Steady-State Performance Evaluation of Totally Open Systems of Markovian Sequential Processes

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Abstract

Totally open systems of Markovian sequential processes are defined as a subclass of stochastic Petri nets. They can be viewed as a generalization of a subclass of queueing networks in which complex sequential servers can be synchronized according to some particular schemes. Structural analysis of these nets is considered for avoiding the state explotion problem of the embedded Markov chain. Some qualitative properties interesting from a performance point of view are presented. In particular, a "potential ergodicity" property is characterized by means of two structural properties: consistency and synchronic distance relation. Necessary and sufficient ergodicity conditions and the computation of steady-state performance measures are studied.

1 Introduction

Stochastic Petri nets constitute an adequate model for the evaluation of performance measures of concurrent systems [Mol 82, ABC 84, FN 85]. The Markovian model obtained from Petri nets by associating exponential distributions to the firing time of transitions has been studied in the literature. Nevertheless, because of the large computational cost originated from the state explosion of the embedded Markov chain, efficient computation methods for the performance measures are still needed.

Non sequential systems constructed from cooperating sequential components communicating with buffers have been modeled with Petri nets (deterministic systems of sequential processes [Rei 82, SB 88]).

Performance evaluation of an interesting subclass of these systems is studied in this article. The continuous-time Markovian case obtained with exponential distributions timing of transitions is considered. The same computation can be applied to discrete time models with geometric distribution functions [Mol 85].

The paper is structured as follows. In Section 2 the notation and terminology are listed for Petri nets and stochastic Petri nets. Stochastic boundedness and ergodicity concepts are presented and a necessary condition for stochastic boundedness of a

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stochastic Petri net is shown. An analogous result of the well known theorem (see, e.g.,[Sil 85]):

"liveness & structural boundedness $\Rightarrow$ consistency & conservativeness"

can be deduced for the stochastic case.

In Section 3, basic definitions and considerations about systems of Markovian sequential processes are presented. The particular case of systems without circuits containing buffers (totally open systems) is studied in Sections 4 and 5. These systems can be seen as a generalization of a subclass of queueing networks (totally open queueing networks) with complex servers in which a particular kind of synchronization is allowed.

Some interesting qualitative properties can be derived from the topology of these nets (for example, reversibility and consistency are equivalent properties). "Structural stochastic" results are presented. In particular, a necessary and sufficient condition for the "potential ergodicity" (i.e. the existence of a stochastic interpretation that makes the system ergodic) is presented: consistency and some synchronic distance relations characterize the potential ergodicity.

The necessary and sufficient ergodicity conditions for a given stochastic interpretation stated in [FN 89] can be computed in an efficient way. Moreover, the steady state performance measures can be found without solving the embedded Markov process, just taking into account the net structure and the rates of exponential distributions.

Due to space constraints the proofs are omitted and can be found in [CS 89].

2 Stochastic Petri nets

2.1 Petri net definitions and notation

A Petri net $\mathcal{N}$ is a 4-tuple $\mathcal{N} = \langle P, T, Pre, Post\rangle$, where P is the set of *places*, T is the set of *transitions* ($P \cap T = \emptyset$, $|P| = n$, $|T| = m$), and Pre ($Post$) is the pre- (post-) incidence function representing the input (output) arcs $Pre\colon P \times T \to I\!N$ ($Post\colon P \times T \to I\!N$).

The *pre-* and *post-set* of a transition $t \in T$ are defined respectively as ${}^\bullet t = \{p \mid Pre(p,t) > 0\}$ and $t^\bullet = \{p \mid Post(p,t) > 0\}$. The *pre-* and *post-set* of a place $p \in P$ are defined respectively as ${}^\bullet p = \{t \mid Post(p,t) > 0\}$ and $p^\bullet = \{t \mid Pre(p,t) > 0\}$.

The *incidence matrix* of the net $C = [c_{ij}]$ ($1 \leq i \leq$n, $1 \leq j \leq$m) is defined by $c_{ij} = Post(p_i,t_j) - Pre(p_i,t_j)$. Similarly the pre- and post-incidence matrices are defined as $PRE = [a_{ij}]$ and $POST = [b_{ij}]$, where $a_{ij} = Pre(p_i,t_j)$ and $b_{ij} = Post(p_i,t_j)$.

A function $M\colon P \to I\!N$ (usually represented by a vector) is called a *marking*. A *marked Petri net* $\langle\mathcal{N}, M_0\rangle$ is a Petri net $\mathcal{N}$ with an *initial marking* M_0.

A transition $t \in T$ is *enabled* at marking M iff $\forall p \in P$, $M(p) \geq Pre(p,t)$. A transition t enabled in M can *fire* yielding a new marking M' defined by $M'(p) = M(p) - Pre(p,t) + Post(p,t), \forall p \in P$ (it is denoted by $M[t\rangle M'$).

A sequence of transitions $\sigma = t_1 t_2 \ldots t_n$ is a *firing sequence* of $\langle\mathcal{N}, M_0\rangle$ iff there exists a sequence of markings such that $M_0[t_1\rangle M_1[t_2\rangle M_2 \ldots [t_n\rangle M_n$. In this case, marking M_n is said to be *reachable* from M_0 by firing σ, and this is denoted by $M_0[\sigma\rangle M_n$. The function $\vec{\sigma}\colon T \to I\!N$ is the *firing count vector* of the firable sequence σ, i.e. $\vec{\sigma}[t]$ represents the number of occurrences of $t \in T$ in σ. If $M_0[\sigma\rangle M$, then we can write in vector form $M = M_0 + C\vec{\sigma}$, which is referred to as the *linear state equation* of the net. A marking M' is said to be *potentially reachable* iff $\exists \vec{X} \geq 0$ such that $M' = M_0 + C\vec{X} \geq 0$.

$L(\mathcal{N}, M)$ is the set of all firing sequences from M. The *reachability set* $R(\mathcal{N}, M_0)$ is the set of all markings reachable from the initial marking. Denoting by $PR(\mathcal{N}, M_0)$ the set of all potentially reachable markings we have the following relation: $R(\mathcal{N}, M_0) \subseteq PR(\mathcal{N}, M_0)$. A place $p \in P$ is said to be *k-bounded* iff $M(p) \le k \ \forall M \in R(\mathcal{N}, M_0)$. A marked net $\langle \mathcal{N}, M_0 \rangle$ is said to be (marking) k-bounded iff each of its places is k-bounded. A 1-bounded net is called *safe*. A marked net $\langle \mathcal{N}, M_0 \rangle$ is bounded if there exists $k \in I\!N$ such that $\langle \mathcal{N}, M_0 \rangle$ is k-bounded. A net $\mathcal{N}$ is *structurally bounded* iff $\forall M_0$ the marked nets $\langle \mathcal{N}, M_0 \rangle$ are k-bounded for some $k \in I\!N$.

A marking $M \in R(\mathcal{N}, M_0)$ is a *home state* iff $\forall M' \in R(\mathcal{N}, M_0) : M \in R(\mathcal{N}, M')$.

A transition $t \in T$ is *live* in $\langle \mathcal{N}, M_0 \rangle$ iff $\forall M \in R(\mathcal{N}, M_0) \ \exists M' \in R(\mathcal{N}, M)$ such that M' enables t. The marked net $\langle \mathcal{N}, M_0 \rangle$ is live iff all its transitions are live (i.e. liveness of the net guarantees the possibility of an infinite activity of all transitions). A net $\mathcal{N}$ is *structurally live* iff $\exists M_0$ such that the marked net $\langle \mathcal{N}, M_0 \rangle$ is live.

Transition t_1 has a shared place p with transition t_2 iff $p \in^\bullet t_1 \cap^\bullet t_2$.

A net is *consistent* iff $\exists X \in (I\!N^+)^m$ such that $CX = \vec{0}$.

2.2 Stochastic Petri nets

In stochastic Petri nets [Mol 82, ABC 84, FN 85], a random variable (enabling time) is associated with each transition. When a transition is enabled by a marking, a value v is randomly chosen from the associated random variable. The duration of the transition enabling period is given by this value, i.e. the firing occurs after v units of time. When several transitions are enabled by a marking, the fired transition is that with the minimum duration.

Definition 2.1 *A stochastic Petri net (SPN for short) is a 3-tuple $\langle \mathcal{N}, \mathcal{W}, M_0 \rangle$, where $\mathcal{N}$ is a Petri net, $\mathcal{W}$ is an application that associates a random variable with each transition, and M_0 is the initial marking. If the associated random variables are independent and exponentially distributed with constant rates, the net is called Markovian.*

A *trajectory* ω of the SPN is a sequence of couples (M_n, τ_n), $\omega = \{(M_0, 0), (M_1, \tau_1), \ldots, (M_n, \tau_n), \ldots\}$ where M_n is the n^{th} marking reached from M_0 and τ_n is the arrival time to M_n. The (stochastic) *marking process* is defined by $M(t, \omega) = M_n$ if $t \in [\tau_n, \tau_{n+1})$. The (stochastic) *firing process* $\vec{\sigma}(t, \omega)$ is the vectorial counting process of firings of transitions, i.e. $M(t, \omega) = M_0 + C\vec{\sigma}(t, \omega)$. The marking process of a Markovian Petri net is a continuous-time homogeneous Markov process.

2.3 Stochastic boundedness and ergodicity of the marking process

The stochastic interpretation $\mathcal{W}$ of a marked Petri net introduces additional constraints on the behaviour of the deterministic model $\langle \mathcal{N}, M_0 \rangle$. So the expected marking behaviour can be bounded even if $\langle \mathcal{N}, M_0 \rangle$ is unbounded. It is interesting to define this kind of stochastic boundedness and to characterize it for at least the subclass of nets in which we are interested.

Definition 2.2 *The marking process $M(t)$ of a SPN $\langle \mathcal{N}, \mathcal{W}, M_0 \rangle$ is stochastically bounded iff $\sup_t E[M(t)] < \infty$. We say that a net is stochastically bounded iff its marking process is stochastically bounded.*

In order to be able to speak about steady-state performance we have to assume that some kind of average behaviour can be estimated on the long run of the system we are studying. The usual assumption in this case is that the system model must be *ergodic*: at the limit when the observation period tends to infinity, the estimates of average values tend (almost surely) to the theoretical expected values of the probability distributions that characterize the performance indexes of interest.

Definition 2.3 [FN 85] *The marking process $M(t)$ of a SPN is ergodic iff*

$$\overline{M} \triangleq \lim_{t\to\infty} \frac{1}{t}\int_0^t M(s)ds = \lim_{t\to\infty} E[M(t)] < \infty, \;\; a.s. \tag{1}$$

The firing process $\vec{\sigma}(t)$ of a SPN is ergodic iff

$$\vec{\sigma}^* \triangleq \lim_{t\to\infty} \frac{\vec{\sigma}(t)}{t} = \lim_{t\to\infty} \frac{E[\vec{\sigma}(t)]}{t} < \infty, \;\; a.s. \tag{2}$$

$\overline{M}$ is the mean marking vector, while $\vec{\sigma}^*$ is called the mean firing rate vector.

Remark 2.1 *Stochastic boundedness is a necessary but non sufficient condition for the ergodicity of the marking process of a net: there exist live and stochastically bounded nets (in fact, structurally bounded) that may lead to non ergodic marking process. This is possible if the embedded Markov chain has one (or more) transient state (the initial marking) and two (or more) different terminal classes (see, e.g.,* [BV 84, Mol 85, CCS 89]*).*

Let us now give a necessary condition for the stochastic boundedness of a live Markovian Petri net.

Theorem 2.1 *Live and stochastically bounded Markovian Petri nets are consistent.*

Let us remark that, for SPNs, the stochastic boundedness concept coincides with the concept of "stochastic conservativeness": there exists a token linear conservative law ($Y^T\overline{M} = Y^T M_0, Y > \vec{0}$) if and only if there exists a bounded behaviour of the marking process (given by $\overline{M}$). Then, the previous result allows to state a stochastic version of the well known result for autonomous Petri nets "liveness & structural boundedness $\Rightarrow$ consistency & conservativeness": stochastic conservativeness collapses with stochastic boundedness nets and if a net is also live then it is consistent.

3 Totally open systems of Markovian sequential processes

Let us consider the subclass of Markovian Petri nets (Section 2) obtained from systems of sequential processes [Rei 82, SB 88] by adding independent (constant rate) exponentially distributed firing times to transitions. This subclass of nets can be used for modelling and analysing distributed systems composed by sequential processes communicating with buffers. Each sequential process is modelled by a *safe state machine*. The communication among them is described by places (buffers) which contain products (tokens) of some processes that are resources for others. Each buffer is *private* for two state machines, in the sense that it is an output place of only one machine and input place of the other. From a queueing network perspective, safe state machines represent "complex servers" while buffers represent queues.

First of all we recall the state machines subclass of Petri nets and some of their basic properties. Then systems of sequential processes are formally defined.

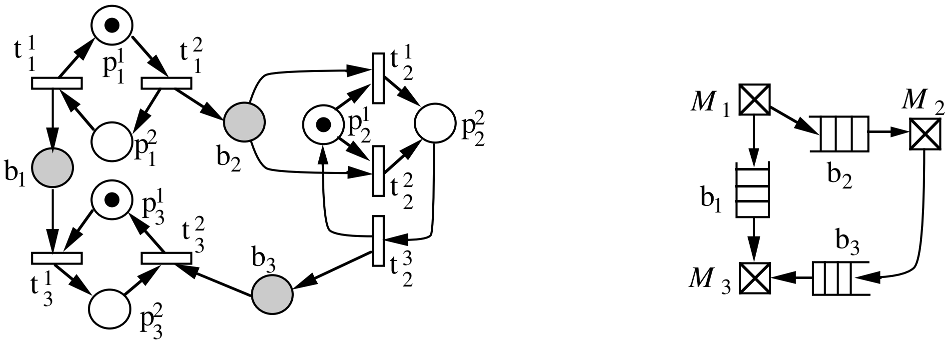


Figure 1: A totally open system of Markovian sequential processes and its queueing network representation.

Definition 3.1 *A Petri net $\mathcal{M} = \langle P, T, Pre, Post\rangle$ is a state machine (SM for short) iff* $\forall t \in T : |{}^\bullet t| = |t^\bullet| = 1$ *and* $Pre({}^\bullet t, t) = Post(t^\bullet, t) = 1$.

Well known results for state machines are grouped in the following property.

Property 3.1 [Sil 85] *Let $\mathcal{M}$ be a state machine. Then $\mathcal{M}$ is structurally bounded. $\mathcal{M}$ is structurally live iff it is strongly connected.*

A marked state machine $\langle \mathcal{M}, M_0\rangle$ is live iff $\mathcal{M}$ is strongly connected and $M_0 \neq \vec{0}$.

The next definition recalls deterministic systems of sequential processes. Each sequential process is a live and safe state machine. The communication among state machines is made by means of private buffers (see Figure 1 where grey places are buffers).

Definition 3.2 [Rei 82] *A marked Petri net $\langle \mathcal{N}, M_0\rangle = \langle P_1 \cup \ldots \cup P_s \cup B, T_1 \cup \ldots \cup T_s, Pre, Post, M_0\rangle$ is a deterministic system of sequential processes (deterministic system, DS for short) iff*

i) $\forall i, j \in \{1, \ldots, s\}$ *s.t.* $i \neq j : P_i \cap P_j = \emptyset, T_i \cap T_j = \emptyset, P_i \cap B = \emptyset$

ii) $\forall i \in \{1, \ldots, s\} : \langle \mathcal{M}_i, M_0|_i\rangle = \langle P_i, T_i, Pre|_i, Post|_i, M_0|_i\rangle$ *are live and safe state machines (where $Pre|_i$,$Post|_i$ and $M_0|_i$ are the restrictions of Pre, $Post$ and M_0 to P_i and T_i)*

iii) *The set of buffers B is such that $\forall b \in B$:*
a) ${}^\bullet b \neq \emptyset$, $b^\bullet \neq \emptyset$
b) $\exists i, j \in \{1, \ldots, s\}, i \neq j$ *such that* ${}^\bullet b \subset T_i$ *and* $b^\bullet \subset T_j$
c) $\forall p \in P_1 \cup \ldots \cup P_s : p^\bullet \cap b^\bullet = \emptyset \vee p^\bullet \subseteq b^\bullet$

The system is called deterministic because the marking of buffers does not disturb the decisions taken by a state machine, i.e. choices in the state machines are free (requirement iii.c of previous definition). This assumption allows a "divide and conquer" structural stochastic analysis: the steady state firing rate vector is *computed separately* for all the state machines and then it is *scaled* because of the buffers influence.

The extension to systems of communicating non safe state machines is not studied here. In any case, most of the results below can be easily extended to those systems if a multiple-server semantics [ABB 89] is consider for the timing of transitions.

Systems of Markovian sequential processes are defined associating (exponentially distributed) random variables to transitions (enabling time).

Definition 3.3 *A Markovian Petri net $\langle \mathcal{N}, \mathcal{W}, M_0 \rangle$ is a system of Markovian sequential processes (Markovian system, MS for short) iff $\langle \mathcal{N}, M_0 \rangle$ is a deterministic system of sequential processes.*

Markovian systems can be seen as a generalization of Markovian queueing networks. Servers are sequential processes (state machines) that can be synchronized by queues (buffers). This kind of servers can be modelled with product form queueing networks but synchronizations between different queues cannot. It is in this sense that Markovian systems generalize Markovian queueing networks (see Figure 1).

Now, totally open systems of Markovian sequential processes are defined as follows.

Definition 3.4 *A totally open deterministic system of sequential processes (open system, OS for short) is a DS without circuits containing buffers. A totally open system of Markovian sequential processes (open Markovian system, OMS for short) is a MS without circuits containing buffers.*

4 Qualitative analysis of totally open systems

Some interesting qualitative results can be derived from the structure of these nets. Liveness of OS and unboundedness of the buffers are presented in Theorem 4.1. In Theorem 4.2, consistency (necessary condition for ergodicity) is shown to collapse with existence of home state for this subclass of nets. The rest of this Section is devoted to the study of necessary and sufficient conditions for the "potential ergodicity": existence of a stochastic interpretation that can lead to ergodic systems.

Theorem 4.1 *Let $\langle \mathcal{N}, M_0 \rangle$ be an OS. Then $\langle \mathcal{N}, M_0 \rangle$ is live and all buffers are unbounded.*

An interesting property of OSs that states a bridge between its behavioural and structural analysis is that all potentially reachable markings are reachable:

Property 4.1 *Let $\langle \mathcal{N}, M_0 \rangle$ be an OS. Then $M \in R(\mathcal{N}, M_0)$ iff $M \in PR(\mathcal{N}, M_0)$. In other words, each vector $\vec{\sigma} \in I\!N^m$ such that $M_0 + C\vec{\sigma} \geq \vec{0}$ corresponds at least to one fireable sequence in $\mathcal{N}$ from M_0.*

The following theorem relates, for OSs, a behavioural property (existence of home state) with a structural one (consistency).

Theorem 4.2 *Let $\langle \mathcal{N}, M_0 \rangle$ be an OS. Then $\mathcal{N}$ is consistent iff M_0 is a home state.*

In Theorem 2.1, a necessary condition for the stochastic boundedness of a live Markovian Petri net is shown. Now, let us remark that there exist non consistent OSs (see Figure 2). Then, in practice, it is convenient to check consistency (a polynomial time computation) of the underlying OS before computing ergodicity conditions for a given stochastic interpretation of an OMS. Taking into account the above remark and Theorem 2.1, the following result with practical interest can be stated.

Theorem 4.3 *There exist OSs $\langle \mathcal{N}, M_0 \rangle$ such that for all stochastic interpretations $\mathcal{W}$ the OMS $\langle \mathcal{N}, \mathcal{W}, M_0 \rangle$ is stochastically unbounded (so non ergodic).*

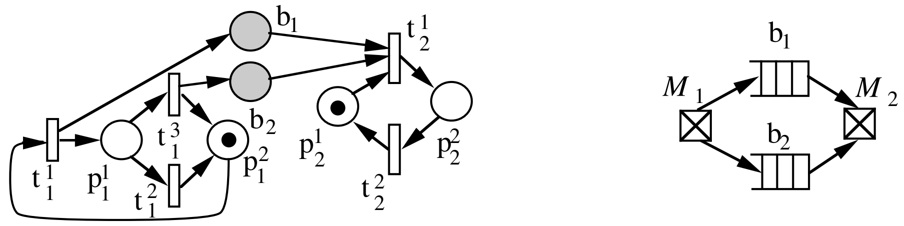
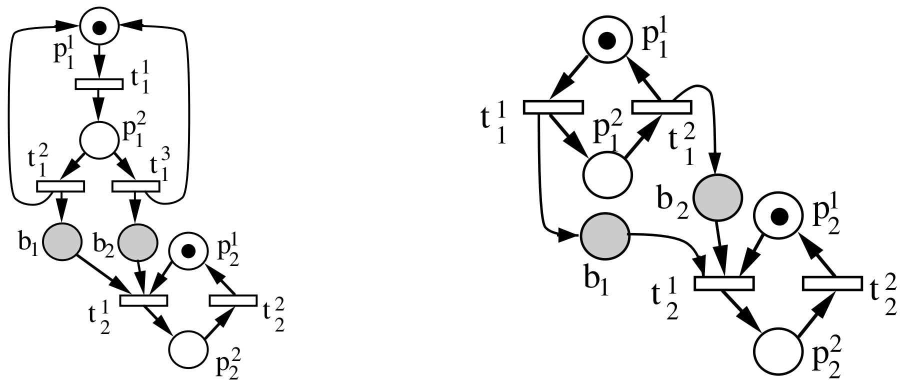


Figure 2: A non consistent OS: $M(b_1) - M(b_2) = \vec{\sigma}(t_1^1) - \vec{\sigma}(t_1^3) = \vec{\sigma}(t_1^1) - [\vec{\sigma}(t_1^1) - \vec{\sigma}(t_1^2) - M(p_1^1)] = M(p_1^1) + \vec{\sigma}(t_1^2)$. Since the OS is live, $\vec{\sigma}(t_1^2) \to \infty \Rightarrow M(b_1) - M(b_2) \to \infty \Rightarrow$ structurally non ergodic net.



(a) Structurally non ergodic
$({}^\bullet b_1, {}^\bullet b_2) \notin$SDR; $(b_1^\bullet, b_2^\bullet) \in$SDR

(b) Potentially ergodic
$({}^\bullet b_1, {}^\bullet b_2) \in$SDR; $(b_1^\bullet, b_2^\bullet) \in$SDR

Figure 3: Consistent OSs with two state machines and two buffers.

Unfortunately, it cannot be stated that if $\langle \mathcal{N}, M_0 \rangle$ is a consistent OS, there exists $\mathcal{W}$ such that $\langle \mathcal{N}, \mathcal{W}, M_0 \rangle$ is ergodic. The net in Figure 3.a is consistent but there does not exist any stochastic interpretation making it ergodic: the case of $\lambda_1^2 = \lambda_1^3$ (of course, only possible in theory !) leads to a null recurrent process and so non ergodic.

Nevertheless, it cannot be stated that if there exists a state machine of the system with more than one input buffer, the Markovian system cannot be ergodic (see Figure 3.b).

Let us now recall the concept of synchronic distance relation. If two subsets of transitions are in synchronic distance relation then it is not possible to fire an infinite number of times some transition of the first subset without firing any transition of the second subset, and vice versa. Even more, if two subsets of transitions are in synchronic distance relation they behave like if they were included in a regulation circuit. This equivalence relation is used below for finding necessary and sufficient conditions for the existence of a stochastic interpretation that makes ergodic the OS.

Definition 4.1 [Sil 87] *Let $\langle \mathcal{N}, M_0 \rangle$ be a Petri net and T_1, T_2 subsets of transitions. T_1, T_2 are in synchronic distance relation, $(T_1, T_2) \in$ SDR, iff $\exists W_1, W_2 \in \mathbb{N}^m$ vectors which express the weights associated with the transitions of the subsets T_1 and T_2 (i.e.*

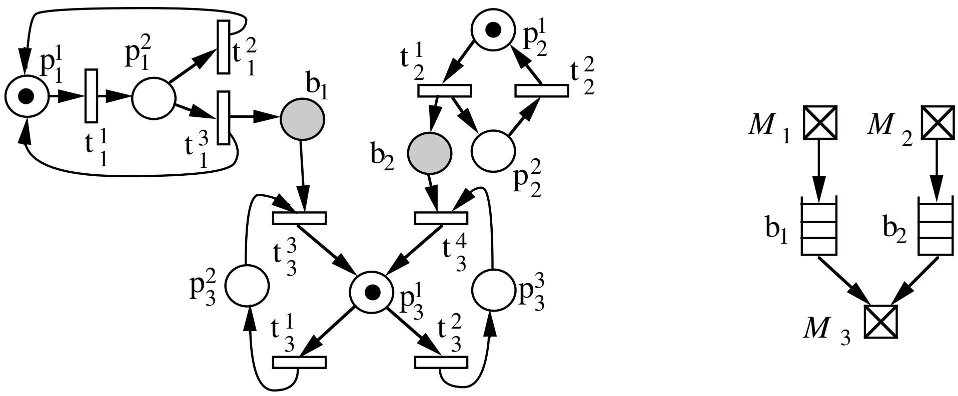


Figure 4: Structurally non ergodic system with three state machines.

support$(W_1) = T_1$ *and support*$(W_2) = T_2$, *where support*$(W) = \{t \in T \mid W(t) \neq 0\}$) *and* $\exists k \in I\!N$ *such that*

$$\text{supremum } \{|(W_1 - W_2)^T \vec{\sigma} \mid \ \textit{with } \sigma \in L(\mathcal{N}, M) \textit{ and } M \in R(\mathcal{N}, M_0)\} \leq k$$

For characterizing the possible existence of a timing interpretation making a given OS ergodic, let us give local rules that will be composed step by step for a large system.

The first result is a negative one. If a given SM receives tokens from two different SMs, one of them without input buffers, the system cannot be ergodic (see Figure 4).

Theorem 4.4 *Let $\langle \mathcal{N}, M_0 \rangle$ be an OS such that for one of their communicating state machines $\mathcal{M}_i$:*

a) *$\exists b_1$ such that $b_1^\bullet \subseteq T_i$ (i.e. it is an input buffer of the machine $\mathcal{M}_i$) and $^\bullet b_1 \subseteq T_j$, where T_j is the set of transitions of another state machine $\mathcal{M}_j$ such that $\nexists b \in B$ satisfying $b^\bullet \subseteq T_j$ (i.e. the input state machine of buffer b_1 has not input buffers), and*

b) *$\exists b_2$ such that $b_2^\bullet \subseteq T_i$ (i.e. another input buffer of the machine $\mathcal{M}_i$) and $^\bullet b_2 \not\subseteq T_j$ (i.e. the input state machine of buffer b_2 is not $\mathcal{M}_j$).*

Then, there is no Markovian interpretation $\mathcal{W}$ such that $\langle \mathcal{N}, \mathcal{W}, M_0 \rangle$ is ergodic.

Now, let us give necessary and sufficient conditions for the existence of a stochastic timing interpretation that makes ergodic a system composed by two state machines (see Figures 2 and 3).

Theorem 4.5 *Let $\langle \mathcal{N}, M_0 \rangle = \langle P_1 \cup P_2 \cup B, T_1 \cup T_2, Pre, Post, M_0 \rangle$ be an OS composed by two SMs and a set of buffers B such that $\forall b \in B : {^\bullet b} \subseteq T_1, b^\bullet \subseteq T_2$.*

There exists a Markovian interpretation $\mathcal{W}$ such that $\langle \mathcal{N}, \mathcal{W}, M_0 \rangle$ is ergodic if and only if: (i) $\mathcal{N}$ is consistent and (ii) $\forall b_i, b_j \in B$: $({^\bullet b_i}, {^\bullet b_j}) \in SDR$ and $(b_i^\bullet, b_j^\bullet) \in SDR$.

Remark 4.1 *In case of OSs composed by two SMs, if (i) and (ii) of Theorem 4.5 hold then the marking of all the buffers can be always computed from the marking of one buffer and the marking of the SMs. With the object of computing ergodicity conditions for a larger system including $\mathcal{N}$ as a subsystem, if (i) and (ii) hold, from the performance point of view, we can suppose without loss of generality that the two state machines are communicating with at most one buffer.*

Let us now give the "transitivity rule" for three state machines communicating with buffers like in Figure 1. This rule completes the stating of necessary and sufficient conditions for the existence of a Markovian timing that makes ergodic a given OS.

Theorem 4.6 *Let $\langle \mathcal{N}, M_0 \rangle = \langle P_1 \cup P_2 \cup P_3 \cup \{b_1, b_2, b_3\}, T_1 \cup T_2 \cup T_3, Pre, Post, M_0 \rangle$ be an OS composed by three state machines and three buffers such that $^\bullet b_1 \subseteq T_1, b_1^\bullet \subseteq T_3$, $^\bullet b_2 \subseteq T_1, b_2^\bullet \subseteq T_2$, $^\bullet b_3 \subseteq T_2$, and $b_3^\bullet \subseteq T_3$.*

There exists a Markovian interpretation $\mathcal{W}$ such that $\langle \mathcal{N}, \mathcal{W}, M_0 \rangle$ is ergodic if and only if: (i) $\mathcal{N}$ is consistent and (ii) $(^\bullet b_1, ^\bullet b_2) \in SDR$, $(b_2^\bullet, ^\bullet b_3) \in SDR$, $(b_1^\bullet, b_3^\bullet) \in SDR$.

Remark 4.2 *If (i) and (ii) of Theorem 4.6 hold, then the marking of b_3 can be always computed from the marking of b_1, b_2 and the marking of the SMs. With the object of computing conditions for a larger system including $\mathcal{N}$ as a subsystem, if (i) and (ii) hold, the state machine $\mathcal{M}_2$ and the buffers b_2, b_3 can be substituted by a unique buffer.*

Theorems 4.5 and 4.6 provide rules (Remarks 4.1 and 4.2) for an iterative reduction of buffers of a totally open system of Markovian sequential process. These rules preserve the possibility of existence of a timing that makes the system ergodic if the necessary and sufficient conditions (stated in the mentioned theorems) hold.

Therefore the existence of a stochastic timing that makes an OMS ergodic is characterized in terms of pure structural conditions: consistency and some synchronic distance relations.

Let us now consider a different problem: given an OMS, once that the previous conditions have been checked, we want to know if ergodicity conditions hold for a given timing. In the next Section we study these conditions and the steady-state throughput of the system under ergodicity assumption.

5 Quantitative analysis of totally open systems

We have seen that for OS the liveness of their isolated components assures liveness of the whole system. Nevertheless, from a performance analysis point of view, after checking the potential ergodicity of the OS (conditions presented in Section 4) it is interesting to quantify the influence of buffers on the firing rate of transitions. In this Section, we study quantitative aspects related with totally open systems of Markovian sequential processes (OMS).

5.1 Necessary and sufficient ergodicity conditions

In [FN 89], an ergodicity theorem is proved for open synchronized queueing networks. Let us introduce now the concept of *saturated net* and the adaptation of the above-mentioned theorem for OMS.

Definition 5.1 *Let $\langle \mathcal{N}, \mathcal{W}, M_0 \rangle$ be an OMS and b one of its buffers. The net obtained from $\langle \mathcal{N}, \mathcal{W}, M_0 \rangle$ by deleting the buffer b and its adjacent arcs is called the saturated system according to b.*

Note that the saturated system according to b behaves like $\langle \mathcal{N}, \mathcal{W}, M_0 \rangle$ in the case in which the buffer b is always marked.

Theorem 5.1 [FN 89] *Let $\langle \mathcal{N}, \mathcal{W}, M_0 \rangle$ be an OMS.*

i) *Let $B' \subseteq B$ be the subset of buffers the marking of which can vary independently. If $\forall b \in B'$, $POST[b]^T \vec{\sigma}^{(b)*} < PRE[b]^T \vec{\sigma}^{(b)*}$ then the associated Markov process is positive recurrent.*

ii) *If there exists a buffer b such that $POST[b]^T \vec{\sigma}^{(b)*} > PRE[b]^T \vec{\sigma}^{(b)*}$ then the associated Markov process is transient.*

Where $\vec{\sigma}^{(b)*}$ denotes the mean firing rate vector of the saturated net according to b.

Condition (i) of Theorem 5.1 means that for each buffer (queue) the input flow (arrival rate) must be less than the service rate of the output state machine.
NUMERICAL COMPUTATION: As it is remarked in [FN 89], the application of this ergodicity criterion requires the computation of the steady-state behaviour of all saturated systems that can be obtained from $\langle \mathcal{N}, \mathcal{W}, M_0 \rangle$. This computation is not possible (so far) for all open synchronized queueing networks (in fact, it is possible just for those nets having at most two unbounded places). However the computation is possible for OMS, because for these nets an efficient (in fact, polynomial in the number of nodes of the net) method for computing the steady-state behaviour exists (see Section 5.2), and all saturated systems of an OMS are OMS again.

Let us illustrate the numerical computation of the ergodicity criterion with the net in Figure 1. The left and right hand side expressions of condition (i) of Theorem 5.1 for buffer b_2 can be computed considering the state machines 1 and 2 in isolation:

$$POST[b_2]^T \vec{\sigma}^{(b_2)*} = \frac{\lambda_1^1 \lambda_1^2}{\lambda_1^1 + \lambda_1^2}; \quad PRE[b_2]^T \vec{\sigma}^{(b_2)*} = \frac{(\lambda_2^1 + \lambda_2^2)\lambda_2^3}{\lambda_2^1 + \lambda_2^2 + \lambda_2^3} \tag{3}$$

The same computation for the buffer b_1 leads to the expresions:

$$POST[b_1]^T \vec{\sigma}^{(b_1)*} = \frac{\lambda_1^1 \lambda_1^2}{\lambda_1^1 + \lambda_1^2}; \quad PRE[b_1]^T \vec{\sigma}^{(b_1)*} = \frac{\lambda_3^1 \lambda_3^2}{\lambda_3^1 + \lambda_3^2} \tag{4}$$

The marking of buffer b_3 linearly depends on the marking of the other buffers (see Remark 4.2), so it must not be considered.

Then, the system is ergodic if and only if:

$$\frac{\lambda_1^1 \lambda_1^2}{\lambda_1^1 + \lambda_1^2} < \min \left(\frac{(\lambda_2^1 + \lambda_2^2)\lambda_2^3}{\lambda_2^1 + \lambda_2^2 + \lambda_2^3}, \frac{\lambda_3^1 \lambda_3^2}{\lambda_3^1 + \lambda_3^2} \right) \tag{5}$$

5.2 Computing the steady-state performance measures

Let us suppose in this Section that the system ergodicity conditions given in Theorem 5.1 are satisfied. The proof of the following theorem [CS 89] gives a method for computing efficiently the steady-state behaviour of the connected machines of an OMS.

Theorem 5.2 *Let $\langle \mathcal{N}, \mathcal{W}, M_0 \rangle$ be an OMS. If its marking process is ergodic then the mean number of tokens in each place and the expected firing rate of transitions in steady-state can be computed in polynomial time.*

As an example, let us consider again the net in Figure 1. In this case, there exists one state machine without input buffers: $\mathcal{M}_1$. Marking invariant and flow equations for this machine have the form:

$$\overline{M}(p_1^1) + \overline{M}(p_1^2) = 1; \quad \vec{\sigma}^*(t_1^1) = \lambda_1^1 \overline{M}(p_1^2); \quad \vec{\sigma}^*(t_1^2) = \lambda_1^2 \overline{M}(p_1^1); \quad \vec{\sigma}^*(t_1^1) = \vec{\sigma}^*(t_1^2) \tag{6}$$

This system can be solved, obtaining:

$$\overline{M}(p_1^1) = \frac{\lambda_1^1}{\lambda_1^1 + \lambda_1^2}; \quad \overline{M}(p_1^2) = \frac{\lambda_1^2}{\lambda_1^1 + \lambda_1^2}; \quad \vec{\sigma}^*(t_1^1) = \vec{\sigma}^*(t_1^2) = \frac{\lambda_1^1 \lambda_1^2}{\lambda_1^1 + \lambda_1^2} \tag{7}$$

Now, for computing the steady-state measures of the other state machines under the assumption of ergodicity (equation (5)), it is necessary to take into account that $\vec{\sigma}^*(t_1^2) = \vec{\sigma}^*(t_2^1) + \vec{\sigma}^*(t_2^2)$ and $\vec{\sigma}^*(t_3^1) = \vec{\sigma}^*(t_1^1)$, i.e. the input flow of tokens to each buffer in steady-state must be equal to the output flow:

$$\begin{aligned}
&\overline{M}(p_2^1) = 1 - \overline{M}(p_2^2); \quad \overline{M}(p_2^2) = \frac{\lambda_1^1 \lambda_1^2}{(\lambda_1^1 + \lambda_1^2)\lambda_2^3} \\
&\vec{\sigma}^*(t_2^1) = \frac{\lambda_1^1 \lambda_1^2 \lambda_2^1}{(\lambda_1^1 + \lambda_1^2)(\lambda_2^1 + \lambda_2^2)}; \quad \vec{\sigma}^*(t_2^2) = \frac{\lambda_1^1 \lambda_1^2 \lambda_2^2}{(\lambda_1^1 + \lambda_1^2)(\lambda_2^1 + \lambda_2^2)} \\
&\vec{\sigma}^*(t_2^3) = \vec{\sigma}^*(t_3^1) = \vec{\sigma}^*(t_3^2) = \frac{\lambda_1^1 \lambda_1^2}{\lambda_1^1 + \lambda_1^2}
\end{aligned} \tag{8}$$

6 Conclusions

Systems of Markovian sequential processes have been defined. This subclass of Petri nets can be seen as an extension of queueing networks: the servers are modelled with state machines and some restricted synchronization schemes are allowed. The extension to systems with servers different from state machines is possible. In general, those servers that only emit tokens (and do not receive) can be modelled with any bounded net, because its reachability graph is a state machine. For those servers that receive tokens, the synchronizations with buffers must preserve the assumption of determinism of the system (buffers have no influence on the decisions taken by a sequential process). Then, processes receiving tokens from buffers can be modelled with bounded nets such that if a given transition has an input arc from a buffer then all transitions that are in conflict or can be concurrently fired with it must have equally weighted input arcs from the same buffer.

Some interesting qualitative properties of totally open systems have been studied making special emphasis in those that assure the possibility of ergodic behaviour of the system. A polynomial complexity algorithm for computing necessary and sufficient ergodicity conditions for these systems has been proposed. Under ergodicity assumptions, steady-state performance measures can be also computed in polynomial time.

Partially open systems of stochastic processes must be studied. Ergodicity conditions introduced here are valid for the open components of these systems. The computation of the exact steady-state performance measures of closed systems (such as a closed producer-consumer system) seems not to be possible in polynomial time. In any case, very efficient techniques for computing bounds for the throughput are being developed.

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