Liveness and Home States in Equal Conflict Systems *

Enrique Teruel   Manuel Silva

Departamento de Ingeniería Eléctrica e Informática
Centro Politécnico Superior de Ingenieros de la Universidad de Zaragoza
María de Luna 3, E-50015 Zaragoza (Spain)
Tel: + 34 76 517274   Fax: + 34 76 512932
e-mail: eteruel@cc.unizar.es

Abstract

Structure theory is a branch of net theory devoted to investigate the relationship between the structure and the behaviour of net models. Many of its powerful results have been derived for some *subclasses* of *ordinary* (i.e. all weights one) Place/Transition net systems. Nevertheless, weights may be very convenient to properly model systems with *bulk* services and arrivals. They can be implemented by means of ordinary subnets, but then the original structure of the net becomes too complex, even in the simplest cases, to be amenable of some interesting structural analysis techniques.

We present here some results regarding the structure theory of a subclass of (weighted) Place/Transition systems that naturally generalizes the ordinary subclass of Free Choice systems: *Equal Conflict systems*. We concentrate on a couple of important results concerning the relationship between the structure (in particular, represented by the state equation) and the behaviour. Actually, we show that liveness of bounded Equal Conflict systems can be decided verifying that there is no solution to a linear system of inequalities in the integer domain, and that live and bounded Equal Conflict systems have home states.

Keywords: Place/Transition Net Systems, Structure Theory, Equal Conflict Systems.

Topics: Analysis and synthesis, structure and behaviour of nets; System design and verification using nets.

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1 Introduction

One of the indigeneous techniques for the analysis of Petri net models is the so called *structural analysis.* This analysis method focuses on the relationship between the net structure and its behaviour. Net structure can be studied using *graph theory*, or *linear algebra*-based arguments using the *incidence matrices.* In general, structural techniques present a good trade-off between computational efficiency and decision power.

From the early seventies, much effort has concentrated on studying limited *subclasses* of Petri nets to obtain a better understanding of more general systems. One of the most studied of these subclasses, presenting an outstanding trade-off between generality and "nice" theory are *Free Choice* systems, introduced and studied in [Hack 72], and further investigated by many researchers (see, for instance, [TV 84, BV 84, Best 87, BCDE 90, ES 91]). The study of subclasses usually considers *ordinary* Place/Transition systems, i.e all arcs weighted one. Nevertheless, *weights* are convenient to properly model systems with *bulk* services and arrivals. They can be implemented by means of ordinary subnets, but then the original structure of the net becomes too complex, even in the simplest cases, to be amenable of some interesting structural analysis techniques. An attempt to generalize the structure theory of a well-studied (ordinary) subclass, namely Marked Graphs, to the weighted case is [TCCS 92].

We mainly consider structural analysis techniques based on linear algebra, using the *state equation* of the net system. Let $(\mathcal{N} = (P, T, W), M_0)$ be a net system, and σ a sequence of transitions firable from M_0:

$$M_0 \xrightarrow{\sigma} M \Rightarrow (M, \vec{\sigma}) \in \mathbb{N}^{|P|+|T|} \text{ is a solution to } M = M_0 + C \cdot \vec{\sigma}$$

where $\vec{\sigma}$ is the Parikh (or firing count) vector of sequence σ, and C is the incidence matrix of the net $\mathcal{N}$. In order to study a property, it could be expressed as a (set of) linear condition(s) and then it would be checked whether the solutions to the state equation fulfill it. To do so, Integer Programming techniques can be used [NW 88]. Unfortunately, in general, the state equation has integer solutions, $(M, \vec{\sigma})$, not reachable on the net system. They are called *spurious* solutions. In general, due to the existence of spurious solutions, only *necessary* or *sufficient* conditions are obtained when this relaxation is used to study behavioural properties like boundedness or deadlock-freeness. Spurious solutions can be removed using different approaches [CS 91], but, in general, it isn't still known whether all of them are removed.

We present here some results regarding the structure theory of a subclass of (weighted) Place/Transition systems that naturally generalizes the ordinary subclass of Free Choice systems: *Equal Conflict systems.* We concentrate on a couple of important results concerning the relationship between the structure (namely the state equation) and the behaviour. Actually, we show that liveness of bounded Equal Conflict systems can be decided verifying that there is no solution to a linear system of inequalities in the integer domain, and that live and bounded Equal Conflict systems have home states, as live and bounded Free Choice systems do [Vogler 89]. Both results are theoretically and practically interesting. On the practical side, they provide an algebraic technique to check liveness, and a strong property of the behaviour which is helpful for the analysis of these systems, respectively. We find them theoretically interesting because they provide a deeper understanding of the basic causes of some of the outstanding results about Free Choice (in fact, only the equality of conflicts is used in the proofs).

The rest of the paper is organized as follows. In section 2, we recall the basic concepts of Petri nets. Equal Conflict systems are defined in section 3. A couple of technical results about the structure of the potential reachability graph are developed in section 4, and they are exploited in sections 5 and 6.

2 Basic Concepts and Notation

In this section we recall the basic concepts and notation that are going to be used. For the sake of readability, whenever a net or system is defined it "inherits" the definition of all the characteristic sets, functions, parameters,... with names conveniently marked to identify whose is which. More comprehensive presentations of Petri net concepts are, for instance, [Silva 85, Murata 89].

Place/Transition net and related concepts. A *Place/Transition net* (P/T-$\mathcal{N}$) is a triple $\mathcal{N} = (P, T, W)$ where P and T are disjoint sets of *places* and *transitions*, with $|P| = n, |T| = m$, and $W : (P \times T) \cup (T \times P) \mapsto \mathbb{N}$ defines the *weighted flow relation*. (Sometimes we write that $(u, v) \in W$ meaning that $W(u, v) > 0$.) Since a P/T-$\mathcal{N}$ can be seen, and drawn, as a bipartite weighted directed graph, several graph concepts, like paths, circuits, strong connectedness, etc., can be extended to nets. In particular, let $v \in P \cup T$; its *preset* and *postset* are given, respectively, by:

$$^\bullet v \stackrel{\text{def}}{=} \{u \in P \cup T \mid W(u, v) > 0\} \quad v^\bullet \stackrel{\text{def}}{=} \{u \in P \cup T \mid W(v, u) > 0\}$$

The preset (postset) of a set of nodes is the union of presets (postsets) of its elements. A place (transition) v such that $|v^\bullet| > 1$ [$|{}^\bullet v| > 1$] is a *choice [attribution] (fork [join])*.

The weighted flow relation can alternatively be defined ($\forall p \in P, \forall t \in T$) by: $Pre(p, t) \stackrel{\text{def}}{=} W(p, t)$, $Post(p, t) \stackrel{\text{def}}{=} W(t, p)$. These functions can be represented by matrices[1]. If $\mathcal{N}$ is *pure* (i.e. $\forall p \in P, \forall t \in T : W(p,t) \cdot W(t,p) = 0$; we assume without loss of generality that all nets are pure), then the weighted flow relation is represented by the *incidence matrix* $C \stackrel{\text{def}}{=} Post - Pre$. *Ordinary* nets are those where $W : (P \times T) \cup (T \times P) \mapsto \{0,1\}$.

Place/Transition system and related concepts. A function $M : P \mapsto \mathbb{N}$ is called *marking*, and can be represented by a vector. A *Place/Transition system* (P/T-Σ) is a pair $(\mathcal{N}, M_0)$ where $\mathcal{N}$ is a P/T-$\mathcal{N}$ and M_0 is the *initial* marking. A transition t is *enabled* at M iff $M \geq Pre[P,t]$. Being enabled, t may *occur* (or *fire*) yielding a new marking $M' = M + C[P,t]$, and this is denoted by $M \xrightarrow{t} M'$ and $M' = M \xrightarrow{t}$.

An *occurrence (or firing) sequence* from M is a sequence $\sigma = t_1 \cdots t_k \in T^*$ such that $M \xrightarrow{t_1} M_1 \ldots M_{k-1} \xrightarrow{t_k} M'$, denoted by $M \xrightarrow{\sigma} M'$ and $M' = M \xrightarrow{\sigma}$. The set of all the occurrence sequences from M_0 (the *language*) is denoted by $L(\mathcal{N}, M_0)$.

The *reachability set* is $R(\mathcal{N}, M_0) \stackrel{\text{def}}{=} \{M_0 \xrightarrow{\sigma} \mid \sigma \in L(\mathcal{N}, M_0)\}$. The *reachability graph* is a labelled directed graph $RG(\mathcal{N}, M_0) = (V, E, l)$ with $l : E \mapsto T$ given by:

- $V = R(\mathcal{N}, M_0)$
- $((M, M') \in E \wedge l(M, M') = t) \Leftrightarrow M \xrightarrow{t} M'$

The set $succ(U)$ of *successors* of a subset of nodes $U \subseteq R(\mathcal{N}, M_0)$, is the minimal set such that $U^\bullet \subseteq succ(U)$ and $succ(U)^\bullet \subseteq succ(U)$. The *subgraph* of (V, E, l) induced by U is defined by $G(U) = (U, (U \times U) \cap E, l|_{(U \times U) \cap E})$ [2] Finally, the *set of labels* of $G(U)$ is $l(U) = \{t \in T \mid \exists e \in (U \times U) \cap E : l(e) = t\}$. We say that U is a *final* iff ($U = \{v\} \wedge v^\bullet = \emptyset$) or ($G(U)$ is strongly connected $\wedge\ U = succ(U)$). Finals of the first kind wil be called *deadlocks* and those of the second *active finals* or, when $l(U) = T$, *live finals.* Observe that there are systems without any final, and that (active) finals could be infinite.

$(\mathcal{N}, M_0)$ is *bounded* iff $\exists max_{p \in P, M \in R(\mathcal{N}, M_0)}\{M[p]\} \in \mathbb{N}$. This property is required if the system has to be implemented. Its structural version, *structural boundedness*, tells us about its fulfilment for *any* initial marking. If $(\mathcal{N}, M_0)$ is bounded, then $R(\mathcal{N}, M_0)$ is finite, and $RG(\mathcal{N}, M_0)$ *has* (finite) finals.

[1] Places and transitions are assumed to be arbitrarily, but fixedly, ordered. Therefore the rows and columns can be indexed by the places and transitions respectively. $A[\Pi, \Theta]$ denotes the submatrix of A corresponding to rows of places in $\Pi \subseteq P$ and columns of transitions in $\Theta \subseteq T$.

[2] $\varphi|_\Sigma$ denotes the restriction of φ to subset Σ.

$(\mathcal{N}, M_0)$ is *live* iff every transition can ultimately occur from any reachable marking. $(\mathcal{N}, M_0)$ is live iff $\forall v \in R(\mathcal{N}, M_0) : l(succ(\{v\})) = T$. A necessary condition for liveness is that every final of $RG(\mathcal{N}, M_0)$ is live. If $(\mathcal{N}, M_0)$ is bounded then it is also sufficient. $(\mathcal{N}, M_0)$ is *deadlock-free* iff at least one transition is enabled at every reachable marking. $(\mathcal{N}, M_0)$ is deadlock-free iff $RG(\mathcal{N}, M_0)$ has no finals being deadlocks. These two properties (the second being a necessary condition for the first) are usually part of the specification of systems. Their *structural* versions tell us about the *existence* of a suitable initial marking such that they hold. We say that $(\mathcal{N}, M_0)$ is *well-behaved* when it is live and bounded. Strong connectedness of the net is a necessary condition for well-behavedness[Murata 89]. As our main interest are well-behaved systems, we will usually deal with strongly connected nets.

M is a *home state* in $(\mathcal{N}, M_0)$ iff it is reachable from every reachable marking. $(\mathcal{N}, M_0)$ is *reversible* iff M_0 is a home state. M is a home state iff there is a unique final in $RG(\mathcal{N}, M_0)$, and this is active and contains M.

The state equation. (See also [CS 91].) Let $(\mathcal{N}, M_0)$ be a P/T-Σ with incidence matrix C. The *state equation* is an algebraic equation that gives a *necessary* condition for a marking to be reachable: a vector $M \in \mathbb{N}^n$ such that $\exists \vec{\sigma} \in \mathbb{N}^m : M = M_0 + C \cdot \vec{\sigma}$ is said to be *potentially reachable*, denoted by $M \in PR(\mathcal{N}, M_0) \supseteq R(\mathcal{N}, M_0)$. The *potential reachability graph* is defined like the reachability graph, but the set of nodes is now $PR(\mathcal{N}, M_0)$ instead of $R(\mathcal{N}, M_0)$. The definitions of successors, subgraphs, and finals are extended to deal with potential reachability graphs. Observe that $RG(\mathcal{N}, M_0)$ is a subgraph of $PRG(\mathcal{N}, M_0)$, defined by the subset of nodes $R(\mathcal{N}, M_0) \subseteq PR(\mathcal{N}, M_0)$.

3 Definition of Equal Conflict Net Systems

We call *syntactical* to subclasses of nets (or systems) that can be defined by imposing certain restrictions on the net, and perhaps the initial marking. Well known examples of syntactical subclasses are Marked Graphs, or Free Choice systems, for instance. Equal Conflict systems are another syntactical subclass of P/T-Σ. As the name suggests, their definition has much to do with conflicts, so let us start by formally defining several *conflict relations* on the set of transitions of a net.

Definition 3.1 (Structural conflicts.)
Let $\mathcal{N}$ be a P/T-$\mathcal{N}$*.*

1. *Two transitions, t and t', are in* Structural Conflict Relation *(denoted by $(t, t') \in \mathrm{SCR}$) iff $t = t'$ or ${}^\bullet t \cap {}^\bullet t' \neq \emptyset$. For instance, every transition that is output of a choice place is in* SCR *with each other. This relation is not transitive.*

2. *Two transitions, t and t', are in* Coupled Conflict Relation *(denoted by $(t, t') \in \mathrm{CCR}$) iff there exists $t_0, \ldots, t_k \in T$ such that $t = t_0$, $t' = t_k$ and for $1 \leq i \leq k : (t_{i-1}, t_i) \in \mathrm{SCR}$. This is an* equivalence relation *on the set of transitions, and each equivalence class is a* Coupled Conflict Set *(*CCS*). A transition that is not in structural conflict with any other is itself a* CCS*. (Such* CCS *will be called* trivial*.)*

3. *Two transitions, t and t', are in* Equal Conflict Relation *(denoted by $(t, t') \in \mathrm{ECR}$) iff $t = t'$ or $Pre[P, t] = Pre[P, t'] \neq 0$. This is also an* equivalence relation *on the set of transitions, and each equivalence class is an* Equal Conflict Set *(*ECS*). Note that whenever two transitions are in* ECR *they are also in* CCR*, but not the other way round.*

The interpretation of these definitions follows. A conflict is a situation in which several actions are enabled but not all of them can occur simultaneously. Structural Conflicts are the "topological construct" making possible the existence of conflicts. Coupled Conflict Sets capture the concept of the set of transitions that might be in conflict, so in order to decide firing one we need not only know that it is enabled but also what happens with the others. An Equal Conflict Set is a particular kind of Coupled Conflict Set such that whenever any transition belonging to it is enabled all of them are.

Equal Conflict systems are defined as those whose net is such that every structural conflict is equal.

Definition 3.2 (Equal Conflict systems.)
A P/T-Σ*, $(\mathcal{N}, M_0)$, is an* Equal Conflict system *(*EC-Σ*) iff $\forall t, t' \in T$*

$$(t, t') \in \mathrm{ECR} \Leftrightarrow (t, t') \in \mathrm{CCR}$$

Let us compare this definition with that for (Extended) Free Choice systems:

Definition 3.3 (Extended Free Choice systems.)
An ordinary P/T-Σ*, $(\mathcal{N}, M_0)$, is an* Extended Free Choice system *iff $\forall t, t' \in T$*

$$^\bullet t \cap {}^\bullet t' \neq \emptyset \Rightarrow {}^\bullet t = {}^\bullet t'$$

Ordinary Equal Conflict systems trivially coincide with Extended Free Choice systems. Nevertheless, if we drop the word "ordinary" from definition 3.3 we get a larger class of weighted P/T-Σ than EC-Σ. (We would call this other class *topological* Extended Free Choice systems or *topological* EC-Σ meaning that they have the same graph, appart from the weights.)

The reason why we say that EC-Σ naturally generalize Free Choice is because they capture a major feature of these, namely that whenever a transition in conflict with others is enabled, *all* of them are, so any one of them is *freely chosen* to occur, what implies that *non-determinism and synchronization are neatly separated.* That all conflicts are equal is indeed a central fact in the properties we will deal with in the next sections.

4 Reachability and the State Equation

This section contains a proof of the *directedness* property of the reachability graph of a live EC-Σ, which generalizes a similar result already known for Free Choice systems [BV 84, Vogler 89]. An important feature of our proof is that it is done at the level of *potential* reachability, that is, we proof not only that the reachability but also the potential reachability graph is directed. The directedness result, as in [BV 84, Vogler 89], is the main fact leading to existence of home states, as will be shown in section 6. One of the advantadges of proving it at the potential reachability level is that it also implies the impossibility of a spurious solution being non-live, which is central in section 5.

We will derive the pursued results from the following rather technical theorem, which says that, in a live EC-Σ, every potentially reachable marking has successors that are reachable.

Theorem 4.1 *Let* $(\mathcal{N}, M_0)$ *be a live* EC-Σ.
If $M \in PR(\mathcal{N}, M_0)$ *then* $R(\mathcal{N}, M_0) \cap R(\mathcal{N}, M) \neq \emptyset$.

Proof Take any $M \in PR(\mathcal{N}, M_0)$, that is, there exists $\vec{\sigma} \in I\!N^m$ such that $M = M_0 + C \cdot \vec{\sigma}$. It must be shown that there exists a common successor of M_0 and M. Let $\vec{\sigma} = \vec{\sigma}' + \vec{\sigma}''$ be such that there exists a sequence σ', eventually empty, with firing count vector $\vec{\sigma}'$ such that $M_0 \xrightarrow{\sigma'} {M_0}'$, and none of transitions in the support[3] of $\vec{\sigma}$" is firable at ${M_0}'$. (It is clear that such decomposition of $\vec{\sigma}$ does exist. It can be constructed starting form σ' empty and $\vec{\sigma}'' = \vec{\sigma}$, and then sucessively firing transitions in the successive $\|\vec{\sigma}''\|$ until no one is firable.) Therefore we can write $M = {M_0}' + C \cdot \vec{\sigma}''$.
Since ${M_0}'$ is reachable from M_0, $(\mathcal{N}, M_0')$ is of course live. If some transition $t \notin \|\vec{\sigma}''\|$ is enabled at ${M_0}'$ (and since ${M_0}'$ is live, there is at least one), it is also enabled at

[3] The set of the non-zero entries of a vector, $\mathcal{V}$, denoted by $\|\mathcal{V}\|$.

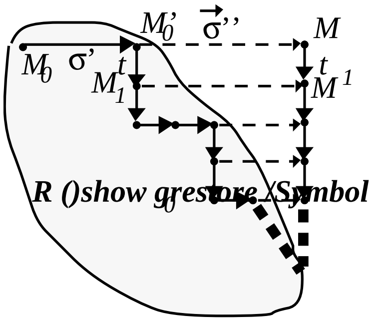

Figure 1: Illustrating the proof of theorem 4.1

M because, due to the EC assumption, in $\|\vec{\sigma}''\|$ there is no transition which takes tokens from input places of t (otherwise such transition in $\|\vec{\sigma}''\|$ being in ECR with t would be enabled too). Let ${M_0}' \xrightarrow{t} M_1$ and $M \xrightarrow{t} M^1$. We can rename M_1 as M_0 and M^1 as M and apply again the same reasoning. That is, we can go on firing transitions from the successive descendants of M_0, and

- if a given transition does not belong to the corresponding $\|\vec{\sigma}''\|$, then it is also firable from the corresponding successor of M (we fire it advancing in parallel)
- if it does belong to the corresponding $\|\vec{\sigma}''\|$, then the new successor of M_0 is "closer" to the corresponding successor of M with respect to $\sum_{t\in T} \vec{\sigma}''[t]$.

Since M_0 is live, every transition in $\|\vec{\sigma}''\|$ can be ultimately fired. This process advances from M_0 and M either "in parallel" by firing a transition not belonging to $\|\vec{\sigma}''\|$ or "reducing" $\|\vec{\sigma}''\|$, so finally a common successor is found. (Figure 1 illustrates this proof: The arrows represent firing sequences; The horizontal dashed arrows pictorially represent successive vectors $\vec{\sigma}''$.)

If every potentially reachable marking has reachable successors, obviously no potentially reachable marking can be a deadlock. In fact, from every potentially reachable marking every transition can ultimately be fired, at least after getting into the reachable subgraph.

Corollary 4.2 (Absence of killing spurious solutions.)
If $(\mathcal{N}, M_0)$ is a live EC-Σ *and $M \in PR(\mathcal{N}, M_0)$, then $(\mathcal{N}, M)$ is live.*

Theorem 4.1 can be generalized: *every* two potentially reachable markings have common successors, which is the usual statement of the directedness property mentioned above.

Theorem 4.3 (Directedness of the potential reachability graph.)
Let $(\mathcal{N}, M_0)$ *be a live* EC-Σ.

$$\forall M_1, M_2 \in PR(\mathcal{N}, M_0) : R(\mathcal{N}, M_1) \cap R(\mathcal{N}, M_2) \neq \emptyset$$

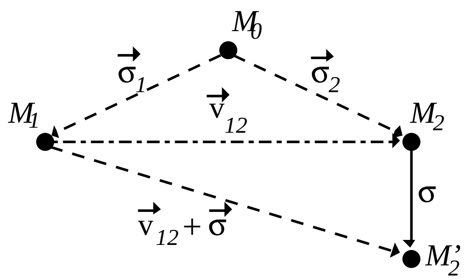

Figure 2: Illustrating the proof of theorem 4.3

Proof Take any $M_1, M_2 \in PR(\mathcal{N}, M_0)$. It must be shown that there exists a common successor of M_1 and M_2. Let $\vec{\sigma}_1, \vec{\sigma}_2 \geq 0$ be vectors such that $M_i = M_0 + C \cdot \vec{\sigma}_i, i = 1, 2$. Let $\vec{v}_{12} = \vec{\sigma}_2 - \vec{\sigma}_1$. Clearly $M_2 = M_1 + C \cdot \vec{v}_{12}$.
The vector $\vec{v}_{12}$ may contain negative coordinates. Since $(\mathcal{N}, M_2)$ is live (by corollary 4.2, because $(\mathcal{N}, M_0)$ is live), there is a successor $M_2' = M_2 \xrightarrow{\sigma}$ satisfying $M_2' = M_1 + C \cdot (\vec{\sigma} + \vec{v}_{12}), \vec{\sigma} + \vec{v}_{12} \geq 0$ (to find such σ it is enough to fire any sequence having at least $-\vec{v}_{12}[t]$ occurrences of t for each negative coordinate t in $\vec{v}_{12}$). It follows that $M_2' \in PR(\mathcal{N}, M_1)$. As $(\mathcal{N}, M_1)$ is live (by corollary 4.2) there exists (by theorem 4.1) a common successor of M_1 and M_2', so a common successor of M_1 and M_2. (Figure 2 illustrates this proof. We have used different dashes for $\vec{v}_{12}$ in order to distinguish it from potentially firable firing count vectors.)

Figure 3 contains several examples showing that the Equal Conflict assumption is needed. The empty marking is isolated in the potential reachability graph of the top-left system. The system below (taken form [CS 91]) shows an ordinary 1-bounded and live system with potentially reachable markings (for instance, two tokens in the leftmost place) without reachable successors, although the reachability graph is directed. The rightmost system (taken from [BV 84]) shows an ordinary 1-bounded and live system where even the reachability graph is not directed (it has two *livelocks*, it doesn't have home states).

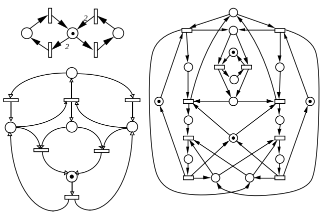

Figure 3: Some live systems whose potential reachability graph does not enjoy the directedness property.

5 Liveness and Deadlock-Freeness

As a first consequence of the results in section 4, liveness of a bounded EC-Σ can be decided proving that a system of linear inequalities hasn't got a solution in the integer domain. To do so, we show first the equivalence of liveness and deadlock-freeness in bounded EC-Σ, which is based on identical arguments to those for Free Choice [Best 87]. Afterwards, we give a sufficient condition for deadlock-freeness, which is an adaptation of the work presented in [Colom 89, TCS 92]. Thanks to corollary 4.2, this technique gives not only a sufficient but a necessary and sufficient condition for deadlock-freeness. Finally, the simplicity of EC-$\mathcal{N}$ ensures that this condition can be written as the check for solution of a single linear system of inequalities in the integer domain.

Theorem 5.1 *Let* $(\mathcal{N}, M_0)$ *be a bounded strongly connected* EC-Σ.

$$(\mathcal{N}, M_0) \textit{ is live} \Leftrightarrow (\mathcal{N}, M_0) \textit{ is deadlock-free}$$

Proof Only "$\Leftarrow$" must be proven. Assume $(\mathcal{N}, M_0)$ is not live. Let t be dead at some reachable marking. By EC, every t' in CCR with t is also dead. Every input place of a dead CCS will retain all the tokens that arrive to it, so boundedness implies that every input transition to these places will die eventually. By strong connectedness, every transition will finally die, hence $(\mathcal{N}, M_0)$ is not deadlock-free.

A general technique to check properties in an algebraic manner (see, for instance, [Silva 85, Colom 89]) consists in relaxing the set of reachable markings by the set of solutions of the state equation. The property to be checked is expressed as some linear inequalities/equations on the marking and firing count variables and then it is checked whether there is a solution to the state equation satisfying the property or not. In particular, the following is a sufficient condition for deadlock-freeness based on the state equation of the net system, and containing $|T|$ "complex conditions", each one concerning the non-firability of a transition of the net.

Proposition 5.2 *Let* $(\mathcal{N}, M_0)$ *be a* P/T-Σ. *If*

$$\nexists\ (M, \vec{\sigma}) \in \mathbb{N}^{n+m} : (M = M_0 + C \cdot \vec{\sigma}) \wedge \left(\bigwedge_{t \in T} \left(\bigvee_{p \in {}^\bullet t} M[p] < W(p,t) \right) \right) \quad (1)$$

then $(\mathcal{N}, M_0)$ *is deadlock-free.*

Proof A system $(\mathcal{N}, M_0)$ is deadlock-free iff

$$\nexists\ M \in \mathbb{N}^n : (M \in R(\mathcal{N}, M_0)) \wedge (\forall t \in T : \exists p \in {}^\bullet t : M[p] < W(p,t)) \quad (2)$$

Since $R(\mathcal{N}, M_0) \subseteq PR(\mathcal{N}, M_0)$, condition (2) is implied by the following one:

$$\nexists\ M \in \mathbb{N}^n : (M \in PR(\mathcal{N}, M_0)) \wedge (\forall t \in T : \exists p \in {}^\bullet t : M[p] < W(p,t)) \quad (3)$$

It is immediate to see that condition (3) is equivalent to condition (1).

The above condition is, in general, only sufficient for deadlock-freeness, since it is based on the set of *potentially* reachable markings. Nevertheless, using the absence of killing spurious solutions (corollary 4.2) and the equivalence between liveness and deadlock-freeness, it is shown next that condition (1) provides a characterization of liveness in bounded strongly connected EC-Σ.

Theorem 5.3 *A bounded strongly connected* EC-Σ, $(\mathcal{N}, M_0)$, *is live iff*

$$\nexists\ (M, \vec{\sigma}) \in \mathbb{N}^{n+m} : (M = M_0 + C \cdot \vec{\sigma}) \wedge \left(\bigwedge_{t \in T} \left(\bigvee_{p \in {}^\bullet t} M[p] < W(p,t) \right) \right) \quad (1)$$

Proof Only necessity needs to be proven. Assume that $(\mathcal{N}, M_0)$ is live and M_d is (the marking part of) a solution of the above condition. It follows that it is a deadlock (i.e. a final consisting of a single node without successors) in the potential reachability graph. Since $M_d \in PR(\mathcal{N}, M_0)$ and $(\mathcal{N}, M_d)$ is not live, $(\mathcal{N}, M_0)$ is not live either, according to (the contrapositive of) corollary 4.2, against the hypothesis.

Up to now we have obtained a characterization of liveness in bounded strongly connected EC-Σ in terms of the set of solutions of the state equation. Let us concentrate now in the expression of condition (1). Since the different linear equations/inequalities are not all linked conjunctively, condition (1) is not a linear system, but a set of linear systems. In fact, we could write it as a (logical) sum of products applying the distributive property to the actual product of sums, leading to:

$$\begin{array}{l} \nexists\ (M, \vec{\sigma}) \in \mathbb{N}^{n+m} : \\ \bigvee_{\alpha : T \mapsto P} \left((M = M_0 + C \cdot \vec{\sigma}) \wedge \left(\bigwedge_{t \in T} M[\alpha(t)] < W(\alpha(t), t)\right)\right) \end{array} \tag{4}$$

where $\alpha : T \mapsto P$ is a mapping that assigns to each transition one of its input places. It follows that evaluating condition (4) involves checking for solution a large number of linear systems, namely $\prod_{t \in T} |{}^{\bullet}t|$ of them. There are several ways to alleviate this problem, from conventional logical simplifications to Petri net based specific rules [TCS 92]. Fortunately, in many interesting particular cases, including that of EC-Σ, condition (1) can be written in terms of a single linear system.

In order to prove that, for EC-Σ, condition (1) results in a single linear system, let us introduce a sufficient condition for the "linearity" of the non-firability condition of a transition. We shall first recall that the *structural bound* of place p in $(\mathcal{N}, M_0)$ is defined as

$$SB(p) \stackrel{\text{def}}{=} \max\{M[p] \mid M \in PR(\mathcal{N}, M_0)\}$$

From now on we assume the net is structurally bounded, so structural bounds are well defined for every place.

Theorem 5.4 *Let $(\mathcal{N}, M_0)$ be a* P/T-Σ*, and let $t \in T$ be a transition such that ${}^{\bullet}t = \pi \cup p' \wedge \forall p \in \pi : SB(p) \leq W(p, t)$. The non-firability condition for transition t can be written as an integer linear inequality as follows:*

$$SB(p') \sum_{p \in \pi} M[p] + M[p'] < SB(p') \sum_{p \in \pi} W(p, t) + W(p', t) \tag{5}$$

Proof Let us rewrite first the original non-firability condition for transition t:

$$\left(\bigvee_{p \in \pi} M[p] < W(p, t)\right) \vee (M[p'] < W(p', t)) \tag{6}$$

It must be shown that, in the integer domain, equation (6) $\Leftrightarrow$ equation (5).

$\Rightarrow$ We distinguish two cases for t disabled:

Case 1: Some $p'' \in \pi$ is such that $M[p''] < W(p'', t)$, so $M[p''] \leq W(p'', t) - 1$. Using also that $\forall p \in P : M[p] \leq SB(p)$ and that $\forall p \in \pi : SB(p) \leq W(p,t)$ we get:

$$\begin{aligned} & SB(p') \sum_{p \in \pi} M[p] + M[p'] \\ = \; & SB(p') \sum_{p \in \pi \setminus \{p''\}} M[p] + SB(p') M[p''] + M[p'] \\ \leq \; & SB(p') \sum_{p \in \pi \setminus \{p''\}} W(p,t) + SB(p')(W(p'',t) - 1) + M[p'] \\ = \; & SB(p') \sum_{p \in \pi \setminus \{p''\}} W(p,t) + SB(p') W(p'',t) - (SB(p') - M[p']) \\ \leq \; & SB(p') \sum_{p \in \pi} W(p,t) < SB(p') \sum_{p \in \pi} W(p,t) + W(p',t) \end{aligned}$$

Case 2: $M[p'] < W(p', t)$. Using also that $M[p] \leq SB[p]$ and that $\forall p \in \pi : SB(p) \leq W(p,t)$ the result follows immediately.

$\Leftarrow$ Let us rewrite first equation (5):

$$SB(p') \sum_{p \in \pi} (M[p] - W(p,t)) < W(p',t) - M[p'] \tag{7}$$

Assume contrary: equation (7) holds and t is enabled, so, in particular, $M[p'] \geq W(p',t)$. Therefore $\sum_{p \in \pi}(M[p] - W(p,t)) < 0$, and, since $\forall p \in \pi : M[p] - W(p,t) \leq 0$, it follows that $\exists p \in \pi : M[p] - W(p,t) < 0$, hence t is not enabled, against the hypothesis.

If it was the case that every transition in the net fulfilled the condition of the above theorem, then every non-firability condition could be written linearly, and therefore condition (1) would be a linear system. What we are going to show next is that any (structurally bounded) EC-Σ can be transformed, preserving deadlock-freeness, into another EC-Σ enjoying such property. Our aim here is to show that it is *possible* to achieve this, disregarding efficiency considerations. (There are indeed more sensible ways, and more rules, to rewrite the original condition reducing the number of linear systems [TCS 92] without adding so many auxiliary variables — places and transitions — as we are going to add here for the sake of simplicity. The interested reader can also find examples of the application of these techniques in the cited work.)

We first perform the classical separation of the synchronization and choice in the "general" (non trivial and having several input places) Equal Conflict

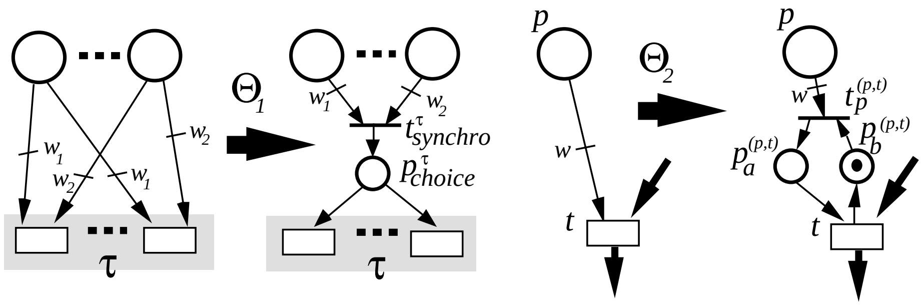

Figure 4: Illustration of transformations Θ_1 and Θ_2.

Sets, identical to the transformation from an Extended Free Choice to a Free Choice, as shown in figure 4, left. This transformation can be more precisely defined as follows:

Definition 5.1 (Transformation Θ_1.)
Let $(\mathcal{N}, M_0)$ *be a* P/T-Σ. *The* P/T-Σ $(\mathcal{N}, M_0)$ transformed by Θ_1, *denoted* $(\mathcal{N}^{\Theta_1}, M_0^{\Theta_1})$ *is obtained as follows:*

$(\mathcal{N}^{\Theta_1}, M_0^{\Theta_1}) := (\mathcal{N}, M_0)$
forall $\tau \subseteq T$ ECS `of` $(\mathcal{N}, M_0)$ `such that` $|\tau| > 1$ `and` $|{}^\bullet\tau| > 1$ **do**
  `Add to` P^{Θ_1} `one place:` p^τ_{choice} $(M_0^{\Theta_1}[p^\tau_{choice}] := 0)$
  `Add to` T^{Θ_1} `one transition:` $t^\tau_{synchro}$
  `Add to` W^{Θ_1} `the following arcs:`
    $\forall p \in {}^\bullet\tau : W^{\Theta_1}(p, t^\tau_{synchro}) := W^{\Theta_1}(p, t)$, `where` $t \in \tau$
    $W^{\Theta_1}(t^\tau_{synchro}, p^\tau_{choice}) := 1$
    $\forall t \in \tau : W^{\Theta_1}(p^\tau_{choice}, t) := 1$
  `Remove from` W^{Θ_1} `the following arcs:`
    $\forall p \in {}^\bullet\tau \backslash \{p^\tau_{choice}\}, \forall t \in \tau : (p, t)$
od

After applying transformation Θ_1 to an EC-Σ, obviously every transition in a non-trivial ECS will have just one input place, so *every transition having more than one input place will be the only output transition of all its input places*, or, in other terms, *every place being input of a join transition will be a non-choice (i.e. persistent) place.* Still it is possible, though, that some join transition has more than one input place having a structural bound bigger than the weight of the arc, violating the condition of theorem 5.4. But since every input place of a join transition has been made non-choice, we can "sequentialize" such synchronizations without affecting the language, by means of the following transformation, illustrated in figure 4, right.

Definition 5.2 (Transformation Θ_2.)

Let $(\mathcal{N}, M_0)$ *be a* P/T-Σ. *The* P/T-Σ $(\mathcal{N}, M_0)$ transformed by Θ_2, *denoted* $(\mathcal{N}^{\Theta_2}, M_0^{\Theta_2})$ *is obtained as follows:*

$(\mathcal{N}^{\Theta_2}, M_0^{\Theta_2}) := (\mathcal{N}, M_0)$
forall `arc` $(p,t) \in W$ `such that` $|{}^\bullet t| > 1$ `and` $p^\bullet = \{t\}$ **do**
  `Add to` P^{Θ_2} `two places:` $p_a^{(p,t)}$ `and` $p_b^{(p,t)}$ `(`$M_0^{\Theta_2}[p_a^{(p,t)}] := 0, M_0^{\Theta_2}[p_b^{(p,t)}] := 1$`)`
  `Add to` T^{Θ_2} `one transition:` $t_p^{(p,t)}$
  `Add to` W^{Θ_2} `the following arcs:`
    $W^{\Theta_2}(p, t_p^{(p,t)}) := W^{\Theta_2}(p,t)$
    $W^{\Theta_2}(t_p^{(p,t)}, p_a^{(p,t)}) := W^{\Theta_2}(p_a^{(p,t)}, t) := W^{\Theta_2}(t, p_b^{(p,t)}) := W^{\Theta_2}(p_b^{(p,t)}, t_p^{(p,t)}) := 1$
  `Remove from` W^{Θ_2} `the arc` (p,t)
od

It is clear that, being $x \in \{1,2\}$, every occurrence sequence in $(\mathcal{N}, M_0)$ is firable in $(\mathcal{N}^{\Theta_x}, M_0^{\Theta_x})$, and viceversa, disregarding the occurrences of the added (silent) transitions. In particular, after the definitions, the equivalence of deadlock-freeness in the original and the transformed net follows immediately.

Property 5.5 *Let* $(\mathcal{N}, M_0)$ *be a* P/T-Σ, *and let* $(\mathcal{N}^{\Theta_x}, M_0^{\Theta_x})$ *be* $(\mathcal{N}, M_0)$ *transformed by* Θ_x, *where* $x \in \{1,2\}$.
$(\mathcal{N}, M_0)$ *is deadlock-free iff* $(\mathcal{N}^{\Theta_x}, M_0^{\Theta_x})$ *is.*

Finally, the announced result can be stated as follows:

Theorem 5.6 *Let* $(\mathcal{N}, M_0)$ *be a structurally bounded* EC-Σ. *Liveness of* $(\mathcal{N}, M_0)$ *can be decided verifying that there is no solution to a single linear system of inequalities in the integer domain.*

Proof Let $(\mathcal{N}', M_0')$ be $(\mathcal{N}, M_0)$ transformed by Θ_1 and Θ_2 (in this order). Both transformations preserve the Equal Conflict property. It is also clear that the structural bound of "$p_a^{(p,t)}$" and "$p_b^{(p,t)}$" places (added by transformation Θ_2) is one, equal to the weight of their output arcs. Therefore, *every transition* in the transformed EC-Σ fulfills the condition of theorem 5.4. (In particular, every input place to the original join transitions and the added "$t^\tau_{synchro}$" is now structurally bounded by one, since all of them are places "$p_a^{(p,t)}$", and the added "$t_p^{(p,t)}$" transitions have only one input place with a structural bound eventually bigger than the corresponding weight, because the other one is a "$p_b^{(p,t)}$" place.) It follows that analysing deadlock-freeness in the transformed EC-Σ involves checking that there isn't any solution of a single linear system. Since this is equivalent to analysing deadlock-freeness of the original system, according to corollary 5.5, and this is equivalent to analysing its liveness, according to theorem 5.1, we are done.

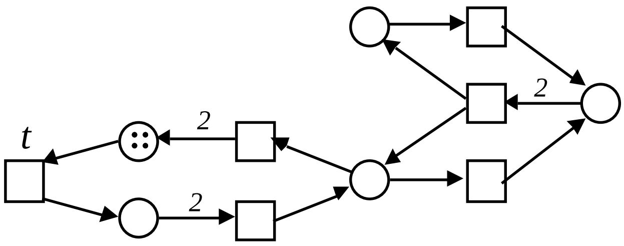

Figure 5: A non reversible live and bounded EC-Σ.

6 Equal Conflict Systems Also Have Home States

The existence of home states in live and bounded EC-Σ basically means that these kind of systems always have states that can be reached after whichever possible evolution. This result generalizes the results appearing in [BV 84] (live and 1-bounded Free Choice systems have home states) and in [Vogler 89] (live and bounded Free Choice systems have home states). The generalization of the result from 1-bounded to just bounded was foreseen by the authors in [BV 84], *"however it is not entirely obvious how the proof might be generalised, because the usual way of transforming bounded systems into 1-bounded does not preserve the Free Choice property"*. An analogous observation applies to our new generalization, because the transformations from weighted to ordinary systems do not preserve the Free Choice property either. In fact we base our proof also in directedness, as in both cited works, but this time we refer to the *potential* reachability graph instead of the reachability graph, what has proved useful for other different purposes in the previous section.

As pointed out in [BV 84], the existence of home states is a strong property, sometimes appearing in the specification of reactive systems: usually they require states to return to from any point, in order to be able to resume their activity. It also facilitates the analysis, because the reachability graph is known to be rather particular: it has only one final, that is active (a strongly connected subgraph). Moreover, many performance analyses or partial simulations are meaningless if there isn't a home space, because it could be the case that the system behaves in completely different ways depending on the initial course of events.

Theorem 6.1 (Well-behaved EC-Σ have home states)
Let $(\mathcal{N}, M_0)$ be a live and bounded EC-Σ. *There exists a nonempty subset of reachable markings $\emptyset \neq HS \subseteq R(\mathcal{N}, M_0)$ such that*

$$\forall M_h \in HS, \forall M \in R(\mathcal{N}, M_0) : M_h \in R(\mathcal{N}, M)$$

(HS is called the home space.*)*

Proof Boundedness ensures the existence of finals in the reachability graph. Theorem 4.3 ensures that there is only one (which is, indeed, a live final). Every marking in this final can be reached from every reachable marking, hence it is the home space.

In [BV 84] the authors conjectured that a firing sequence containing every transition at least once would necessarily produce a home state in a live and bounded Free Choice, which was proven in [BCDE 90]. This is not true for EC-Σ, as the example in figure 5 shows (t must occur at least twice in order to reach the home space). We conjecture that an occurence sequence σ with vector $\vec{\sigma}$ bigger than or equal to *every* minimal T-semiflow of the (consistent) net, leads necessarily to the home space.

7 Conclusion and Future Work

We have reported here some results regarding the structure theory of a subclass of (weighted) Place/Transition systems that generalizes the ordinary subclass of Extended Free Choice systems: *Equal Conflict systems.* The basic technical results concern the directedness of the potential reachability graph of live EC-Σ. From this fact it has been shown that liveness of bounded EC-Σ can be decided checking the existence of solutions to a linear system inequalities in the integer domain. Secondly, the existence of a home space in well-behaved EC-Σ has been proved, generalizing with rather different arguments the result in [BV 84, Vogler 89]. These results are both theoretically and practically interesting. On the practical side, they provide an algebraic technique to check liveness, and a strong property which is helpful for the analysis of these systems, respectively. We find them theoretically interesting because they provide a deeper understanding of the basic causes of some of the outstanding results about Free Choice-like systems.

We conjecture that many other important results of the structure theory of Free Choice systems have their counterpart for EC-Σ (for instance the beautiful decomposition theory, naturally after adapting the concept of *components*), and that their generalization will not only be an extension but will also shed new light on the understanding of the relationships between the structure and the behaviour of concurrent systems. Some of our ongoing research is devoted to this topic.

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