\delta(s)}$ then $\langle x, z\rangle \in A$ iff $\langle x, z\rangle \in A_s$, and , as observed above, the latter can be decided in $O(2^{3\delta(s)})$ steps. Otherwise, by case assumption and by choice of y_x, $0^{\delta(s)} \leq x < 0^{\delta(s+1)}$ and $\langle x, z\rangle \in A$ iff z is a proper initial segment of y_x. But y_x can be computed in *poly*$(|x|)$ steps, whence in this case $\langle x, z\rangle \in A$ can be decided in *poly*$(\delta(s))$ steps.

By a straightforward modification of the above proof, we can show that, for $k \geq 2$, there is a set $A \in DTIME(2^{3n})$ which is *p-k-T*-complete for $\mathbf{E}$ and such that $\mathbf{P}_{(2^k-2)\text{-}tt}(A) \cap BIIMUNE(2^{3n}) = \emptyset$. Hence weak *p-k-T*-completeness does not imply weak *p*-$(2^k - 2)$-*tt*-completeness (cf. (2.5)).

In our final separation result, we compare disjunctive with conjunctive reductions.

Theorem 6.3 *There is a set A such that*

(6.6) A is p-2-d-complete for $\mathbf{E}$ *(hence weakly p-2-d-complete for* $\mathbf{E}$ *and* $\mathbf{E}_2$*) and*

(6.7) A is neither weakly p-c-complete for $\mathbf{E}$ *nor weakly p-c-complete for* $\mathbf{E}_2$.

Note that for the complement $\overline{A}$ of a set A as in Theorem 6.3 we obtain the corresponding result with the roles of the disjunctive and conjunctive reducibilities interchanged. So, for any $k \geq 2$, this gives a pairwise separation of weak *p-k-c*-completeness, weak *p-k-d*-completeness and weak *p-k- tt*-completeness.

Similar to the previous separation results in this paper, Theorem 6.3 follows from the following lemma.

Lemma 6.4 *There is a set $A \in DTIME(2^{3n})$ such that (6.6) holds and*

(6.8) $\mathbf{P}_c(A) \cap BIIMUNE(2^{3n}) = \emptyset$.

We omit the proof of the lemma which follows the same pattern as the proofs of Lemmas 5.2 and 6.2.

7 Further Results and Open Problems

Resource-bounded Baire category concepts provide another useful tool for the study of the structure of complexity classes, and weak completeness has been studied in this setting too (see Ambos-Spies [1] for details). Since, as shown in [3], the AFH-n^k-generic sets of Ambos-Spies, Fleischhack and Huwig [2], share the fundamental properties of the n^k-random sets used in this paper (cf. Lemmas 3.7 and 3.8), the collapse results of Section 4 can be duplicated in the Baire category setting induced by this genericity concept and the separation results of Sections 5 and 6 immediately carry over to this setting by our technical lemmas. In fact, for the same reason, our separation results also hold for the weak completeness notions for stronger Baire category concepts, like the concepts of Ambos-Spies [1] based on the so called general and extended genericity concepts.

An interesting question which we did not discuss in this paper concerns the relations between the corresponding completeness and weak completeness notions (in the measure setting; in the category setting these questions have been completely answered). Since Ambos-Spies et al. [4] have shown that, for any n^k-random set A, A is weakly p-m-complete for $\mathbf{E}$ but not p-btt-complete for $\mathbf{E}$, it follows from Proposition 3.10 that, for any bounded query reducibility r, weak p-r-completeness is a proper generalization of p-r-completeness. For unbounded queries, however, the corresponding questions are still open.

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