Unforced continuous Petri nets and positive systems

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Abstract. Petri nets (PNs) are a well-known family of formalisms whose definition immediately sets them, in a broad sense, as positive systems. Although they are originally discrete event models, their relaxation through continuization transforms them in continuous models. In this paper one of the most relevant timing interpretations of continuous PNs, unforced infinite servers semantics continuous PNs, is compared with linear positive systems and compartmental models.

1 Introduction

Continuization is one of the possible relaxations applicable to Petri Nets (PNs), a well known family of discrete event dynamic formalisms [6, 7]. This technique is particularly well suited to deal with heavy traffic (or highly populated) systems. Although not all PN systems allow such kind of approximate modelling [8], this relaxation is possible in many practical cases, leading to a continuous-time formalism that generalizes in several aspects other classical models of positive (continuous) systems theory. Among the structural aspects that differentiate PNs from other models, are the possible existence of attributions, choices, forks and joins, and the absence of "strict" (material) conservation rules. Moreover, timed interpretations lead to different firing/flow policies. This paper explores some of the differences and similarities among unforced infinite servers semantics continuous PNs, linear positive systems and compartmental models. The PN models under consideration lead to piecewise linear, state-homothetic behaviours. Not only superposition principles do not hold, but also many kind of non monotonous behaviours may appear. Nevertheless, monotonicity can be observed in some PN subclasses, while linearity asks for even more constrained net subclasses, namely join free systems (JF). The transformation of continuous JF nets into P-graphs allows, by some weights removing operations, to deal with directed graphs (DG), the underlying structure of classical compartmental models.

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Positive and compartmental systems will be introduced in Section 2. Section 3 presents basic PN concepts. Continuous PNs under infinite servers semantics will be studied in Section 4. This kind of models will be compared with compartmental systems in Section 5. In Section 6 the comparison will concentrate on the subclass of JF nets, which can be considered to be quite close to classical linear positive systems.

2 Positive and compartmental systems

In a broad sense, a system is said to be positive if none of its variables takes negative values. In a more restricted sense, according to Luenberger in [4], *a positive system automatically preserves the non-negativity of the state variables.* In other words, if non-negativity constraints on the state are added, they are redundant. More formally, let $\Sigma : \dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t)$ be a linear system.

Definition 1. *[3] Σ is said to be positive iff for every nonnegative initial state and for every nonnegative input its state is nonnegative .*

If $\mathbf{B} = \mathbf{0}$, the system is said to be *uncontrolled or unforced* (in automatic control it is also called autonomous; since autonomous has a quite different meaning for PNs [7], this term will not be used here). For the particular semantics that apply to positive systems, unforced means input-flow closed. A *closed* system is both input and output-flow closed.

A matrix (vector) is positive if all its elements are nonnegative. A square matrix is Metzler if non-diagonal elements are nonnegative.

Property 1. [3] A linear system $\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t)$ is positive, iff $\mathbf{A}$ is a Metzler matrix and $\mathbf{B} \geq \mathbf{0}$.

A particular case of positive systems are *compartmental systems*. Compartmental systems are composed of a finite number of subsystems (compartments), interacting by exchanging material among the compartments and with the environment [2, 11]. A compartmental system can be represented as a graph (with compartments as nodes) with an associated interpretation (*compartmental networks*). The level of each compartment, x_i, changes according to the input and output flows through the arcs, i.e., $\dot{x}_i = \sum_k f_{ki} - \sum_j f_{ij}$.

Inside compartmental systems generation of matter is forbidden. This means, in a system $\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t) + \mathbf{B} \cdot \mathbf{u}(t)$, that for every column of $\mathbf{A}$ the sum of its elements is non positive, i.e., $a_{ii} + \sum_{j \neq i} a_{ji} \leq 0$, or $\mathbf{1} \cdot \mathbf{A} \leq \mathbf{0}$. Hence, all the eigenvalues have a non-positive real part [3], and so the systems are either asymptotically or marginally stable. If it is an unforced system, then $\dot{\mathbf{x}}(t) = \mathbf{A} \cdot \mathbf{x}(t)$. Therefore, $\mathbf{1} \cdot \dot{\mathbf{x}}(t) \leq 0$, that is, $\mathbf{1} \cdot \mathbf{x}(t)$ is bounded. If it is a closed system, then $\mathbf{1} \cdot \mathbf{A} = \mathbf{0}$ (or equivalently, $\mathbf{A}$ is singular), and then $\mathbf{1} \cdot \mathbf{x}(t)$

is constant. That is, the system is strictly conservative. Otherwise, there are loses (evaporation, ...) in the system.

The flows can be defined according to different semantics [11]: (pure) *donor controlled* when f_{ij} depends only on x_i ($f_{ij} = a_{ij} \cdot x_i$ in the linear case), (pure) *recipient controlled* when f_{ij} depends only on x_j ($f_{ij} = b_{ij} \cdot x_j$ in the linear case), *donor and recipient controlled* if f_{ij} depends on both, x_i and x_j (for example, $f_{ij} = c_{ij} \cdot x_i \cdot x_j$), ... Pure recipient controlled systems are *not* positive systems according to Definition 1. In general any kind of control can be defined, i.e., the flows may depend (global function) on any set of variables. In a linear donor controlled system, $\mathbf{A}$ is a Metzler matrix. According to Property 1, for any non-negative initial state the variables are always non-negative, i.e., $\mathbf{x} \geq \mathbf{0}$ is a redundant constraint.

3 Basics of continuous Petri nets

A *(discrete)* PN *system* is a pair $\langle \mathcal{N}, \mathbf{m_0} \rangle$, where $\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post} \rangle$ is a PN net (P and T are disjoint (finite) sets of *places* and *transitions*, and $\mathbf{Pre}$ and $\mathbf{Post}$ are $|P| \times |T|$ sized, natural valued, *incidence matrices*), and $\mathbf{m_0}$ is the *initial marking* (a $|P|$ sized, non-negative, *integer* valued, vector). The PN structure has also a graphical interpretation as a bipartite graph in which places and transitions are represented as circles and bars, respectively.

In *continuous* PN *systems* the restriction on the integrality of firings is removed. Hence, the marking becomes a non-negative real number. In the sequel, unless otherwise stated, only continuous net systems will be considered.

For $v \in P \cup T$, the set of its input and output nodes are denoted as $^\bullet v$, and $v^\bullet$, respectively. A transition t is *enabled* at $\mathbf{m}$ iff for every $p \in {}^\bullet t$, $\mathbf{m}[p] > 0$. The *enabling degree* of a transition measures the maximal amount in which the transition can be fired in one go, i.e. $\mathbf{e}(\mathbf{m})[t] = \min_{p \in {}^\bullet t}\{\mathbf{m}[p]/\mathbf{Pre}[p,t]\}$. The firing of t in a certain amount $\alpha \leq \mathbf{e}(\mathbf{m})[t]$ leads to a new marking $\mathbf{m}' = \mathbf{m} + \alpha \cdot \mathbf{C}[P,t]$, where $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$, the token-flow matrix. This is denoted as $\mathbf{m} \xrightarrow{\alpha t} \mathbf{m}'$. A certain marking $\mathbf{m}'$ is *reachable* from $\mathbf{m}$ if a fireable sequence exists leading from $\mathbf{m}$ to $\mathbf{m}'$. If $\mathbf{m}$ is reachable from $\mathbf{m_0}$ through a sequence σ, a state (or fundamental) equation can be written: $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma} \in \mathbb{R}^{|T|}$ is a vector with the number of times each transition is fired.

Right and left natural annullers of the token-flow matrix are called T- and P-semiflows, respectively. When $\mathbf{y} \cdot \mathbf{C} = \mathbf{0}$, $\mathbf{y} > \mathbf{0}$ ($\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$) the net is said to be *conservative*(*consistent*). Some basic subclasses of nets are for example: *choice free* (CF) nets, each place has at most one output transition; *join free* (JF) nets, each transition has at most one input place; *P-graphs*, each transition has one input and one output place.

The formalism introduced up to this moment is autonomous (in the PN sense!): *which*, *when* and *how much* the enabled transitions will fire is not defined (in other words, the net model is *fully non-deterministic*). In particular, it has no notion of time. In order to introduce it, and looking for coherence, let us take *discrete* PNs as inspiration. A simple and interesting way to introduce time in discrete PNs is to assume that all the transitions are timed with an exponential probability distribution function [5]. For a basic timing interpretation of *continuous* PN systems a first order (or deterministic) approximation of the discrete case [8] can be used, assuming that the delays associated to the firing of transitions can be approximated by their mean values. Let $\langle \mathcal{N}, \mathbf{m_0}, \boldsymbol{\lambda} \rangle$ denote the timed net, with $\boldsymbol{\lambda} \in \mathbb{Q}^T$ the speeds of the transitions. Now the state equation has an explicit dependence on time $\mathbf{m}(\tau) = \mathbf{m_0} + \mathbf{C} \cdot \boldsymbol{\sigma}(\tau)$. Deriving with respect to time, $\dot{\mathbf{m}}(\tau) = \mathbf{C} \cdot \dot{\boldsymbol{\sigma}}(\tau)$ is obtained. Let us denote $\mathbf{f} = \dot{\boldsymbol{\sigma}}$, since it represents the *flow* through the transitions. Observe that in the steady state $\dot{\mathbf{m}}(\tau) = 0$, and so, from the state equation, $\mathbf{C} \cdot \mathbf{f} = \mathbf{0}$. Since $\mathbf{f} \geq \mathbf{0}$, the flow in the steady state is a T-semiflow.

Extrapolating from discrete markovian PNs [5], the most important semantics for the purpose of this work is *infinite servers* (*variable speed*) [1, 8]. In that case, assuming the net system is unforced, the flow through a transition t is $\mathbf{f}[t] = \boldsymbol{\lambda}[t] \cdot \mathbf{e}(\mathbf{m})[t] = \boldsymbol{\lambda}[t] \cdot \min_{p \in {}^\bullet t}\{\mathbf{m}[p]/\mathbf{Pre}[p,t]\}$, that is, the speed of a server times the number of "active servers". Putting all together:

$$\begin{array}{ll} \dot{\mathbf{m}} = \mathbf{C} \cdot \mathbf{f} & \\ \mathbf{f}[t] = \boldsymbol{\lambda}[t] \cdot \min_{p \in {}^\bullet t}\{\mathbf{m}[p]/\mathbf{Pre}[p,t]\} & \text{for every } t \in T \\ \mathbf{m}(0) = \mathbf{m_0} & \end{array} \tag{1}$$

or, equivalently,

$$\begin{array}{ll} \dot{\mathbf{m}} = \mathbf{C} \cdot \boldsymbol{\Lambda} \cdot \mathbf{e}(\mathbf{m}) & \\ \mathbf{e}(\mathbf{m})[t] = \min_{p \in {}^\bullet t}\{\mathbf{m}[p]/\mathbf{Pre}[p,t]\} & \text{for every } t \in T \\ \mathbf{m}(0) = \mathbf{m_0} & \end{array} \tag{2}$$

where $\boldsymbol{\Lambda} = \text{diag}(\boldsymbol{\lambda}[t])$ is a diagonal matrix with the rates of the transitions.

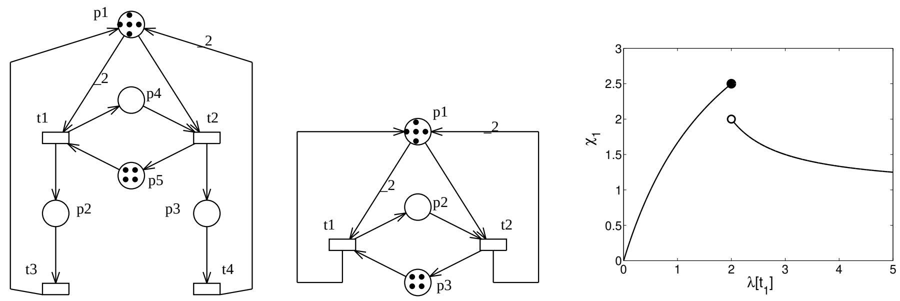
Observe that the above system non linear (due to the "min", its evolution is defined by a set of piecewise linear differential equations).

4 Some properties of unforced continuous Petri nets under infinite servers semantics

One of the characteristics of infinite servers semantics continuous PNs is that they are positive systems. There is no need of explicitly imposing it, since it is guaranteed by (1). Another characteristic is that, unlike the discrete models, continuous PNs preserve their properties under scaling of the marking.

Property 2. [8] Let $\langle \mathcal{N}, \mathbf{m_0}, \boldsymbol{\lambda} \rangle$ be a continuous infinite servers semantics PN.

- For every $\tau \geq 0$, it holds $\mathbf{f}(\tau) \geq \mathbf{0}$ and $\mathbf{m}(\tau) \geq \mathbf{0}$.
- For any marking $\mathbf{m}$ that can be reached in $\langle \mathcal{N}, \mathbf{m_0} \rangle$, $k \cdot \mathbf{m}$ can be reached in $\langle \mathcal{N}, k \cdot \mathbf{m_0} \rangle$. Moreover, for any timing $\boldsymbol{\lambda}$, if $\mathbf{f}(\tau)$ is the flow in $\langle \mathcal{N}, \mathbf{m_0}, \boldsymbol{\lambda} \rangle$, then $k \cdot \mathbf{f}(\tau)$ is the flow in $\langle \mathcal{N}, k \cdot \mathbf{m_0}, \boldsymbol{\lambda} \rangle$.



(a) (b) (c)

Fig. 1. (a) A net system whose throughput as continuous is not an upper bound for the throughput as discrete, with $\boldsymbol{\lambda} = [3, 1, 1, 10]$, (b), (c) For this net system, with $\boldsymbol{\lambda}[t_2] = 1$, increasing the rate of t_1 does not necessarily increase the throughput

Superposition properties do not hold, and quite intricate behaviours may appear. Even more, the flow of a continuous net system does not fulfill in general any *significant* monotonicity property, neither w.r.t. the *initial marking*, nor w.r.t. the *structure* of the net, nor w.r.t. the *transitions rates*.

For example, in the net system in Fig. 1 (a), if the marking of p_5 is augmented to 5, it deadlocks, i.e., the throughput goes down to 0. However, the throughput increases from 0.535 to 1.071 if $\mathbf{m_0}[p_5]$ is reduced to 3! Notice that this token (i.e., resource) reduction is equivalent to adding a place "parallel" to p_5 (i.e., with an input arc from t_2 and an output arc to t_1), marked with 3 tokens. Hence, adding constraints may increase the throughput.

Finally, an increase in a transition rate (for example, due to a replacement by a faster machine) may also lead to a decrease in the global throughput. In other words, local improvements do not necessarily lead to global improvements. For example, Fig. 1 (b) shows how the throughput of the net system in Fig. 1 (c) changes if the rate of t_1 varies from 0 to 5, assuming $\boldsymbol{\lambda}[t_2] = 1$. Notice that even a *discontinuity* (!) appears at $\boldsymbol{\lambda}[t_1] = 2$.

Hopefully, monotonicity properties do hold for some subclasses.

5 Continuous Petri nets vs. positive and compartmental systems: A structural point of view

An immediate similarity between PNs and compartmental systems is that both allow a representation based on graphs. However, PNs are bipartite graphs, while compartmental models have a single kind of nodes. In discrete PNs there are two kinds of nodes: OR nodes (attributions/choices) and AND nodes (joins/forks). Nevertheless, in continuous PNs with infinite servers semantics the forward OR node is transformed into a "+" operation: choices

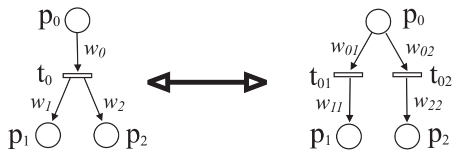


Fig. 2. In continuous PNs, under infinity servers semantics, forks can be transformed into equal choices and viceversa.

can be seen as flow splitters. Analogously, forks are also "+" operations, and can be immediately transformed into choices.

Let p_1 and p_2 be output places of t_0 (to simplify, let us assume there are just two output places), see Figure 2. Then, t_0 can be replaced by two transitions t_{01} and t_{02}, connected as follows: $w_{01} = w_{02} = w_0/2$, $w_{11} = w_1$, $w_{22} = w_2$, $\boldsymbol{\lambda}[t_{01}] = \boldsymbol{\lambda}[t_{02}] = \boldsymbol{\lambda}[t_0]/2$. Notice that although these arc weights may be rational, they can be transformed into integer numbers multiplying by two all the arc weights and the marking. Hence, a continuous net system can be transformed into an equivalent one *without forks*. By equivalent it is meant that they have the same markings (i.e., states) and flows, up to scaling factors or duplication of some variables. Similarly, choices (with the same input set) can be transformed into forks. Let t_{01} and t_{02} be the output transitions of p_0. These two transitions can be merged as follows: $w_0 = \dfrac{(\boldsymbol{\lambda}[t_{01}] + \boldsymbol{\lambda}[t_{02}]) \cdot w_{01} \cdot w_{02}}{\boldsymbol{\lambda}[t_{01}] \cdot w_{02} + \boldsymbol{\lambda}[t_{02}] \cdot w_{01}}$, $w_1 = \dfrac{w_{11} \cdot w_{02} \cdot \boldsymbol{\lambda}[t_{01}]}{\boldsymbol{\lambda}[t_{01}] \cdot w_{02} + \boldsymbol{\lambda}[t_{02}] \cdot w_{01}}$, define analogously w_2, and $\boldsymbol{\lambda}[t_0] = \boldsymbol{\lambda}[t_{01}] + \boldsymbol{\lambda}[t_{02}]$. Now, multiply the arcs and the initial marking by the adequate constant so that the arcs are integers.

Any linear compartmental system based on donor controller can be "naturally" simulated by means of a PN. However, to our knowledge there is no way to simulate a pure recipient controlled system with any of the policies that are defined in continuous PNs. A basic element in the evolution of PNs is that behaviour is of the consumption/production type, but that is not the idea in a recipient controlled system. Some mixed donor and recipient controls may appear in a natural way through *decoloration* (see [8]).

Another difference between PNs and compartmental systems is with respect to matter conservation. In compartmental systems there is a strong "strict" conservation law: matter is not created, although it may "evaporate" and disappear if the system is not (output) closed. In PNs such kind of constraint does not exist. However, conservativeness (i.e., $\mathbf{y} > \mathbf{0}$ exists such that $\mathbf{y} \cdot \mathbf{C} = \mathbf{0}$) is a similar property that is often required. From this and the state equation, $\mathbf{y} \cdot \dot{\mathbf{m}} = 0$ can be deduced and so, $\mathbf{y} \cdot \mathbf{m} = \mathbf{y} \cdot \mathbf{m_0} =$ constant. Since the tokens do not exactly represent a flow of matter and may change their meaning along the net, it guarantees not a strict matter conservation, but a weighted conservation. This is a structural property, and is independent of the transitions speeds. Conservativeness in nets is related to the notion of closeness in compartmental systems.

6 Join Free and Linear Positive Systems

Joins are associated to "min" functions and are at the base of important non-linearities of continuous PN systems. The PN subclass more closely related to linear positive systems is the class of (connected) join free (JF) nets. In that class all the transitions have just one input place, and hence the "minimum operator" problem disappears. Under infinite server semantics, the evolution of a JF system can be represented as a linear positive system:

$$\dot{\mathbf{m}} = \mathbf{C} \cdot \boldsymbol{\Lambda} \cdot \widetilde{\mathbf{Pre}} \cdot \mathbf{m}$$

where $\widetilde{\mathbf{Pre}}[t,p] = 1/\mathbf{Pre}[p,t]$ (thus, for ordinary (unweighted) nets, $\widetilde{\mathbf{Pre}} = \mathbf{Pre}^T$). Observe that $\mathbf{A}$ (in $\dot{\mathbf{x}} = \mathbf{A} \cdot \mathbf{x}$) is a node–to–node matrix (*adyacence*), while $\mathbf{C}$ is node–to–arc (*incidence*).

We may also assume that the nets are also choice free (CF), since choices can be transformed into forks (see Figure 2). Now, if the net is JF and CF, $|P| = |T|$, and so $\mathbf{Pre}, \mathbf{Post}, \mathbf{C}$ are square matrices. In that case, since $\mathbf{Pre}$ is diagonal, $\widetilde{\mathbf{Pre}}[t,p] = \mathbf{Pre}[p,t]^{-1}$, and so

$$\dot{\mathbf{m}} = \mathbf{C} \cdot \boldsymbol{\Lambda} \cdot \mathbf{Pre}^{-1} \cdot \mathbf{m}$$

Obviously, $\mathbf{C} \cdot \boldsymbol{\Lambda} \cdot \mathbf{Pre}^{-1} = (\mathbf{Post} \cdot \mathbf{Pre}^{-1} - \mathbf{I}) \cdot \boldsymbol{\Lambda}$ is a Metzler matrix. Nevertheless, this system is not necessarily a compartmental system because flow conservation is not guaranteed.

Let us assume that it is a conservative system, which as we said, is "close" to the matter conservation law of compartmental systems. Strongly connected and conservative JF PNs are also consistent, i.e., $\mathbf{x} > \mathbf{0}$ exists such that $\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$ [10]. Moreover, in JF and CF nets this annuller is unique but for a multiplying factor. If $\mathbf{m}_{ss}$ is a steady state marking, i.e., an equilibrium point, then $\dot{\mathbf{m}}_{ss} = \mathbf{0}$, and so $\mathbf{C} \cdot \mathbf{f} = \mathbf{C} \cdot \boldsymbol{\Lambda} \cdot \mathbf{Pre}^{-1} \cdot \mathbf{m}_{ss} = \mathbf{0}$. Then, the following property can be deduced.

Property 3. In JF and CF nets the equilibrium point is uniquely defined by: the token conservation law, $\mathbf{y} \cdot \mathbf{m}_{ss} = \mathbf{y} \cdot \mathbf{m_0}$, and $\mathbf{C} \cdot \mathbf{f} = \mathbf{0}$, $\mathbf{f} = \boldsymbol{\Lambda} \cdot \mathbf{Pre}^{-1} \cdot \mathbf{m}_{ss}$.

Another difference between PNs and the graphs associated to compartmental systems is the arc weights. However, this is not significant, since for any conservative JF net, an equivalent one exists with arc weights one.

Doing as in Figure 2, the JF net can be transformed so that each transition has just one input and one output place, i.e, it is transformed in a P-graph. Then, the following result can be applied.

Property 4. [9] Let $\langle \mathcal{N}, \mathbf{m_0}, \boldsymbol{\lambda} \rangle$ be a conservative P-graph, and define $\mathcal{N}' = \langle P', T', \mathbf{Pre}' \, \mathbf{Post}' \rangle$, an ordinary net with the same topological structure ($P' = P, T' = T, \mathbf{Pre}'[p,t] = 1$ iff $\mathbf{Pre}[p,t] > 0$ and $\mathbf{Post}'[p,t] = 1$ iff $\mathbf{Post}[p,t] > 0$). An initial marking $\mathbf{m_0}'$ exists such that $\langle \mathcal{N}, \mathbf{m_0}, \boldsymbol{\lambda} \rangle$ and $\langle \mathcal{N}', \mathbf{m_0}', \boldsymbol{\lambda} \rangle$ express equivalent behaviour under infinite servers semantics.

Therefore, strongly connected and conservative JF PNs are equivalent, from the modelling point of view, to closed linear compartmental systems.

7 Conclusions

In a broad sense, PNs are positive systems. In particular, under so called infinite servers semantics, continuous PNs are positive systems in the sense of Luenberger. In this paper continuous PNs are compared with compartmental systems. The structure in PNs is richer than the directed graphs of compartmental systems. This is reflected in two main aspects: the existence of synchronizations, which implies that the system is in general non-linear due to the "min" operators; and the arc weights.

It has been seen that PNs without synchronizations, i.e., JF subnets, can be transformed into P-graphs. Moreover, if the net is conservative the P-graph is equivalent to a state machine, i.e., a directed graph without arc weights. In other words, weights in JF models are a modelling convenience (i.e., do not add theoretical expressive power). However, that is not always the case with more general net subclasses, and there exist nets in which weights are not so easily removed.

On the other hand, there is a problem to represent recipient controlled compartmental systems in a "natural" way, since the common practice in PNs is to define the flow according to the enabling, i.e., the input places.

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