Petri Nets and Integrality Relaxations: A View of Continuous Petri Net Models

Manuel Silva and Laura Recalde

***Abstract*—Petri nets are formalisms for the modeling of discrete event dynamic systems (DEDS). The integrality of the marking and of the transitions firing counters is a clear reflection of this. To reduce the computational complexity of the analysis or synthesis of Petri nets, two relaxations have been introduced at two different levels: (1) at *net* level, leading to continuous [1]–[3] net systems; (2) at *state equation* (or *fundamental equation*) *level*, which has allowed to obtain systems of linear inequalities, or linear programming problems [4]–[6]. These relaxations are mainly related to the fractional firing of transitions, which implies the existence of non-integer markings. For instance, a first-order continuization transforms*a discrete and stochastic model, into a continuous and deterministic "approximated" model*.**

The purpose of our work is to give an overview of this emerging field. It is focused on the relationship between the properties of (discrete) PNs and the corresponding properties of their continuous approximation. Through the interleaving of qualitative and quantitative techniques, surprising results can be obtained from the analysis of these continuous systems. For these approximations to be "acceptable," it is necessary that large markings (populations) exist. It will also be seen, however, that not every populated net system can be continuized. In fact, there exist systems with "large" populations for which continuization does not make sense. The possibility of expressing non-linear behaviors may lead to deterministic continuous differential systems with complex behaviors (orbits, limit cycles, different attractors, and chaos).

***Index Terms*—Continuization, integer relaxation, nonlinear-linear differential systems, petri net.**

I. Petri Nets: A Paradigm For the Modelling of Discrete Event Systems

In a discrete event dynamic system (DEDS), states can be encoded by a set of variables defined in the integers, usually in the natural numbers, $\mathbb{N}$. Provided the existence of events to which the system is receptive, state changes occur at fixed instants.

Petri nets (PNs) constitute a formal paradigm for the modeling, analysis, synthesis, and implementation of systems that "can be seen" as discrete. Here, the word paradigm is used in the conceptual sense of T. Kuhn, as "total pattern of perceiving, conceptualising, acting, validating, and valuing associated with a particular image of reality that prevails in a science or branch of science." From the sociology of science and technology's point of view, PNs are also a paradigm since communities of researchers and engineers exist, grouped under common or complementary interests, using it as a family of formalisms.

In a PN, the state variables are the places, while their value is the marking. The marking vector is a state vector. The marking is changed by the firing of transitions, thanks to the occurrence of their associated events. See for instance [7], [8] for an introduction of the basic concepts and notations of PNs. We will just remark that a system is a structure $\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post} \rangle$ ($\mathbf{Pre}$ and $\mathbf{Post}$ represent the static structure of the model, from which the token flow matrix $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$ can be deduced) provided with an initial marking over P, $\mathbf{m_0}$ (analogous to the initial condition of a differential equation). If all arc weights are one, the net is called *ordinary*. Starting from a PN system, if $\mathbf{m}$ is reachable from $\mathbf{m_0}$ through a sequence σ, a state (or fundamental) equation can be written:

$$\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma},$$

where $\boldsymbol{\sigma} \in \mathbb{N}^{|T|}$ is the firing count vector, and $\mathbf{m} \in \mathbb{N}^{|P|}$.

The solution of a system in the integers, in general, is NP-hard. Another difficulty has to be added: the existence of integer solutions of the state equation that cannot be reached from $\langle \mathcal{N}, \mathbf{m_0} \rangle$. They are the so-called *spurious solutions* [6], [9]. It is known that there exists some subclasses of systems that do not have spurious solutions (e.g., live marked graphs), or even if spurious solutions did exist, they are not as problematic (for instance, if a bounded free-choice net system has a spurious deadlock, it has a reachable deadlock as well [10]). Some techniques have been developed to diminish the number of spurious solutions although in general, they are unable to completely remove them [9].

All the previous comments concern autonomous completely non-dterministic-deterministic discrete systems, for which no rule has been set to decide *which* transition will fire and *when*. The PN paradigm contains a whole set of autonomous formalisms with different *abstraction levels*: condition/event, elementary, place/transition, colored, and predicate/transition.

Place/transition level is considered here as the reference level.

A model of a system cannot be purely non-dterministic-deterministic. It must be possible to describe the inputs, outputs, and their temporal interactions with the system. *Interpretative extensions* [11] (interpretations in the sequel) are *extensions* of the abstract formalism that provide information about I/O behavior. This can be done by associating guards that depend on external events to the firing of transitions, or by means of a temporal framework that associates a certain timing to events.

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The authors are with the Departamento de Informática e Ingeniería de Sistemas, Centro Politécnico Superior, Universidad de Zaragoza, María de Luna 3, E-50015 Zaragoza, Spain (e-mail: silva@posta.unizar.es, lrecalde@posta.unizar.es).

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Temporal interpretations may be considered some of the most interesting interpretations. They often use the notion of stochastic processes (the deterministic case is a very particular one), thus allowing the representation of performance evaluation models [12], [13]. Other interpretations are, for instance, based on fuzzy sets and logic (see [14] for a survey of these works). Here, we will be concerned with stochastic timed interpretations, with the purpose of obtaining performance measures (throughput, cycle time, and mean number of tokens).

Once the model has been set, it is important to *analyze* logic, or qualitative, properties (for instance, is it deadlock-free?), and quantitative or performance-related properties (the mean throughput, for example). It is particularly interesting to approach the analysis of the model in an integrated way, interleaving the use of qualitative and quantitative analysis techniques [15]. Sometimes, it may occur, that its synthesis has to be obtained from a partial model. This happens, for instance, in supervisory control [16], [17], or when trying to obtain the optimal initial marking for a certain net system (for example: finding an initial marking with minimal cost that guarantees a certain cycle time, "minimal initial marking problem" [18]–[21]). In other cases, the synthesis may concern the definition of a scheduling policy (for example, a control to minimize a cycle time, trying to define the routing, and the firing date).

The above concerns the modeling, analysis, and synthesis of DEDS by means of discrete event formalisms. However, if the system contains large populations, i.e., groups of elements with common properties (for example, a large number of foxes and rabbits, or a large numbers of parts of a certain type), it may be the case that relaxing integrality into non-negative reals allows the system to obtain a "reasonably" approximated model whose study might be easier. Relaxation of the integrality constraints (continuization) is a classical technique that in the case of PNs can be done at net-level or at equational-level. This issue is addressed in Section II.

Continuous PNs have been proposed in the literature to obtain an approximation of discrete behavior [22], [23]. However, little effort has been devoted to compare these discrete and continuous behaviors. Although it seems reasonable to expect that the continuized model and the original discrete models have a close behavior, it is not always true. Section III shows some examples that prove the existence of non-continuizable PN models, in the sense that important properties of the discrete model are lost. Some basic properties that continuous models exhibit are presented also in this section, some rather counter-intuitive.

To clarify the basic semantics used in continuous PNs, Section IV revisits the so-called *infinite* and *finite* server-semantics, as used in discrete PNs. Their relation with variable and constant speed semantics, commonly used in continuous PNs, are shown. It is also proven that in some net subclasses, the computation of the exact value of the steady-state behavior for both semantics reduces to a linear programming problem. Therefore, it is computable in polynomial time.

In all these sections, the starting point is the discrete PN system, and it is studied as a mathematical object to see how continuization changes its basic properties. Section V studies continuous PN systems from a modeling point of view. It focuses on the relationships that exist between population systems and net objects and properties, emphasizing the continuization of models obtained through "decoloration," and showing that not every DEDS with a large population can be continuized, even if its elements exhibit extremely symmetric behavior.

II. Relaxations of Discrete Event Dynamic Systems

In general, analysis and synthesis problems in DEDS are computationally complex, since the set of reachable states can become extremely large (*state explosion problem*) resulting in the need for relaxations. Initially, we will not restrict our attention to a net sub-class, but we will relax the conditions on the whole set of net models.

Section II-A sets the integrality relaxation techniques among the traditional relaxation techniques of formal models. The application of continuization techniques to PNs is briefly sketched in Section II-B.

A. Relaxations for the Analysis and Synthesis of DEDS

Although we do not intend to systematically present all possible relaxation methods and techniques, some of the most interesting are:

1) *Reduction of the most "disagreeable" restrictions*, when this does not denature the behaviors that have to be analyzed/synthetized. This can be done directly, by just removing them or, in an indirect way, replacing subsystems of constraints by others that can be more easily handled.
2) *Decomposition techniques* ("divide and conquer" approach). This can be applied, for instance, at a temporal level (the fastest modes are separated from the slowest ones) or at spatial level (division into subsystems with relatively little interactions; i.e., loosely coupled).
3) *Lagrangian relaxations*, based on duality properties of dynamic systems, are especially useful when it is an optimization problem.
4) *Relaxation of the integrality constraints*, the approach that will be considered in this work. The relative error that is introduced by the relaxation of natural numbers into non-negative real numbers may be admissible under heavy traffic (which happens when the populations in the system are relatively large) but, it will be seen later that sometimes continuization is not possible, even if large populations do exist (see Section III-A and Section V-A).

Techniques for the relaxation of the integrality constraints of discrete models have been employed in very different fields:

- *Population dynamics* (Biology, Ecology, Sociology, etc.). The classical model of Volterra-Lotka is of paradigmatic simplicity (see, for example, [24]).
- *Manufacturing systems* [23], [25], [26], in which continuization fits particularly well when heavily loaded, long production (possibly interleaved) lines are considered.
- *Communication systems* [27].
- *Traffic systems*. It is necessary to underline here that, in this particular application domain, the usual macroscopic models are often separable variables, which leads to partial differential equations (see, for instance, [28]).

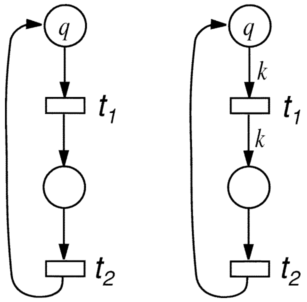

Fig. 1. Two bounded net systems. The one on the right is live as discrete if and only if $q \geq k$, but it is live as continuous simply if $q > 0$.

B. Continuization of Petri Nets

In view of the previous techniques, it is natural that relaxation of the integrality constraints (continuization) appear also in the PN paradigm. They allow PNs to go from discrete models, that may be relatively detailed but computationally difficult to manage, to "coarser" models, where computations can be done in a "more reasonable" way. The prize payed to have computationally less complex models concern errors in certain quantitative measures, and in some qualitative properties.

In 1987, at the "European Workshop on Petri Nets Applications and Theory" held in Zaragoza, two papers, with different direct motivations yet indirectly similiar, considered relaxations of a continuization type. In [1] (revisited afterwards in [2], [3]) continuization was introduced at *net level*, autonomous or timed, while in [4] (revisited in [5], [6]) the *state equation* was relaxed to find linear programming problems for the efficient computation of non-temporal properties. The use of linear programming techniques, to the computation of performance bounds is presented in [29], [30]. We will concentrate here on the first approach and study continuous PNs.

Definition 1: A continuous PN system is a pair $\langle \mathcal{N}, \mathbf{m_0} \rangle$, where $\mathcal{N}$ is a plain PN ($\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post} \rangle$, with P and T disjoint sets of places and transitions, and $\mathbf{Pre}$ and $\mathbf{Post}$ $|P| \times |T|$ sized, natural valued, incidence matrices), and $\mathbf{m_0} \in (\mathbb{R}^+)^{|P|}$ is a *continuous marking*.

A transition t is *enabled* at $\mathbf{m}$ if for every $p \in {}^\bullet t$, $\mathbf{m}[p] > 0$. In other words, the enabling condition of continuous systems is the same as the enabling condition of discrete ordinary systems: "every input place is marked." As in discrete systems, the *enabling degree* of a transition measures the maximal amount in which the transition can be fired in one shot.

Definition 2: The enabling degree of a transition t at a certain marking $\mathbf{m}$ is defined as $\mathbf{e}(\mathbf{m})[t] = \min_{p \in {}^\bullet t}\{\mathbf{m}[p]/\mathbf{Pre}[p, t]\}$.

The main drawback of continuization techniques is that synchronizations are "strongly weakened." For instance, it is not difficult to see that the two net systems in Fig. 1 have different behaviors as discrete, while as continuous they behave in a similar way: if the enabling degree of t_1 in the net system on the left is q, its enabling degree in the net system on the right is q/k (if $q/k < 1$, t_1 cannot fire if the model is considered as discrete).

Behind the aforementioned approaches lies the idea of a *total continuization* (over the full set of places and transitions). Partial continuizations lead to hybrid models [3], this topic, is out of the scope of this present work.

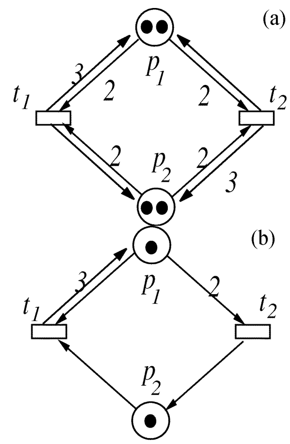

Fig. 2. Two bounded and strongly connected net systems which behave in very different ways if they are considered as discrete or as continuous: (a) is nonlive as discrete, but never gets completely blocked as continuous unless an infinitely long sequence is considered; (b) is live as discrete, but nonlive as continuous.

III. Autonomous Continuous Petri Nets

In this section, autonomous (that is, fully non-deterministic) net models will be considered. Timed models will be taken into account in Section IV. It is well known that not every dynamic system admits a linearization. The goal of Section III-A is to show that analogously not every PN can be continuized in a "reasonable" way. Section III-B presents some properties of continuous PNs, adapted and generalized from discrete PNs, making use of structural techniques based on the state equation.

A. NonContinuizable Petri Nets

If a continuous model has to be an approximation of a discrete model, it looks sensible to expect that the basic qualitative properties (deadlock-freeness, liveness, existence of home states) are preserved. The fact that this is not always true [31], puts a limit to this relaxation approach. The two net systems in Fig. 2 show that *deadlock-freeness in the discrete PN is neither a necessary, nor sufficient condition for deadlock-freeness of its continuous relaxation*. Net (a) deadlocks as discrete if t_1 or t_2 is fired, but it is not blocked as continuous because the places are not empty. Net (b) is live as discrete because it is not possible to empty p_1 when its initial marking is odd, but as continuous p_1 can be emptied. It should be noticed that the "correct" continuization of autonomous PNs requires that qualitative/logical properties are preserved if the initial marking is multiplied by a natural number k. That is, if $\langle \mathcal{N}, \mathbf{m_0} \rangle$ verifies a certain property Π (for example, deadlock-freeness), then $\langle \mathcal{N}, k\mathbf{m_0} \rangle$ should verify Π as well. This marking scaling monotonicity property (even a stronger one) holds, for instance, for the subclass of equal conflict net systems [10].

We will not consider here the most technical aspects, but it must be remarked that using the classical definitions, the continuous PN in Fig. 3 does not deadlock, since arbitrarily long-firing sequences can be found. (If a time-base is added, the problem is similar to the discharge of an R-C electric circuit.) To avoid getting trapped in nonsense, in [31], the idea of a marking being reachable at the limit (lim *-reachability*) was introduced.

Definition 3: Let $\langle \mathcal{N}, \mathbf{m_0} \rangle$ be a continuous system. We say that a marking $\mathbf{m} \in (\mathbb{R}^+)^{|P|}$ is *limit reachable*,

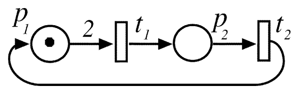

Fig. 3. Zeno PN: While it is structurally nonlive as discrete, it never deadlocks as continuous if only finite sequences are considered (after k firings of t_1 and t_2, $1/2^k$ "tokens" still remain).

$\mathbf{m} \in \mathrm{RS}_C(\mathcal{N}, \mathbf{m_0})$, if a sequence of reachable markings $\{\mathbf{m}_i\}_{i \geq 1}$ exists verifying

$$\mathbf{m}_0 \xrightarrow{\sigma_1} \mathbf{m}_1 \xrightarrow{\sigma_2} \mathbf{m}_2 \cdots \mathbf{m}_{i-1} \xrightarrow{\sigma_i} \mathbf{m}_i \cdots$$

and $\lim_{i \to \infty} \mathbf{m}_i = \mathbf{m}$.

With this definition, the net system Fig. 3 reaches a deadlock in the limit. After introducing this concept, let us clarify some properties of autonomous continuous PNs.

B. Some Properties of Autonomous Continuous Petri Net Models

A first group of properties of continuous models that can be deduced are:

Property 1—[31]: $\mathrm{RS}_c(\mathcal{N}, \mathbf{M_0})$ is a convex set.

That is, any linear combination of two reachable markings is reachable too. This property does not hold in discrete systems.

Definition 4: The set of solutions of the relaxed state equation will be denoted as $\mathrm{LRS}_C(\mathcal{N}, \mathbf{m_0})$. That is, $\mathrm{LRS}_C(\mathcal{N}, \mathbf{m_0}) = \{\mathbf{m} | \mathbf{m} = \mathbf{m}_0 + \mathbf{C} \cdot \boldsymbol{\sigma} \geq \mathbf{0}, \boldsymbol{\sigma} \geq \mathbf{0}\}$].

Property 2—[31]: $\mathrm{RS}_C(\mathcal{N}, \mathbf{m}_0) \subseteq \mathrm{LRS}_C(\mathcal{N}, \mathbf{m}_0)$ In words, as in discrete PNs, the state equation relaxation can add spurious solutions to the relaxation made at the net-level.

In most practical cases, however, these sets are equal contrary to what happens in the discrete case.

Property 3—[31]: If $\mathcal{N}$ is consistent (i.e., $\exists \mathbf{x} > \mathbf{0} : \mathbf{C} \cdot \mathbf{x} = \mathbf{0}$) and every transition can be fired, $\exists \boldsymbol{\sigma} > \mathbf{0}, \mathbf{m_0} \xrightarrow{\sigma}$ (or, equivalently, $\exists \mathbf{m} > \mathbf{0}, \mathbf{m} \in \mathrm{RS}_C(\mathcal{N}, \mathbf{m}_0)$), then

$$\mathrm{RS}_C(\mathcal{N}, \mathbf{m}_0) = \mathrm{LRS}_C(\mathcal{N}, \mathbf{m}_0).$$

Since the conditions of Property 3 are very weak, meaning *in practice* the relaxation at net-level is "equivalent" to the relaxation at the state equation level, there do not exist *spurious* solutions to the state equation. Observe also that the conditions can be checked in polynomial time, the first using a linear programming problem, and the second applying a simple algorithm [31]. The idea of the algorithm is to fire each enabled transition in half of its enabling degree, thus, never decreasing the set of enabled transitions. If it does not increase, a point is reached, in which the firing of the enabled transitions cannot lead to the enabling of any other. Hence, not every transition can be fired. Since the number transitions is finite, the algorithm stops when all have been considered.

Fig. 4 shows examples in which one of the conditions of Property 3 does not hold and the property is false. Markings [0, 1, 0, 0] in the system on the left and [0, 1, 0, 0, 1] in the system on the right belong to the LRS_C but not to the RS_C.

Property 4—[31]: If every transition can be fired, then for every $\mathbf{x} \geq \mathbf{0}$ such that $\mathbf{C} \cdot \mathbf{x} \geq \mathbf{0}$ a marking $\mathbf{m} \in \mathrm{RS}_C(\mathcal{N}, \mathbf{m}_0)$ exists such that $\mathbf{m} \xrightarrow{\sigma}$ and $\boldsymbol{\sigma} = \alpha \mathbf{x}$ with $\alpha > 0$. Moreover, if the net is consistent, both properties are *equivalent*.

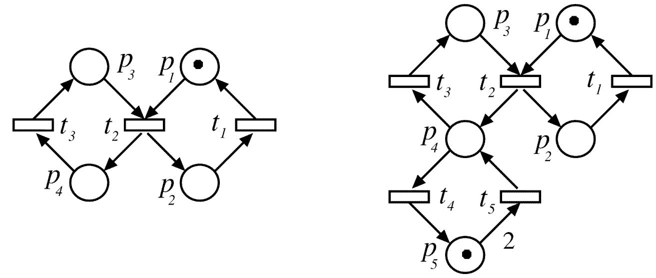

Fig. 4. Two continuous systems for which either not every transition is fireable (left) or the net is not consistent (right), and $\mathrm{RS}_C(\mathcal{N}, \mathbf{m}_0) \neq \mathrm{LRS}_C(\mathcal{N}, \mathbf{m}_0)$.

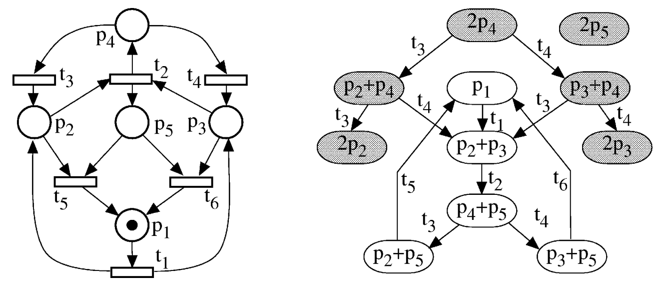

Fig. 5. A live system and its LRG.

In other words, if every transition is fireable, then every T-semiflow $\mathbf{x}$ ($\mathbf{x} \geq \mathbf{0}, \mathbf{C} \cdot \mathbf{x} = \mathbf{0}$) can be fired. This has an important consequence: *behavioral and structural synchronic relations [32] coincide*. For example, given the following linear programming problem:

$$B(p) = \max \{\mathbf{m}[p] \mid \mathbf{m} = \mathbf{m}_0 + \mathbf{C} \cdot \boldsymbol{\sigma}, \boldsymbol{\sigma} \geq \mathbf{0}\}$$

then, $B(p)$ is the *reachable* upper bound of $\mathbf{m}[p]$ in the continuous system.

A (structural) *trap* in a (discrete) PN is a set of places $\Theta \subseteq P$ such that $\Theta^\bullet \subseteq {}^\bullet\Theta$ (i.e., output transitions are also input transitions). A structural trap in a discrete PN fulfills an interesting behavioral property: If it is marked at $\mathbf{m}$, $\sum_{p \in \Theta} \mathbf{m}[p] > 0$, then for any successor it remains marked. This is the reason for its name, since at least a token remains "trapped." However, in continuous PNs, (structural) traps are not (behavioral) traps. For instance, Fig. 5, $\{p_1, p_2, p_3, p_4\}$ is a trap that is emptied by the firing of

$$t_1 t_2 \frac{1}{2} t_3 \frac{1}{2} t_4 \frac{1}{2} t_2 \ \frac{1}{4} t_3 \frac{1}{4} t_4 \frac{1}{4} t_2 \ \frac{1}{8} t_3 \frac{1}{8} t_4 \frac{1}{8} t_2 \ldots .$$

Property 5—[31]: A *trap* can eventually be emptied at the limit.

Let us define liveness and deadlock-freeness properties at the limit:

Definition 5: Let $\langle \mathcal{N}, \mathbf{m}_0 \rangle$ be a continuous PN system.

- $\langle \mathcal{N}, \mathbf{m}_0 \rangle$ lim-*deadlocks* if a marking $\mathbf{m} \in \lim -\mathrm{RS}_C(\mathcal{N}, \mathbf{m}_0)$ exists such that $\mathbf{e}(\mathbf{m})[t] = 0$ for every transition t.
- $\langle \mathcal{N}, \mathbf{m}_0 \rangle$ is lim-*live* if for every transition t and for any marking $\mathbf{m} \in \lim -\mathrm{RS}_C(\mathcal{N}, \mathbf{m}_0)$ a successor $\mathbf{m}'$ exists such that $\mathbf{e}(\mathbf{m}')[t] > 0$.

Under the weak conditions of Property 3, a simple necessary and sufficient condition for deadlock-freeness can be obtained for continuous PNs: a solution of the state equation exists in which no transition is enabled. See, for instance the PN in Fig. 6(b) It is deadlock-free if and only if the following (nonlinear) system has no solution:

$$\begin{aligned} \mathbf{m} = \mathbf{m}_0 + \mathbf{C} \cdot \boldsymbol{\sigma} &\geq \mathbf{0} \\ \min\left(\mathbf{m}\left[p_1\right], \mathbf{m}\left[p_4\right]\right) &= 0 \\ \left\{\text{or } \mathbf{m}\left[p_1\right] \cdot \mathbf{m}\left[p_4\right] \right. &= 0 \equiv \left.\mathbf{e}(\mathbf{m})[t1] = 0\right\} \\ \min\left(\mathbf{m}\left[p_1\right], \mathbf{m}\left[p_5\right]\right) &= 0 \\ \left\{\text{or } \mathbf{m}\left[p_1\right] \cdot \mathbf{m}\left[p_5\right] \right. &= 0 \equiv \left.\mathbf{e}(\mathbf{m})[t2] = 0\right\} \\ \min\left(\mathbf{m}\left[p_2\right], \mathbf{m}\left[p_3\right]\right) &= 0 \\ \left\{\text{or } \mathbf{m}\left[p_2\right] \cdot \mathbf{m}\left[p_3\right] \right. &= 0 \equiv \left.\mathbf{e}(\mathbf{m})[t3] = 0\right\}. \end{aligned}$$

To solve this system, numerical techniques can be applied. Another possibility is to define some boolean variables $\overline{\mu(i)} = 0 \Leftrightarrow \mathbf{m}[p_i] = 0$, and so a deadlock exists if and only if $(\overline{\mu(1)} + \overline{\mu(4)}) \cdot (\overline{\mu(1)} + \overline{\mu(5)}) \cdot (\overline{\mu(2)} + \overline{\mu(3)}) = 1$. Since $\mathbf{m}[p_4] + \mathbf{m}[p_5] = 1$, then $\mu(4) + \mu(5) = 1$, and the boolean expression can be simplified. Hence, the system deadlocks if and only if $\overline{\mu(1)} \cdot (\overline{\mu(2)} + \overline{\mu(3)}) = 1$. Let

$$\Sigma_1 = \begin{cases} \mathbf{m} = \mathbf{m}_0 + \mathbf{C} \cdot \boldsymbol{\sigma} \geq \mathbf{0}, & \boldsymbol{\sigma} \geq \mathbf{0} \\ \mathbf{m}\left[p_1\right] = \mathbf{m}\left[p_2\right] = 0 \end{cases}$$

$$\Sigma_2 = \begin{cases} \mathbf{m} = \mathbf{m}_0 + \mathbf{C} \cdot \boldsymbol{\sigma} \geq \mathbf{0}, & \boldsymbol{\sigma} \geq \mathbf{0} \\ \mathbf{m}\left[p_1\right] = \mathbf{m}\left[p_3\right] = 0. \end{cases}$$

Then, the continuous PN system in Fig. 6(b) does not deadlock if and only if Σ_1 and Σ_2 do not have a solution, and this happens when $k > 1$.

Finally, two conditions for lim-liveness, one necessary and one sufficient, can be obtained. Let us first introduce some definitions:

Two transitions t and t' are in (structural) conflict relation if ${}^\bullet t \cap {}^\bullet t' \neq \emptyset$. The structural conflict relation is not transitive; let the *coupled conflict* relation be its transitive closure. The quotient set that this closure defines is called Set of Coupled Conflict Sets (SCCS). When $\mathbf{Pre}[P, t] = \mathbf{Pre}[P, t'] \neq \mathbf{0}$, t and t' are in *equal conflict* (EQ) relation, meaning that they are both enabled whenever one is. By defining that a transition is always in EQ relation with itself, this is an *equivalence* relation on the set of transitions. The quotient set, i.e., the Set of all the Equal Conflict Sets, is denoted as (SEQS). An *equal conflict (EQ) net* is a PN in which all conflicts are equal, that is, SCCS $=$ SEQS (it is a weighted generalization of free choice [10]). As an example, in the PN in Fig. 6(a) SEQS $=$ SCCS $= \{\{t_1, t_2\}, \{t_3\}\}$, while in Fig. 6(b) SEQS $= \{\{t_1\}, \{t_2\}, \{t_3\}\}$ and SCCS $= \{\{t_1, t_2\}, \{t_3\}\}$. EQ systems allow the modeling of cooperation and competition relationships, but under some constraints.

A net is structurally bounded if it is bounded *for every* initial marking. It is structurally live if a marking exists for which the net system is live. For discrete PNs, necessary and sufficient conditions exist that relate (structural) liveness of a system with the rank of the incidence matrix [33].

- A live and bounded PN is consistent ($\exists \mathbf{x} > \mathbf{0} : \mathbf{C} \cdot \mathbf{x} = \mathbf{0}$), conservative ($\exists \mathbf{y} > \mathbf{0} : \mathbf{y} \cdot \mathbf{C} = \mathbf{0}$), and the rank of its incidence matrix is at most $|\text{SEQS}| - 1$.

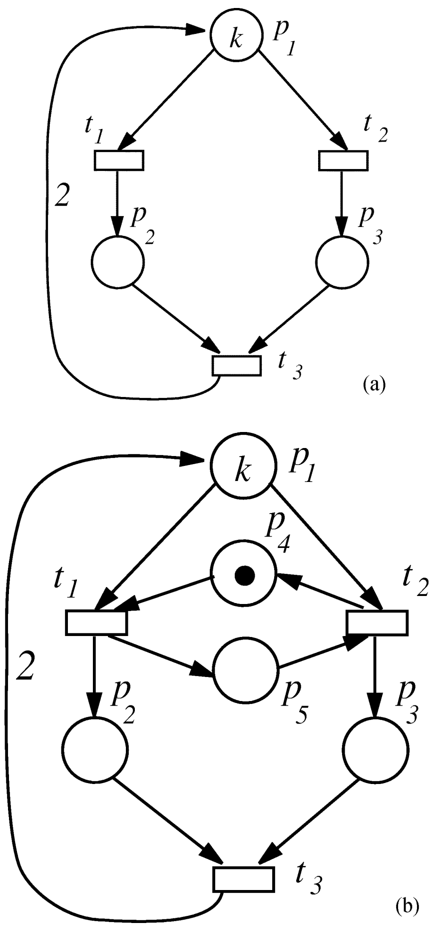

Fig. 6. A structurally nonlive PN (a) and a structurally live PN (b).

- If a net is consistent, conservative, and the rank of its incidence matrix is $|\text{SCCS}| - 1$ then it is structurally live. For EQ nets SEQS $=$ SCCS, and the condition is also necessary.

These *rank theorems* are of polynomial-time computational complexity, and can be extended to continuous systems.

Property 6—[31]: Let $\langle \mathcal{N}, \mathbf{m}_0 \rangle$ be a lim-live and bounded continuous PN system. Then $\mathcal{N}$ is structurally live and structurally bounded as discrete and so, it is consistent ($\exists \mathbf{x} > \mathbf{0} : \mathbf{C} \cdot \mathbf{x} = \mathbf{0}$), conservative ($\exists \mathbf{y} > \mathbf{0} : \mathbf{y} \cdot \mathbf{C} = \mathbf{0}$), and rank$(\mathbf{C}) \leq |\text{SEQS}| - 1$.

The previous property provides a bridge between the properties of nets, seen as discrete and continuous.

Property 7—[31]: If $\mathcal{N}$ is consistent ($\exists \mathbf{x} > \mathbf{0} : \mathbf{C} \cdot \mathbf{x} = \mathbf{0}$), conservative ($\exists \mathbf{y} > \mathbf{0} : \mathbf{y} \cdot \mathbf{C} = \mathbf{0}$), and rank$(\mathbf{C}) = |\text{SCCS}| - 1$, and $\mathbf{m}_0$ is such that no conservative component is empty ($\nexists \mathbf{y} \geq \mathbf{0} : \mathbf{y} \cdot \mathbf{C} = \mathbf{0}, \mathbf{y} \cdot \mathbf{m}_0 = 0$ and $\mathbf{y} \neq \mathbf{0}$), then $\langle \mathcal{N}, \mathbf{m}_0 \rangle$ is bounded and lim-live. Moreover, if we consider just EQ nets, the condition is both necessary and sufficient.

Observe that the property can be checked in polynomial-time. As an example, Fig. 6(a) shows an EQ net. It is consistent ($\mathbf{x} = [1, 1, 1]$), conservative ($\mathbf{y} = [1, 1, 1]$), rank$(\mathbf{C}) = 2$ and $|\text{SEQS}| = 2$, hence the continuous PN is not live and there exists no $\mathbf{m}_0$ for which it may be live. The PN in Fig. 6(b) is consistent ($\mathbf{x} = [1, 1, 1]$), conservative ($\mathbf{y} = [1, 1, 1, 1, 1]$), rank$(\mathbf{C}) = 2$, $|\text{SEQS}| = 3$, and $|\text{SCCS}| = 2$. Being a non- EQ net, for any (non-null) $\mathbf{m}_0$, we cannot conclude about liveness of $\langle \mathcal{N}, \mathbf{m}_0 \rangle$.

IV. Timed Continuous EQ Petri Nets: Infinite and Finite Server-Semantics

All along the previous discussion, the PN model, either seen as discrete or as continuous, has been *autonomous*, and in particular, did not contain any notion of time. A simple and interesting way to introduce time in discrete PNs is to assume that all the transitions are timed with an exponential probability distribution function (pdf). This way, a purely markovian performance model is obtained. Due to the memoryless property, the states of the underlying Markov chain are the markings of the autonomous PN. As a consequence, *the Markov chain of the stochastic system and the reachability graph of the autonomous PN are isomorphous* [34]. Hence, the timed model is *ergodic* (respectively, it is *irreducible*) if and only if, the autonomous model has *home states* (respectively, it is *reversible*) [15].

This interpretation is often enriched by the addition of *immediate* transitions with (marking and time independent) routing rates for the modeling of conflicts. This allows for the separation of routings (conflicts resolutions) and services (work at the stations). For the subset of immediate transitions $\{t_1, \ldots, t_k\} \subset T$ being in conflict at a reachable marking, constants $r_1, \ldots, r_k \in \mathbb{R}^+$ are explicitly defined and when $t_1, \ldots, t_k$ are simultaneously enabled, transition t_i fires with probability $r_i/(\sum_{j=1}^{k} r_j)$. All these rates define a routing matrix, $\mathbf{R}$.

Independently of the timing interpretation, since the marking represents the global state of the discrete stochastic system, the equation $\mathbf{m}(\tau) = \mathbf{m}(0) + \mathbf{C} \cdot \boldsymbol{\sigma}(\tau), \boldsymbol{\sigma}(\tau) \geq \mathbf{0}$ contains a lot of information (here it is made explicit that $\mathbf{m}$ and $\boldsymbol{\sigma}$ depend on τ, the actual time). If we interpret this equation as a description of the timed evolution of the continuous PN, and derive afterwards, we see that

$$\dot{\mathbf{m}}(\tau) = \mathbf{C} \cdot \dot{\boldsymbol{\sigma}}(\tau) \stackrel{\text{def}}{=} \mathbf{C} \cdot \mathbf{f}(\tau)$$

where $\mathbf{f}(\tau)$ is the flow obtained by firing the transitions. Hence, if $\mathbf{f}(\tau)$ is defined by an *interpretative extension*, the timed evolution of the continuous PN can be obtained. In the time extensions that we will study, $\mathbf{f}(\tau)$ will be purely deterministic. More precisely, we will consider a *first-order* approximation of the timed models, based on the mean of the pdf associated to the firing of the transitions. Since continuization makes sense for systems under heavy traffic, the first-order approximation can be quite reasonable (always being a relaxation!). This way, *from a discrete and stochastic model, an approximated continuous and deterministic model is obtained*.

In this paper, we will just consider the following case, whenever two immediate transitions are in conflict relation, it is an equal conflict relation. In particular, the weight of input and output arcs of immediate transitions must always be equal, hence, immediate transitions will be used essentially for routing. Let us also assume, for the moment, that *timed* transitions may never be in conflict. All timed transitions are persistent: a transition is said to be *persistent* if once enabled; the only way to disable it is firing it (it is structurally persistent if it is not in conflict with any other transition).

If the pdfs of persistent transitions are not exponential, but have a rational Laplace transform, using a Cox expansion we can construct a simulation with only exponential transitions. And, since places and transitions are local objects, macro-expansion techniques allow to express the simulation in PN terms [35]. Nevertheless, in a first-order continuization the expansion will not have effect in the permanent (or steady-state), but only in the transient behavior.

A different perspective for the approximation of stochastic PNs is the so-called fluid PN, used by K. Trivedi and his colleagues [36], [37]. The continuization, in this case, is *partial* (just one or a few places), but preserves part of the *stochastic* character of the original model. This approach, despite its interest, is not followed in this particular work since it leads to hybrid models. (Unfortunately, the denomination of fluid may lead to some misunderstanding, but the formalism is not completely fluid.)

A. Time and Semantics Interpretations for Continuous Petri Nets

Starting from discrete and markovian PNs, two particularly interesting semantics appear [38].

1) Infinite Server-Semantics: In this case, transitions are fired with: $\mathbf{f}(\tau)[t_i] = \boldsymbol{\lambda}[t_i] \cdot \mathbf{e}(\mathbf{m},\tau)[t_i]$, where $\mathbf{e}(\mathbf{m},\tau)[t_i] = \min_{p \in {}^\bullet t_i} \{(\mathbf{m}[p])/(\mathbf{Pre}[p,t_i])\}$ is the enabling degree of t_i, and $\boldsymbol{\lambda}[t_i]$ is the firing rate associated to t_i. That is, $\mathbf{e}(\mathbf{m},\tau)[t_i]$ represents the number of active *servers* in the station (transition) t_i, at instant τ.

2) Finite Server-Semantics: In discrete PNs, the constraint on the number of servers can be made evident by elementary self-loops around each transition t_i marked with k_{t_i} tokens, as many as the number of servers. The meaning of the "servers tokens" and the "client tokens" becomes very different now, since the latter represents large populations while the former count as units. This immediately suggests that the speed $\mathbf{f}(\tau)][t_i]$ has just an upper bound, $\boldsymbol{\lambda}[t_i]$ (k_{t_i} times the speed of a server, $\mathbf{F}[t_i]$). Then $\mathbf{f}(\tau)[t_i] \leq \boldsymbol{\lambda}[t_i] = k_{t_i} \cdot \mathbf{F}[t_i]$ (knowing that at least a transition will be in saturation, that is, its utilization will be equal to 1).

In continuous PNs terminology, infinite server-semantics is *variable speed*, while finite server-semantics is named *constant speed* (see for instance [3]). Notice that, under infinite server-semantics, the definition of $\mathbf{f}(\tau)$ is applicable also to PNs with conflicts among timed transitions. However, the above extension is not so simple in the case of finite server-semantics, because there is no clear rule about how the flow splitting should be done.

Let us consider the net system in Fig. 7. If the transitions are interpreted using infinite server-semantics, the flow vector $\mathbf{f}(\tau)$ ($\geq \mathbf{0}$ because $\boldsymbol{\lambda} \geq \mathbf{0}$ and $\mathbf{m}_0(\tau) \geq \mathbf{0}$) will be

$$\begin{aligned}
\mathbf{f}(\tau)[t_1] &= \frac{\boldsymbol{\lambda}[t_1] \cdot \mathbf{m}(\tau)[p_1]}{2} \\
\mathbf{f}(\tau)[t_2] &= \boldsymbol{\lambda}[t_2] \cdot \mathbf{m}(\tau)[p_2] \\
\mathbf{f}(\tau)[t_3] &= \boldsymbol{\lambda}[t_3] \cdot \mathbf{m}(\tau)[p_3] \\
\mathbf{f}(\tau)[t_4] &= \boldsymbol{\lambda}[t_4] \cdot \mathbf{m}(\tau)[p_4] \\
\mathbf{f}(\tau)[t_5] &= \boldsymbol{\lambda}[t_5] \cdot \min\left(\frac{\mathbf{m}(\tau)[p_5]}{4}, \mathbf{m}(\tau)[p_6]\right) \\
\mathbf{f}(\tau)[t_6] &= \frac{\boldsymbol{\lambda}[t_6] \cdot \mathbf{m}(\tau)[p_7]}{2}.
\end{aligned}$$

Observe that t_5 has an associated flow, $\mathbf{f}(\tau)[t_5]$, defined through a *minimum* (that represents the synchronization). If $\mathbf{f}(\tau)$ is introduced in the state equation, $\dot{\mathbf{m}}(\tau) = \mathbf{C} \cdot \mathbf{f}(\tau)$, the system dynamics is positive (non-negative strictly speaking) and piecewise linear, and may be described by a set (two in this case) of *switching positive systems of linear differential equations* (due to the "min"). At this point, it should be remarked that (discrete) PNs can also be provided with an alternative, interpretative time-extension based on *differential algebraic equations* [40]. In this case, switching systems of differential algebraic equations are obtained, and we have a *hybrid* formalism.

The same net system, under a finite server-semantics interpretation, with $\boldsymbol{\lambda} = \mathbf{1}$, has an initial flow vector:

$$\begin{aligned}
\mathbf{f}(0)[t_1] &= \mathbf{f}(0)[t_2] = \mathbf{f}(0)[t_4] = 1\\
\mathbf{f}(0)[t_3] &= \min\{1, \mathbf{f}(0)[t_6]\} = \frac{7}{8}\\
\mathbf{f}(0)[t_5] &= \min\left\{1, \frac{(2^*\mathbf{f}(0)[t_1] + \mathbf{f}(0)[t_2])}{4}\right\} = \frac{3}{4}\\
\mathbf{f}(0)[t_6] &= \min\left\{1, \frac{(\mathbf{f}(0)[t_4] + \mathbf{f}(0)[t_5])}{2}\right\} = \frac{7}{8}.
\end{aligned}$$

This flow is maintained until $\tau = 8$, when p_4 empties. The flow vector then changes to [1, 1, 3/4, 3/4, 3/4, 3/4] until $\tau = 12$, when p_2 empties. From then on the flow is equal to [1, 2/3, 2/3, 2/3, 2/3, 2/3].

Until now, we have only studied nets without *conflicts*. Let us again consider the PN in Fig. 6(a) and assume that t_1 and t_2 are immediate transitions with rates r_1 and r_2, respectively. For denotational simplicity, let us take an initial marking of [0, k, k]. It is interesting to notice, once more, that the approximation by continuization leads from a *discrete and stochastic* model to a *continuous and deterministic* one. Since the discrete PN is (structurally) bounded, the probability of getting into a deadlock state, (0, $2k$, 0) or (0, 0, $2k$), is "1" even if the mean time to reach one of them may be very long. The particular problem under consideration belongs to the classical "gambler's ruin problem" [41]. It can be seen that the mean number of firings of t_3 till one of these states is reached is: $(k \cdot |r_1^{2k} - (1-r_1)^{2k}|)/(|1-2r_1| \cdot (r_1^{2k} + (1-r_1)^{2k}))$. Therefore, $2k^2$ if $r_1 = r_2 = 1/2$, and approximately $k/|2r_1 - 1|$ if k is large and $r_1 \neq r_2$.

Observe that if $r_1 = r_2$ the mean time for deadlock is quadratic w.r.t. k, while it is approximately linear in other case. However, if $r_1 = r_2$ the discrete nonlive PN is transformed into a live continuous PN, which may be interpreted as a "very large" transient to deadlock.

A classical concept in queueing network theory is the visit ratio. The *visit ratio* of t_j with respect to t_i, $\mathbf{v}^{(i)}[t_j]$, is the mean number of times that t_j is visited (fired) per visit to t_i. Observe that $\mathbf{v}^{(i)}$ is a "normalization" of the flow vector in the steady state, i.e., $\mathbf{v}^{(i)} = \lim_{\tau\to\infty}(\mathbf{f}[t_j](\tau))/(\mathbf{f}[t_i](\tau))$. The visit ratio vector of a bounded and live system has to be a T-semiflow: $\mathbf{C} \cdot \mathbf{v}^{(i)} = \mathbf{0}$, $\mathbf{v}^{(i)} \geq \mathbf{0}$. Moreover, if it is an EQ net and all conflicts are solved using immediate transitions, $\mathbf{R} \cdot \mathbf{v}^{(i)} = \mathbf{0}$, where $\mathbf{R}$ is the routing matrix.

In summary, the following linear system has to be fulfilled:

$$\mathbf{C} \cdot \mathbf{v}^{(i)} = \mathbf{0},\ \mathbf{R} \cdot \mathbf{v}^{(i)} = \mathbf{0},\ \mathbf{v}^{(i)}[t_i] = 1.$$

The previous system has just one solution, which is independent of $\mathbf{m}_0$ and the service rates, for live and bounded equal conflict nets (it is a more general property, but it does not hold for arbitrary nets). This provides a different perspective of the so-called rank theorems (Property 6 and Property 7). From a historical perspective, it should be recognized that the computation of the visit ratio of free choice nets leads to the introduction of the first rank theorems [30].

B. Some Properties of Timed Continuous EQ Models

Let us concentrate on EQ net systems, and assume that conflicts are solved using immediate transitions and routing rates.

If the net system is seen as continuous, under infinite server-semantics we can write that

$$\dot{\mathbf{m}}(\tau) = \mathbf{C} \cdot \mathbf{f}(\tau)$$

with: $\mathbf{f}(\tau)[t_i] = \boldsymbol{\lambda}[t_i] \cdot \min_{p\in{}^\bullet t_i}\{\mathbf{m}[p]/\mathbf{Pre}[p, t_i]\}$ for the timed transitions, and $\mathbf{R} \cdot \mathbf{f}(\tau) = \mathbf{0}$ for the transitions being in conflict.

Then, the continuized model is a *set of switching systems of linear differential equations* with constant coefficients. Some elementary properties of arbitrary continuous net systems are:

1) $\mathbf{f}(\tau) \geq \mathbf{0}$ and $\mathbf{m}(\tau) \geq \mathbf{0}$, because $\mathbf{m}(0) \geq \mathbf{0}$. That is, the usual restriction on the non-negativity of the marking appears in a natural way with this timing interpretation.
2) Let $k \in \mathbb{N}$, if $\mathbf{m}'(0) = k \cdot \mathbf{m}(0)$, then $\mathbf{m}'(\tau) = k \cdot \mathbf{m}(\tau)$ and $\mathbf{f}'(\tau) = k \cdot \mathbf{f}(\tau)$ (that is, behavior is preserved under scaling of the initial marking).

For discrete EQ net systems, the following monotonicity properties also hold.

- Let $\boldsymbol{\chi}_k[t]$ be the throughput of t in the steady state, when the initial marking is $\mathbf{m}_k(0) = k \cdot \mathbf{m}(0)$. Then $(\boldsymbol{\chi}_k/k) \geq \boldsymbol{\chi}_1$. Intuitively, multiplying the resources in the system is more efficient than having several copies of the system run independently and parallel.
- The previous sequence is upper bounded by the throughput of the continuous system, i.e., $(\boldsymbol{\chi}_k/k) \leq \boldsymbol{\chi}_{\text{cont}}$.

We conjecture that stronger results hold (see Table I): for every k, $(\boldsymbol{\chi}_k/k) \leq (\boldsymbol{\chi}_{k+1})/(k+1)$, and $\lim_{k\to\infty}(\boldsymbol{\chi}_k/k) = \boldsymbol{\chi}_{\text{cont}}$, i.e., the throughput per copy of $\mathbf{m}_0$ is monotonically non-decreasing and tends to $\boldsymbol{\chi}_{\text{cont}}$.

For discrete EQ net systems (and in some other subclasses) an upper bound of the throughput can be obtained by means of a linear programming problem [30]. It can be seen that this bound is equal to the throughput in the continuous case (which can therefore be computed in polynomial time).

Let $\boldsymbol{\chi}_{\text{cont}}[t_i]$ and $\boldsymbol{\Gamma}_{\text{cont}}[t_i]$ be respectively the throughput and the cycle time of the continuous system in the steady state normalized for t_i (i.e., $\mathbf{v}^{(i)}[t_i] = 1$). Defining $\mathbf{D}^{(i)}$ as the vector of *average service demands*, i.e., $\mathbf{D}^{(i)}[t] = \mathbf{v}^{(i)}[t]/\boldsymbol{\lambda}[t]$, the following can be stated:

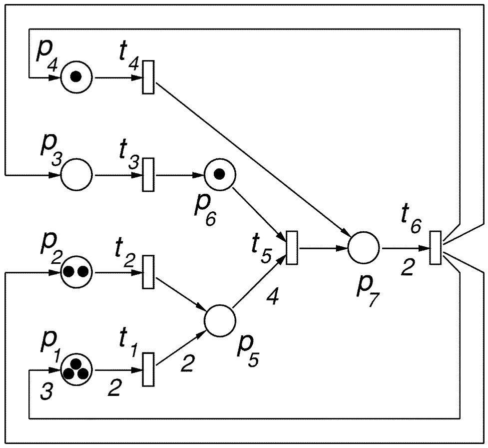

Fig. 7. Choice-free system modeling a manufacturing application [39].

TABLE I
THROUGHPUT OF t_6 IN THE NET IN FIG. 7 WITH AN INITIAL MARKING EQUAL TO $k \cdot \mathbf{m}_0$

k	Size of the reachability set	Infinite servers $\chi_I[t_6]$	$\chi_I[t_6]/k$	Single server $\chi_F[t_6]$
1	54	0.2522	0.2522	0.2401
2	1685	0.6493	0.3246	0.4757
3	10354	1.0567	0.3522	0.5805
4	37722	1.4623	0.3656	0.6309
5	103914	1.8671	0.3734	0.6563
...	...	...	...	...
∞			LPP$_\infty$=2/5	LPP$_F$=2/3

Property 8—[38]: Let $\langle \mathcal{N}, \mathbf{m}_0 \rangle$ be a live and bounded EQ continuous system with infinite server-semantics. The flow of the system in the steady state can be computed with the following linear programming problem:

$$\text{LPP}_\infty : \frac{1}{\boldsymbol{\chi}_{\text{cont}}[t_i]} = \boldsymbol{\Gamma}_{\text{cont}}[t_i] = \max\Big\{ \mathbf{y} \cdot \mathbf{Pre} \cdot \mathbf{D}^{(i)} \mid \mathbf{y} \cdot \mathbf{C} = \mathbf{0}, \; \mathbf{y} \cdot \mathbf{m}_0 = 1, \mathbf{y} \geq \mathbf{0} \Big\}.$$

For discrete PNs the solution of LPP$_\infty$ is, in general, just an upper bound of the throughput [30], [42]. This linear programming problem basically looks for the slowest isolated subnet among those generated by the elementary P-semiflows ($\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \cdot \mathbf{C} = \mathbf{0}$). In other words, the bound is obtained looking at the bottleneck P-semiflow. As an example, for the continuous model in Fig. 7 (Table I), $\boldsymbol{\chi}_{\text{cont}}[t_6] = 2/5$, and the bottleneck P-semiflow is [0, 0, 1, 1, 0, 1, 1].

It can be easily observed that if an empty P-semiflow exists ($\mathbf{y} \cdot \mathbf{C} = \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{y} \neq \mathbf{0}, \mathbf{y} \cdot \mathbf{m}_0 = 0$), $\boldsymbol{\Gamma}_{\text{cont}}[t_i]$ is infinite and $\boldsymbol{\chi}_{\text{cont}}[t_i]$ is zero, thus t_i is nonlive and the net system reaches a deadlock. (This condition is close to the one that appears in Property 7, asking for the existence of tokens in every P-semiflow.) Under liveness, the solution of LPP$_\infty$ is a finite value. Thus, its dual LPP provides the same value (strong duality theorem in LP):

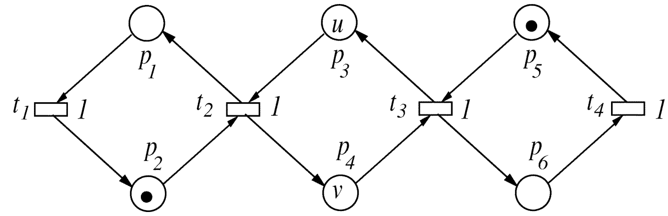

Fig. 8. In this live and bounded marked graph, with $\boldsymbol{\lambda} = \mathbf{1}$, the throughput on the steady state depends only on $k = u + v$. However, if $k > 1$ the marking in the steady state is not completely defined by the throughput vector. (Note: if $k < 1$, then $u = v$).

$$D - \text{LPP}_\infty = \frac{1}{\boldsymbol{\chi}_{\text{cont}}[t_i]} = \boldsymbol{\Gamma}_{\text{cont}}[t_i] = \min\Big\{ \gamma \mid \mathbf{C} \cdot \mathbf{z} + \gamma \cdot \mathbf{m}_0 \geq \mathbf{Pre} \cdot \mathbf{D}^{(i)} \Big\}.$$

LPP$_\infty$ (or $D -$ LPP$_\infty$) computes the cycle time ($\boldsymbol{\Gamma}_{\text{cont}}[t_i]$) for a given $\mathbf{m}_0$. If $\mathbf{m}_0$ is not defined, but is a decision variable constrained by some linear inequalities that represent bounds on resources ($\mathbf{W} \cdot \mathbf{m}_0 \leq \mathbf{k}$), replacing γ by $1/\alpha$, and adding the new constraints, the following property is derived:

Property 9: Let $\mathcal{N}$ be a structurally live and structurally bounded EQ net with infinite server-semantics. Given a set of constraints on the initial marking, $\mathbf{W} \cdot \mathbf{m}_0 \leq \mathbf{k}$, the optimal flow of the system in the steady state can be computed with the following linear programming problem:

$$\text{LPP}'_\infty = \boldsymbol{\Gamma}_{\text{cont}}[t_i] = \max\Big\{ \alpha \mid \mathbf{C} \cdot \mathbf{u} - \alpha \cdot \mathbf{Pre} \cdot \mathbf{D}^{(i)} + \mathbf{m_0} \geq \mathbf{0}, \; \mathbf{W} \cdot \mathbf{m}_0 \leq \mathbf{k}, \; \mathbf{m}_0 \geq \mathbf{0} \Big\}$$

where $\mathbf{u} = \alpha \cdot \mathbf{z}$.

As previously mentioned, in discrete systems, Coxian pdfs of persistent transitions can be simulated applying a local transformation, using exponential transitions. It can be proved that for continuous systems, the steady state in both cases is the same. Observe that the P-semiflows of the transformed system can be easily deduced from the P-semiflows of the original system, and that the product $\mathbf{y} \cdot \mathbf{Pre} \cdot \mathbf{D}^{(i)}$ does not change. Hence, the solution of LPP$_\infty$ in both cases, is the same.

Finally, it should be pointed out that the previous linear programming problems allows to obtain the throughput in steady state. This does not mean, however, that the steady state vector is completely defined (see Fig. 8).

Up until now, we have just considered the computation of the throughput of a continuous EQ net system under infinite server-semantics for a given $\mathbf{m}_0$, or the maximum throughput that can be obtained for any $\mathbf{m}_0$, subject to some constraints that can be stated as linear inequalities. We also obtain the initial marking that should be put in the net to get a certain throughput with the minimal cost $\mathbf{w} \cdot \mathbf{m}_0$ (a problem addressed in [19], [21] for discrete net systems). The computation turns out to be a linear programming problem, as well. In fact, it is based on $D -$ LPP$_\infty$:

Property 10—[38]: Let $\mathcal{N}$ be an EQ net with infinite server-semantics. Given a cost weighting vector $\mathbf{w}$ and a cycle time Γ, the following linear programming problem provides an initial marking to have a cycle time of Γ with minimal cost with respect to $\mathbf{w}$:

$$\textbf{Optimal cost}: \min\Big\{\mathbf{w}\cdot\mathbf{m}_0 \mid \mathbf{C}\cdot\mathbf{z}+\Gamma\cdot\mathbf{m}_0 \geq \mathbf{Pre}\cdot\mathbf{D}^{(i)}, \mathbf{m}_0 \geq \mathbf{0}\Big\}.$$

In a sense, these properties show that the continuization of a PN is a "strong" relaxation of its behavior; in fact, it may be "reasonable enough," only if the system is heavily loaded ("bottleneck analysis").

The throughput under *finite* server-semantics can also be obtained using a linear programming problem, which may be deduced from LPP_∞. Informally the idea is (1) the original initial marking is multiplied by a "large enough" constant to distinguish the "divisible" tokens from the "indivisible" servers, and (2) finite-server semantics is imposed by adding a self-loop place per transition, marked with as many tokens as servers. We have two kinds of P-semiflows now, the "original" ones, and the ones related to self-loops. Since the first kind of P-semiflows have a large amount of tokens, the "bottleneck" P-semiflow will be one of those associated to the self-loops. After some algebraic manipulations, the following statement is obtained:

Property 11—[38]: Let $\langle\mathcal{N},\mathbf{m}_0\rangle$ be a live and bounded EQ continuous system with finite server-semantics. The throughput of transition t_i in the steady state is

$$\mathrm{LPP}_F: \boldsymbol{\chi}_{\mathrm{cont}}[t_i] = \max\Big\{\alpha \mid \alpha\cdot\mathbf{v}^{(i)} \leq \boldsymbol{\lambda}\Big\}.$$

Observe that the solution does not depend on $\mathbf{m}_0$, but on the topology of the net, the routing, and the maximal rates of the transitions. It can be seen that the Coxian transformation does not make sense in the continuous system if a finite server-semantics is considered.

V. Continuous Petri Nets and System Modelling

Up to now, continuous PNs have been studied from a mathematical point of view, comparing the properties of the transformed continuous model with the ones of the original discrete model. In this section we will completely change the perspective, and continuous PNs will be considered from a modeling point of view. In Section V-A, we will observe that for continuization to make sense, it is necessary not only to have large populations, but that the individuals or items of the population are *indistinguishable* with respect to their behavior, the use of resources. In Section V-B, we will study the classical predator/prey problem from a PN perspective. It will be seen that, besides infinite and finite server-semantics, a transition firing rate defined as the product of the marking of the input places may appear in a natural way, if the model is obtained through decoloration. As a consequence, classical chaotic models can be expressed.

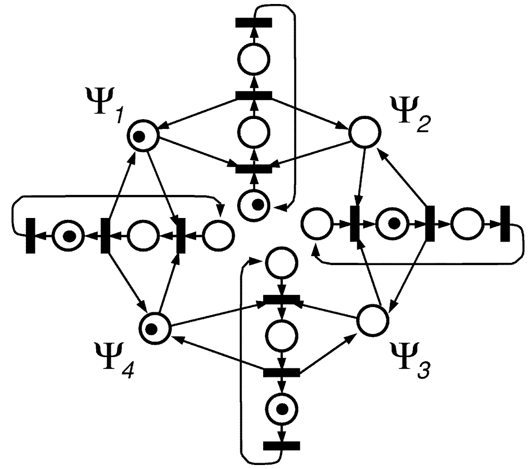

Fig. 9. Place/transition net system of the four philosophers problem. Left and right forks (Ψ_i) are taken at the same time.

A. Marking Scaling and "Reality"

In Section III-A, it was observed that the continuization of the marking of certain PN models could hide remarkable qualitative properties of the underlying discrete PN. Otherwise stated, it did not seem reasonable to continuize these net systems since the result could not be considered as an approximation. Furthermore, in Section IV-A we saw in Fig. 6(a), if $r_1 = r_2$ the timed continuous system will not deadlock, while a deadlock can always be observed if the system is seen as discrete. All this points out that practical *limits* exist for first-order continuization of stochastic discrete systems.

In this section, we show another kind of limitation. This one is not related to PNs as a formalism, but to the fact that multiplying the marking by a large number, in some cases, is not a way to *represent* a scale growth of the population. Fig. 9 shows a place/transition model of the classical philosophers problem, due to Dijkstra [43]. This is a well-known, extremely symmetric problem where sequential processes (the philosophers) share in a very particular way a set of resources (the forks). However, if the initial marking is multiplied by k, it does not represent the problem of 4k philosophers. To have a model of the 4k philosophers, the whole structure of the net has to be modified, and not only its marking. By not having the possibility of identifying big populations (set of individuals with identical properties with respect to the resources, i.e., the forks), there can be no reasonable hope for obtaining good results using continuization. Moreover, the discrete and deterministic timed model of 4 philosophers shows a risk of *monopoly* (or, as seen from the other side, *starvation*) [44], [45], while the continuous model is not able to detect this property. Notice, that this property is in strong relationship with the notion of disjunctive resources, and therefore cannot be tackled if the integrality constraint is relaxed.

In summary, marking scaling does not always represent "population scaling." Even if it were technically possible to continuize a PN (in the sense that its properties are preserved under scaling), it may be the case that the continuous model does not represent the underlying reality.

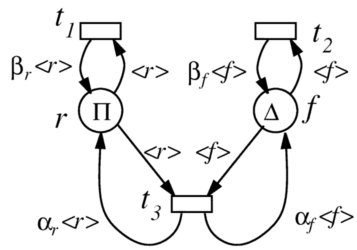

Fig. 10. Colored PN model of a predator/prey system.

B. Decoloration and Interpretation

In the previous sections, we have supported the idea that for continuization to make sense, populations of a "big" size are necessary. Therefore, it is interesting to consider "classical" problems of population dynamics. Among them, the predator/prey model of Volterra-Lotka [46] one of the simplest and most paradigmatic. This model describes the dynamics of two populations (foxes and rabbits, for example), with one predating the other. The population of predators increases with a growth rate dependent on the rate of predation. The rate of predation is proportional to the amount of prey and the amount of predators available to eat them. The predator population also declines due to natural mortality, at a rate proportional to population size. Preys reproduces at a rate proportional to their population and are removed only be predation.

The colored PN in Fig. 10 represents the problem using a discrete model. (More realistic hypothesis could be introduced in a simple way, but providing a more elaborated model is not our goal at this point). The use of colored PNs, in this case, is simply methodological to reveal the existence of individuals that can be grouped in homogenous populations. That means that the model has to be *decolored* (a technical aspect which is still quite open [47]) and the firing rates of the new transitions have to be obtained. To decolor means to aggregate indistinguishable elements (processes, objects, resources, etc.). Let $\mathbf{m}[f]$ and $\mathbf{m}[r]$ be the number of predators and preys. If we consider the colored transition t_3 at a certain instant, it is enabled in $\mathbf{m}[r] \cdot \mathbf{m}[f]$ differently colored ways. For this reason in the decolored (discrete) model (see Fig. 11(a)) t_3 has an associated firing rate equal to $\boldsymbol{\lambda}[t_3] \cdot \mathbf{m}[r] \cdot \mathbf{m}[f]$. This (discrete and stochastic) net system is nonbounded and nonlive. In fact, it has two absorbent states: in both of them $\mathbf{m}[f] = 0$, and either $\mathbf{m}[r] = 0$ or $\mathbf{m}[r] = \omega$ (ω is an arbitrarily large number). Only $\mathbf{m}[r] = \mathbf{m}[f] = 0$ is a steady state (this can be found in [48], [49]).

It is important to observe that with $\mathbf{m}[r] \cdot \mathbf{m}[f]$, the *product of variables* has been introduced as a rate. Now, if the model is considered as continuous, *the net system is not just a set of switching linear systems any more*. In this particular example, we have *one differential nonlinear system*. In general, we may obtain a set of *switching nonlinear systems*.

If the constants in Fig. 10 (death and birth rates) are defined as $\alpha_r = 0$, $\alpha_f = \alpha$, $\beta_r = 2$, $\beta_f = 0$, the equations that correspond to the continuous and decolored PN are the classical Volterra-Lotka equations [Fig. 11(a)]:

$$\dot{\mathbf{m}}[r] = \boldsymbol{\lambda}[t_1] \cdot \mathbf{m}[r] - \boldsymbol{\lambda}[t_3] \cdot \mathbf{m}[r] \cdot \mathbf{m}[f]$$
$$\dot{\mathbf{m}}[f] = -\boldsymbol{\lambda}[t_2] \cdot \mathbf{m}[f] + (\alpha - 1) \cdot \boldsymbol{\lambda}[t_3] \cdot \mathbf{m}[r] \cdot \mathbf{m}[f].$$

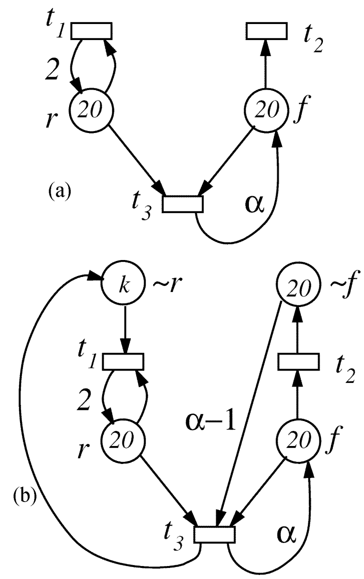

Fig. 11. Place/transition net systems that correspond to the colored PN in Fig. 10, with $\alpha_r = 0$, $\alpha_f = \alpha$, $\beta_r = 2$. $\beta_f = 0$),and $|\Pi| = |\Delta| = 20$. They can be seen either as discrete or as continuous PNs.

For $\dot{\mathbf{m}}[r] = \dot{\mathbf{m}}[f] = 0$ the two classical equilibrium solutions are found: $\mathbf{m}[r] = \mathbf{m}[f] = 0$ and $\mathbf{m}[r] = \boldsymbol{\lambda}[t_2]/(\boldsymbol{\lambda}[t_3](\alpha - 1))$, $\mathbf{m}[f] = \boldsymbol{\lambda}[t_1]/\boldsymbol{\lambda}[t_3]$. However, it must be noticed that according to this model, the system does not have equilibrium solutions, but oscillates in orbits defined by the initial populations. Some important remarks:

- Using *decoloration* a nonbounded, and nonlive colored PN system has been transformed into a nonbounded, and nonlive place/transition net system.
- Decoloration forced the introduction of a *product* as a transition rate at the timing interpretation level, instead of the enabling degree: that is, we have $\boldsymbol{\lambda}[t_3] \cdot \mathbf{m}[r] \cdot \mathbf{m}[f]$ instead of $\boldsymbol{\lambda}[t_3] \cdot \min\{\mathbf{m}[r], \mathbf{m}[f]\}$. Luckily, *the minimum is zero in the same situations than the product*, and negative flows are not possible (since $\mathbf{m}[r] \geq 0$ and $\mathbf{m}[f] \geq 0$).
- The existence of rates that are products of the makings may lead (and leads here) to *oscillatory behaviors*, that prevent the model to reach one of the equilibrium points.

The discrete PNs (colored or not) of our example are *stochastic nonbounded and nonlive* models. The continuous PN is a *deterministic, bounded and live* model! One could imagine that boundedness and liveness are due to a "certain equilibrium" between the nonboundedness and the deadlocks of the discrete system. To go deeper into this question, the discrete PN in Fig. 11(a) has been transformed into a bounded net system, just adding complementary places to $r(\sim r)$ and $f(\sim f)$ [see Fig. 11(b)]. If the initial markings of $\sim r$ and $\sim f$ are large enough, they will never pose a constraint to the evolution of the timed system. However, seen as discrete, this system is bounded and contains deadlocks. The underlying stochastic process will sooner or latter enter into one of the deadlocks ($\mathbf{m}[f] = 0$, with $\mathbf{m}[r] = 0$ or $\mathbf{m}[r] = k + 20$)!

Just as an exercise, Fig. 12 shows the trajectories for the case of having a maximal number of preys of $20 + k$. Since, for the given $\mathbf{m}_0$, the place $\sim f$ never restricts the enabling of t_3, the equation of f in the steady state is: $\boldsymbol{\lambda}[t_3] \cdot \mathbf{m}[r] \cdot \mathbf{m}[f] - \boldsymbol{\lambda}[t_2] \cdot \mathbf{m}[f] = 0$, and so: $\mathbf{m}[r] = \boldsymbol{\lambda}[t_2]/\boldsymbol{\lambda}[t_3] = 80/3$. This

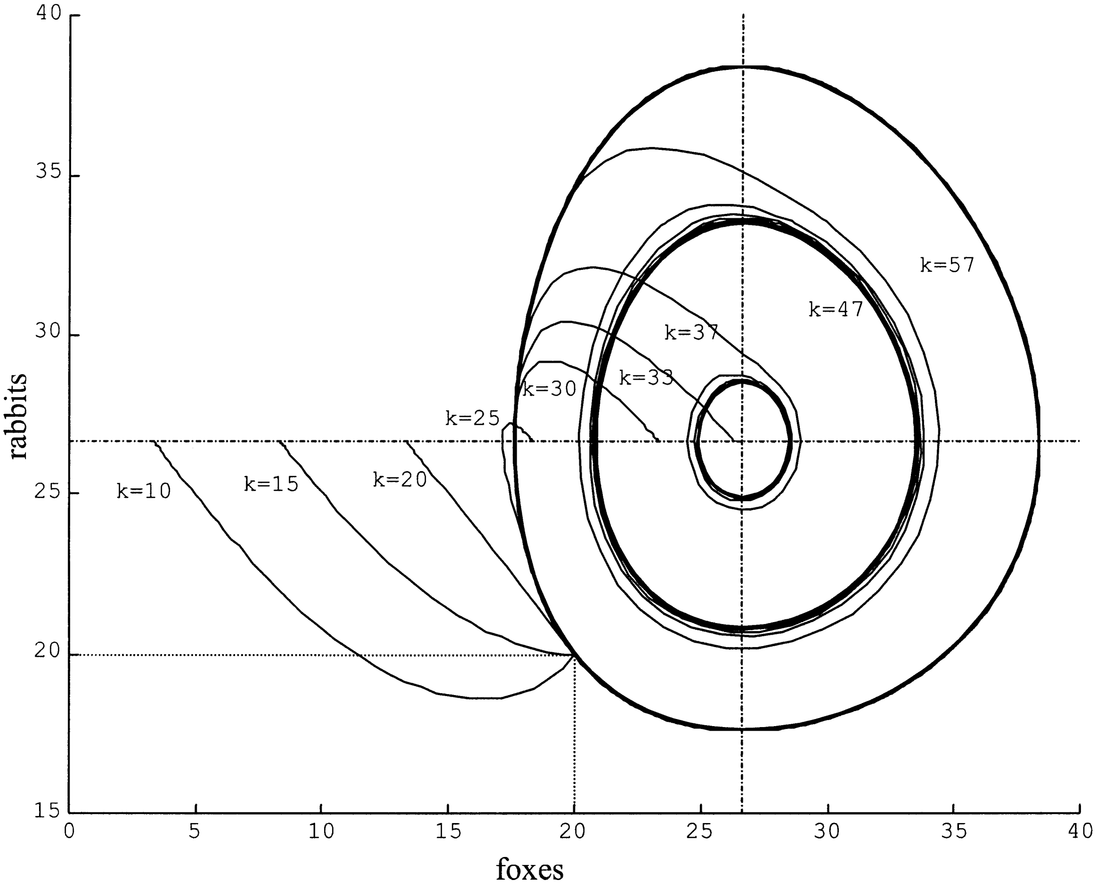

Fig. 12. Trajectories obtained with $\boldsymbol{\lambda}[t_1] = \boldsymbol{\lambda}[t_2] = 20, \boldsymbol{\lambda}[t_3] = 0.75,\ \alpha = 2, \mathbf{m}_0[r] = \mathbf{m}_0[f] = 20, \mathbf{m}_0[\sim f] = 40$ and $\mathbf{m}_0[\sim r] = k$.

value is effectively reached if k is small enough (for instance $k \leq 33$), in other cases the equilibrium state is never reached in the nonlinear system: its behavior is oscillatory but stable. The behavior of the system when k decreases shows that in a first transitory phase, the limitation has the effect of placing the system in an orbit closer to the non-null equilibrium point. From a critical value, the evolution does not lead the system to an orbit, but it directly goes to an equilibrium point, with $\mathbf{m}[r] = 80/3$ (obtained from the equation $\dot{\mathbf{m}}[f] = 0$). For $k = 20$, the main constraint for t_1 is always $\sim r$, hence the flow at t_1 is $20 \cdot \mathbf{m}[\sim r]$ which (by conservation) must be equal to the flow at t_2, $20 \cdot \mathbf{m}[f]$; and so $\mathbf{m}[f] = \mathbf{m}[\sim r]$. Therefore $\mathbf{m}[r] + \mathbf{m}[\sim r] = 20 + k = 40 \Rightarrow \mathbf{m}[\sim r] = 40 - 80/3 = 40/3$, and so $\mathbf{m}[f] = 40/3$. (In fact, for $k = 20$, $\dot{\mathbf{m}}[r] = -\dot{\mathbf{m}}[f]$, from which the straight line in the figure.)

Two final remarks:

- If the starting point is a colored net, it is necessary *first to decolor* (from which large populations can be obtained) and *later to continuize*. The reverse, first continuize and then decolor, does not make sense, especially considering that "addition" and "min" operations do not commute.
- We have seen that the introduction of the product may affect the firing speeds of the transitions. If functions that depend on the *global state* of the system are allowed in $\mathbf{f}(\tau)[t]$, *chaotic* behaviors (even the classical of Lorenz [50]) may be represented with continuous PNs. Since continuization is a relatively strong relaxation, the chaotic trajectories may not be *very representative*. Hence, it is possible that just some of their *qualitative* properties make sense (see [50], [51] for some reflections about this).

In summary, the diversity of behaviors that can be described with a PN is enormous. Even if we consider only first-order (deterministic) continuization, *some behaviors may appear which are difficult to comprehend in a quantitative way*. Furthermore, "the door" that communicates the problems and concerns of discrete event systems researchers and those of continuous nonlinear systems (and stochastic if a second-order approximation is used) researchers is just in its beginnings.

Finally, the predator/prey model admits a decoloration, even if some difficulties have appeared. More problematic is the net system in Fig. 13. It shows a colored model of the previously mentioned philosophers problem, a system that is extremely symmetric and parameterisable (when it is seen at the colored PN level). However, in spite of these nice properties, we do not know how to *decolor* it, because the resources (forks), even if they are always described with the same function $(r + l)$, concern different instances. Hence, it is not obvious how to obtain an uncolored but parameterisable population of philosophers, and so, continuization cannot be applied.

With regard to the relationship between continuous PNs and formalisms used to model continuous systems, some links with Forrester Diagrams have been observed (see [52] for details). *Forrester Diagrams (FD) (also called Dynamo Diagrams or Schematic Diagrams [53], [54])* are a tool for modeling population dynamics problems (in fact it is often used in such domains as socio-economic, biologic, ecologic, etc.) that has a graphical representation.

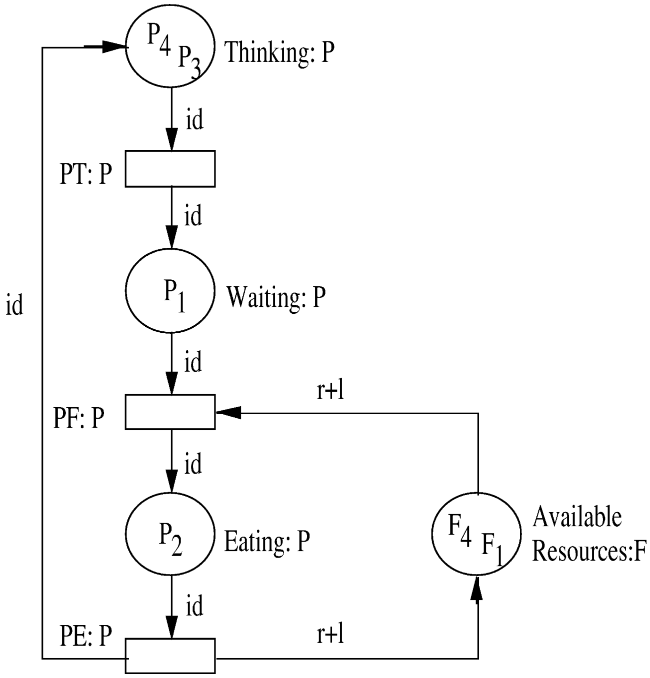

Fig. 13. Colored PN that corresponds to the basic strategy of taking the two resources at the same time.

VI. Conclusions

A common practice in many fields in which systems with large state-spaces appear, is to relax the description by removing integrality constraints. In PN systems, one approach to this idea has led to the definition of continuous PN systems. A distinctive feature in this work is that it focuses on the problems that may appear when continuizing (discrete) PNs.

Among the qualitative properties that we have observed that may change when continuizing a PN (Section III) are as follows.

- Deadlock-freeness: A deadlock-free discrete system may deadlock as continuous (and vice versa).
- Liveness: A live discrete system may be nonlive as continuous (and vice versa).
- Mutex relationships: They cannot be observed in continuous systems, since this property is based on the notion of disjunctive resources, which is lost in the continuous models. This clearly extends to monopoly and fairness situations, for example.
- Boundedness: Any bounded continuous system is also structurally bounded (assuming every transition can be fired). Since this property does not hold in discrete systems, there exist bounded discrete systems which are unbounded as continuous.

Nevertheless, fortunately some properties are preserved. For example, a live and bounded continuous system is structurally live and structurally bounded as discrete. The relationship between discrete and continuous systems is particularly strong for some subclasses. For example, in bounded equal conflict (EQ) systems structural liveness as discrete and as continuous are equivalent [31]. A clear parallelism can be observed between the continuization of (discrete) PNs and the linearization of nonlinear differential equation systems: not every system can be reasonably linearized, however it is well known that some subclasses of systems can be linearised without any problem.

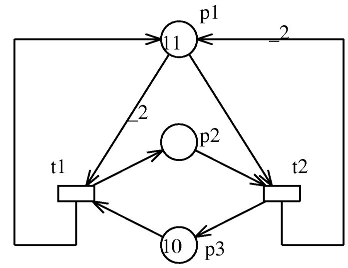

Fig. 14. If $\boldsymbol{\lambda}[t_1] = 2.25$ and $\boldsymbol{\lambda}[t_2] = 1$, the throughput as continuous is not an upper bound of the throughput as discrete.

Difficulties may also arise if a temporal interpretation is added to the system (Section IV). For example, in the classical Volterra-Lotka example, the nonpreservation of deadlocks and unboundedness of the discrete system is reflected in a very different temporal behavior of the discrete system (which is nonrepetitive) and the continuous one (orbits). On the other hand, under an infinite-server semantics timing interpretation, the steady-state throughput of a continuous EQ system is an upper bound of the discrete one (and can be obtained by means of a linear programming problem).

Section V changed the perspective and dealt with modeling. Conceptual problems have also appeared in the continuization of some systems. Technically speaking, the continuous system can be a good approximation of the discrete one when the tokens represent a large number of *indistinguishable* individuals/parts. In other words, it is not enough that they have a similar or symmetric behavior, like in the example of the philosophers.

So, as we conclude this work, here's some advice: take care when using a continuous approximation of a discrete system with cooperation and competition relationships, because it is not guaranteed that their behaviors are similar. Further work is needed before being able to say which kind of properties should be asked to a system in order to be continuizable. As mentioned in Section III-A and Section IV-B, at least marking scaling monotonicity is necessary. In some sense this last property means that the behavior of the continuous model does not depend on the "volumetric units" used to quantify the marking (liters, $m^3 \ldots$).

Another problem that remains open is how to return back from the solution of the continuous system to the solution of the discrete one. Other problems require further effort, for example, the development of decoloration techniques, and the study of the quality of the continuous approximation. This is particularly true because although, in many cases continuization is optimistic, it may also be a *pessimistic* approximation! For example, the throughput of the net in Fig. 14, with rates $\boldsymbol{\lambda}[t_1] = 2.25$ and $\boldsymbol{\lambda}[t_2] = 1$, is 2.53 if it is seen as discrete with exponential pdfs, while the throughput as continuous is 1.8. So, counter-intuitively, even nets with a single T-semiflow may show worse performance under continuization. Luckily, we know that continuization of EQ net models is optimistic. Sufficient or necessary conditions to guarantee optimistic (or pessimistic) approximations are also required.

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Manuel Silva received the Industrial-Chemical Eng. degree from the University of Sevilla, Sevilla, Spain, in 1974 and the postgraduate and Ph.D. degrees in control engineering from the Institut National Polytechnique de Grenoble, Grenoble, France, in 1975 and 1978, respectively.

From 1975 to 1978, he was with the Centre National de la Recherche Scientifique, Laboratoire d'Automatique de Grenoble. In 1978, he started the group of Systems Engineering and Computer Science at the University of Zaragoza, Zaragosa, Spain, where he was named Professor of systems engineering and automatic control in 1982. His research interests include modeling, validation, performance evaluation, and implementation of distributed concurrent systems using Petri nets, binary decision graphs, and robotic programming and control. He is author of the book *Las Redes de Petri en la Automática y la Informática* (AC, 1985) and coauthor of the book *Practice of Petri Nets in Manufacturing* (London, UK: Chapman & Hall, 1993), He was Dean of the Centro Politécnico Superior, University of Zaragoza, from 1986 to 1992 and President of the Aragonese Research Council (CONAI) and of the Research and Innovation Committee of the French–Spanish Comisión de Trabajo de los Pirineos (CTP) from 1993 to 1995. He was an associate editor of the *European Journal of Control.*

Dr. Silva is an advisory member of the *IEICE Transactions on Fundamentals on Electronics, Communications and Computer Sciences,* a member of the Steering Committee of the International Conference on Application and Theory of Petri Nets, a founder member of the Asociación Española de Robótica, and member of IFAC and EATCS. Interested in the history of technology and engineering, he received a medal from Lille and from the Association of Telecommunication Engineers of Aragón. He is a member of the National Academy of Engineering of Spain and was an associate editor of the IEEE TRANSACTIONS ON ROBOTICS AND AUTOMATION and co-editor of the special issue on CIM of that TRANSACTIONS.

Laura Recalde received the M.S. degree in mathematics and the Ph.D. degree in computer engineering from the University of Zaragoza, Zaragosa, Spain, in 1995 and 1998, respectively.

She is currently an Assistant Professor with the Departamento de Informática e Ingeniería de Sistemas, University of Zaragoza, where she is in charge of courses in systems theory and discrete event systems. Her main research interest is the modeling and analysis of distributed concurrent systems using Petri nets.