Hybrid Approximations of Markovian Petri Nets ⋆

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Abstract: A Markovian Petri net is a stochastic discrete event system (DES) frequently used for analysis and performance evaluation purposes. In the past, the fluidification has been proposed in different DES's as a relaxation technique for avoiding the "state explosion problem". Following the same approach, in this paper a hybrid Petri net model is defined as a *partial* relaxation of an original Markovian Petri net. It is shown through a simple example that such partial relaxation can be worse than a full relaxation (given by a fully continuous Petri net). Therefore, the rest of the paper is devoted to obtain sufficient conditions for guaranteeing the approximation of the hybrid Petri net model to the original discrete system.

Keywords: Petri-Nets, Hybrid, Markov models, Approximate analysis.

1. INTRODUCTION

Different authors have proposed hybrid models based on Petri nets (PN). Alla & David (1998) deal with fluid and hybrid Petri nets with constant and variable speeds, for which they have explored their modeling capabilities. The topic is revisited by Recalde & Silva, (2001). Following a different approach, Trivedi and his group introduced the so called Fluid Stochastic Petri Nets (Trivedi & Kulkarni (1993)): stochastically timed hybrid models, which are analyzed in their probability spaces. Alternative approaches are provided by Valette et al. (1998) and Demongodin & Koussoulas (1998), by adding continuous modeling capabilities to PN's. For instance, in the Valette's approach, differential equations associated to places are introduced (as a generalization to hybrid automata).

In this paper, an approach similar to that introduced by Alla and David is considered. In (Silva & Recalde (2004); Jiménez et al. (2004)) fluid and hybrid Petri nets are introduced as a relaxation of an original discrete PN, rather than considering them as models *per se*. In this way, the analysis of the relaxed version of the system can provide information about the original one, but avoiding the so called "state explosion problem", which frequently appears in discrete event systems. In particular, here we are interested in keeping quantitative information, meaning that the average marking of the relaxed model should approximate that of the original one. Moreover, in the relaxed model it appears an interesting advantage: techniques from both PN's and Control theories can be applied for analysis and design purposes (Vázquez, Ramírez, Recalde & Silva (2008); Mahulea et al. (2008)).

Under such approach, the approximation of Markovian Petri nets MPN (i.e., stochastic Petri nets under exponential services assumption) by the corresponding fully continuous PN was studied in (Vázquez, Recalde & Silva (2008)). In the present paper, those results are extended to partially relaxed models. The goal is to provide sufficiency rules for guaranteing the approximation of a MPN system by the corresponding hybrid relaxation.

This paper is structured as follows: In Section 2 some basic concepts on continuous and Markovian Petri nets are introduced. After that, results related to the approximation of timed continuous Petri nets to Markovian Petri nets are recalled in Section 3. The hybrid Petri net model under study is introduced in Section 4. The approximation to MPN by the hybrid relaxation is analyzed in Section 5. Finally, sufficient conditions for the approximation are provided as conclusions in Section 6.

2. BASIC CONCEPTS ON CONTINUOUS AND MARKOVIAN PETRI NETS

We assume that the reader is familiar with PN's (see for instance Silva (1993)). The structure $\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post} \rangle$ of *continuous Petri nets* ($ContPN$) is the same as the structure of discrete PNs. That is, P is a finite set of places, T is a finite set of transitions with $P \cap T = \emptyset$, **Pre** and **Post** are $|P| \times |T|$ sized, natural valued, *pre- and post- incidence matrices*. We assume that $\mathcal{N}$ is connected and that every place has a successor, i.e., $|p^\bullet| \geq 1$. The usual PN system, $\langle \mathcal{N}, \mathbf{M}_0 \rangle$ with $\mathbf{M}_0 \in \mathbb{N}^{|P|}$, will be said to be *discrete* so as to distinguish it from a *continuous* PN system $\langle \mathcal{N}, \mathbf{m}_0 \rangle$, in which $\mathbf{m}_0 \in \mathbb{R}^{|P|}_{\geq 0}$. In the following, the marking of a $ContPN$ will be denoted in lower case $\mathbf{m}$, while the marking of the corresponding *discrete* one will be denoted in upper case $\mathbf{M}$. The main

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difference between both formalisms is in the evolution rule, since in *continuous* PNs firing is not restricted to be done in integer amounts. As a consequence the marking is not forced to be integer. More precisely, a transition t is *enabled* at $\mathbf{m}$ iff for every $p \in^{\bullet} t$, $\mathbf{m}(p) > 0$, and its *enabling degree* is $enab(t, \mathbf{m}) = \min_{p\in^{\bullet}t}\{\mathbf{m}(p)/\mathbf{Pre}(p,t)\}$. The firing of t in a certain amount $\alpha \leq enab(t, \mathbf{m})$ leads to a new marking $\mathbf{m'} = \mathbf{m} + \alpha \cdot \mathbf{C}$, where $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$ is the token-flow matrix. As in discrete systems, right and left integer annullers of the token flow matrix are called *T-* and *P-flows*, respectively. When they are non-negative, they are called *T-* and *P-semiflows*. If there exists $\mathbf{y} > \mathbf{0}$ such that $\mathbf{y} \cdot \mathbf{C} = \mathbf{0}$, the net is said to be *conservative*, and if there exists $\mathbf{x} > \mathbf{0}$ such that $\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$ the net is said to be *consistent*. Here, we consider net systems whose initial marking marks all *P-semiflows*.

A *Markovian Stochastic Petri Net* system *(MPN)* is a *discrete* system in which the transitions fire at independent exponentially distributed random time delays, where conflicts are solved with a race policy. Then, the firing time of each transition is characterized by its firing rate. In this way, a MPN is a tuple $\langle \mathcal{N}, \mathbf{M}_0, \boldsymbol{\lambda}\rangle$, where $\boldsymbol{\lambda} \in \mathbb{R}^{|T|}_{>0}$ represents the transition rates. Transitions (like stations in queueing networks) are the meeting points of clients and servers. In this paper, we will assume infinite server semantics for all transitions. Then, the time to fire a transition t_i, at a given marking $\mathbf{M}$, is an exponentially distributed random variable with parameter $\lambda_i \cdot Enab(t_i, \mathbf{M})$, where the integer enabling degree is $Enab(t_i, \mathbf{M}) = \min_{p\in^{\bullet}t_i}\{\lfloor \mathbf{M}(p)/\mathbf{Pre}(p, t_i)\rfloor\}$. $Enab(t_i, \mathbf{M})$ also represents the number of active servers of t_i at marking $\mathbf{M}$. We suppose that a unique steady-state behavior exists, and we restrict our study to bounded in average and reversible (therefore ergodic) MPN systems.

3. TIMED CONTINUOUS PETRI NETS AS AN APPROXIMATION TO MPN

The approximation of a MPN by means of the corresponding *ContPN* was studied in (Vázquez, Recalde & Silva (2008)). There, the analysis is focused in the marking, rather than the throughput, since it constitutes an *state* representation of the system. The main ideas are recalled next.

3.1 Fundamental equation for Markovian Petri nets

Consider a MPN system with structure $\mathcal{N}$, timing rates $\boldsymbol{\lambda}$, and initial marking $\mathbf{M}_0$. Denote the initial time as τ_0 and consider a particular transition t_i. By definition, at any marking the time to fire each active server of t_i is characterized by a random variable (r.v.) having an exponential probability distribution function (p.d.f.) with parameter λ_i. Now, consider a fixed time interval $\Delta\tau$. If a server remains active during $\Delta\tau$ then the number of its firings (the number of jobs done) during $\Delta\tau$ is characterized by a r.v. having a Poisson p.d.f. with parameter $\lambda_i \cdot \Delta\tau$. Furthermore, since we are considering *infinite server semantics*, the number of firings of t_i during $\Delta\tau$ is the sum of the number of firings of each of its servers during this time interval. If $\Delta\tau$ is small enough then the number of active servers of t_i during this time interval remains almost constant. Therefore, the number of firings of t_i, during the time interval $(\tau_0, \tau_0+\Delta\tau)$, can be approximated by a r.v. $\Delta\boldsymbol{\sigma}_i(\Delta\mathbf{F}(t_i, \tau_0))$ having a Poisson p.d.f. with parameter $\Delta\mathbf{F}(t_i, \tau_0) = \Delta\tau \cdot \lambda_i \cdot Enab(t_i, \mathbf{M}_0)$, where $Enab(t_i, \mathbf{M}_0)$ is the number of active servers of t_i at $\mathbf{M}_0$ (the sum of independent Poisson distributed r.v.'s is also a Poisson distributed r.v., whose parameter is the sum of the parameters of the summands).

Now, considering the firing count vector $\Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(\tau_0))$, whose elements are the corresponding r.v.'s $\Delta\boldsymbol{\sigma}_i(\Delta\mathbf{F}(t_i, \tau_0))$ of each transition, the marking at time $\tau_0 + \Delta\tau$ can be approximated by using the fundamental equation, i.e.

$$\mathbf{M}(\tau_0 + \Delta\tau) \simeq \mathbf{M}_0 + \mathbf{C} \cdot \Delta\boldsymbol{\sigma}(\Delta\mathbf{F}(\tau_0))$$

which can be generalized as:

$$\mathbf{M}_{k+1} \simeq \mathbf{M}_k + \mathbf{C} \cdot \Delta\boldsymbol{\sigma}(\Delta\mathbf{F}_k) \quad (1)$$

where $\mathbf{M}_k$ and $\Delta\mathbf{F}_k$ denote $\mathbf{M}$ and $\Delta\mathbf{F}$ at time $\tau_0 + k\Delta\tau$, respectively. The parameters are given by $\Delta\mathbf{F}_k(t_i) = \Delta\tau \cdot \lambda_i \cdot Enab(t_i, \mathbf{M}(k))$. This equation constitutes a useful representation of the MPN.

3.2 Timed Continuous Petri nets

A *Timed Continuous Petri Net (TCPN)* is a *continuous* PN together with a vector $\boldsymbol{\lambda} \in \mathbb{R}^{|T|}_{>0}$. Different semantics have been defined for timed *continuous* transitions, the two most important being *infinite server* or *variable speed*, and *finite server* or *constant speed*. Here *infinite server semantics* will be considered. Like in purely Markovian *discrete* net models, under *infinite server semantics* the flow through a timed transition t_i is the product of the rate, λ_i, and $enab(t_i, \mathbf{m})$, the instantaneous enabling of the transition, i.e., $\mathbf{f}_i(\mathbf{m}) = \lambda_i \cdot enab(t_i, \mathbf{m}) = \lambda_i \cdot \min_{p\in^{\bullet}t_i}\{\mathbf{m}_p/\mathbf{Pre}(p, t_i)\}$. Observe that $Enab(t_i, \mathbf{M}) \in \mathbb{N}$ while $enab(t_i, \mathbf{m}) \in \mathbb{R}_{\geq 0}$. For the flow to be well defined, every transition must have at least one input place, hence in the following we will assume $\forall t \in T, |^{\bullet}t| \geq 1$. The "min" in the definition leads to the concept of *configurations*: a configuration assigns to each transition one place that, for some markings, will control its firing speed. An upper bound for the number of configurations is $\prod_{t\in T} |^{\bullet}t|$. The reachability space is divided into *regions* according to the *configurations*. These *regions* are polyhedrons (in bounded systems), and are disjoint, except on the borders.

The flow through the transitions can be written in a vectorial form as $\mathbf{f}(\mathbf{m}) = \boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbf{m}$, where $\boldsymbol{\Lambda}$ is a diagonal matrix whose elements are those of $\boldsymbol{\lambda}$, and $\boldsymbol{\Pi}(\mathbf{m})$ is the configuration operator matrix at $\mathbf{m}$, which is defined such that the i-th entry of the vector $\boldsymbol{\Pi}(\mathbf{m})\mathbf{m}$ is equal to the enabling degree of transition t_i (more details can be found, for instance, in Mahulea et al. (2008)). Therefore, the state equation of a $TCPN$ system, which is linear inside each *region*, is given by:

$$\dot{\mathbf{m}} = \mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbf{m} \quad (2)$$

3.3 Approximation of MPN by TCPN

In order to study the approximation of the MPN by means of the $TCPN$, in (Vázquez, Recalde & Silva (2008)) the continuous system was analyzed in discrete-time, obtaining the following difference equation:

$$\mathbf{m}_{k+1} \simeq \mathbf{m}_k + \mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m}_k)\mathbf{m}_k\Delta\tau \quad (3)$$

In that paper, it was proved that given $\mathbf{m}_0 = \mathbf{M}_0$, the marking of a $TCPN$ system $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$ (3) approximates the expected value of the marking of the MPN $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{M}_0 \rangle$ (1), during the time interval $(\tau_0, \tau_0 + n\Delta\tau)$, if the following conditions are fulfilled at $\mathbf{M}_k$ for any time step k in the interval $(\tau_0, \tau_0 + n\Delta\tau)$:
Condition 1) The probability that each transition of the MPN is enabled is near to one.
Condition 2) The probability that the marking is outside the *region* of $\mathbf{M}_0$ is near to zero.

Even if the quality of the approximation decreases when a change of *regions* occurs (i.e., Condition 2 does not hold during certain time) and/or the transitions are not enabled during certain time period (Condition 1), the approximation could be good enough for analysis and control purposes. Then, both Conditions should be consider just as sufficient for the mean value approximation.

In order to improve the approximation when Condition 2 does not hold, a noise column vector $\mathbf{v}_k$ is added to the flow of the $TCPN$ model, leading to a *Markovian continous* Petri net ($MCPN$). The noise vector has as entries independent normally distributed random variables with mean and covariance matrix:

$$E\{\mathbf{v}_k\} = \mathbf{0}, \qquad \boldsymbol{\Sigma}_{\mathbf{v}_k} = diag[\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m}_k)\mathbf{m}_k\Delta\tau] \tag{4}$$

Then the $MCPN$ model is defined as:

$$\mathbf{m}_{k+1} = \mathbf{m}_k + \mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m}_k)\mathbf{m}_k\Delta\tau + \mathbf{C}\mathbf{v}_k \tag{5}$$

By analyzing the moments of this system and applying the Central Limit Theorem, it was shown that the first two moments (mean value and covariance) of the marking of the $MCPN$ system (5) *approximate* those of the marking of the corresponding MPN (1) during a time interval $(\tau_0, \tau_0 + n\Delta\tau)$, if $\mathbf{m}_0 = \mathbf{M}_0$ and Condition 1 is fulfilled (i.e, Condition 2 is no longer required).

4. MARKOVIAN HYBRID PETRI NET MODEL

According to the results of the previous section, if some transitions are not enabled during all the time with a probability near to 1 (Condition 1) then significant errors may appear in the continuous approximation. In such case, it makes sense to fluidify only those transitions for which Condition 1 holds, obtaining thus a *hybrid* Petri net model.

Hybrid Petri nets were introduced by Alla & David (1998). There, the *discrete* part of the *hybrid PN* model is defined as a timed PN (i.e., with constant delays at the transitions), while the *continuous* part is a *continuous PN* with constant speed (finite server semantics). In order to be consistent with the MPN model, the *hybrid PN* system considered in this paper must include the random behavior of the MPN at the *discrete* transitions, and the infinite server semantics in the *continuous* part. Therefore, the following hybrid model is proposed as a *Markovian* timing for the autonomous hybrid PN already introduced in (Silva & Recalde (2004)).

Definition 1. A *Markovian hybrid* Petri net ($MHPN$), under infinite server semantics, is a tuple $\langle \mathcal{N}, \mathbf{M}_0, \boldsymbol{\lambda} \rangle$. $\mathcal{N}$ is the structure of the PN, in which the set of places P (transitions T) is partitioned into the set of continuous P^c (T^c) and discrete P^d (T^d) ones (i.e., $P = P^c \cup P^d$, $P^c \cap P^d = \emptyset$ and $T = T^c \cup T^d$, $T^c \cap T^d = \emptyset$). Since the fluidification is introduced through transitions, it is imposed in the model that fluid transitions only can have input or output fluid places, and each fluid place must have at least one input or output fluid transition, i.e., $P^c =^\bullet T^c \cup T^{c\bullet}$ (it is possible to make all the places fluid by fluidifying only some transitions). $\mathbf{M}_0 \in \mathbb{N}^{|P|}$ represents the initial marking, and $\boldsymbol{\lambda} \in \mathbb{R}^{|T|}_{>0}$ represents the transition rates. Each discrete transition $t_i \in T^d$ fires in discrete amounts with exponentially distributed random time delays, with parameter $\lambda_i \cdot Enab(t_i, \mathbf{M})$, as in the MPN model. Each continuous transition $t_i \in T^c$ fires with the flow $\mathbf{f}_i(\mathbf{m}) = \lambda_i \cdot enab(t_i, \mathbf{m})$, as in the $TCPN$ model.

Under this definition, the fundamental equation introduced in subsection 3.1 can be used for representing the behavior of the discrete part of the system (the firing of discrete transitions), while (3) can be used for describing the continuous behavior. Without loss of generality, let us suppose that the first columns of matrix $\mathbf{C}$ are related to the discrete transitions, while the last columns correspond to fluid ones. Then the $MHPN$ can be represented as:

$$\mathbf{M}_{k+1} \simeq \mathbf{M}_k + \begin{bmatrix} \mathbf{C}^d & \mathbf{C}^c \end{bmatrix} \cdot \begin{bmatrix} \Delta\boldsymbol{\sigma}(\Delta\mathbf{F}_k) \\ \boldsymbol{\Lambda}^c\boldsymbol{\Pi}(\mathbf{M}_k)\mathbf{M}_k\Delta\tau \end{bmatrix}$$

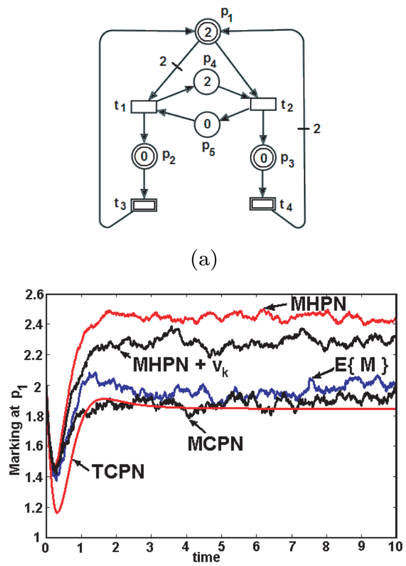
where $\mathbf{M}_k$ represents the whole marking and $\mathbf{C}^d$ ($\mathbf{C}^c$) represents the restriction of $\mathbf{C}$ to the discrete (continuous) transitions (i.e., $\mathbf{C} = [\mathbf{C}^d \quad \mathbf{C}^c]$). In the same way, the firing rate matrix $\boldsymbol{\Lambda}$ is divided into a matrix for the discrete transitions ($\boldsymbol{\Lambda}^d$) and other one for the continuous transitions ($\boldsymbol{\Lambda}^c$). The firing count vector $\boldsymbol{\sigma}(\Delta\mathbf{F}_k)$, having as elements random variables with Poisson p.d.f. with parameters $\Delta\mathbf{F}_k = \boldsymbol{\Lambda}^d \cdot Enab(\mathbf{M}_k) \cdot \Delta\tau$, is defined just for the discrete transitions, while the configuration matrix $\boldsymbol{\Pi}(\cdot)$ is defined just for fluid transitions.

Now, let us suppose that the first rows of the incidence matrix corresponds to the discrete places, while the last rows to fluid ones. Then, the marking can be represented as $\mathbf{M}_k = [\mu_k^T \quad \mathbf{m}_k^T]^T$, where μ_k ($\mathbf{m}_k$) corresponds to the marking of the discrete (fluid) places. In the same way, the incidence matrices can be written as $\mathbf{C}^d = [({}^d\mathbf{C}^d)^T \quad ({}^c\mathbf{C}^d)^T]^T$ and $\mathbf{C}^c = [({}^d\mathbf{C}^c)^T \quad ({}^c\mathbf{C}^c)^T]^T$, where ${}^d\mathbf{C}^d$ (${}^c\mathbf{C}^d$) represents the restriction of $\mathbf{C}^d$ to the discrete (continuous) places, and both ${}^d\mathbf{C}^c$ and ${}^c\mathbf{C}^c$ are defined in a similar way. However, since $P^d \cap ({}^\bullet T^c \cup T^{c\bullet}) = \emptyset$ then ${}^d\mathbf{C}^c = \mathbf{0}$. Therefore, the $MHPN$ can be rewritten as two different systems but connected:

$$\begin{aligned} \mu_{\mathbf{k+1}} &\simeq \mu_{\mathbf{k}} +^d \mathbf{C}^d \cdot \Delta\boldsymbol{\sigma}(\Delta\mathbf{F}_k) \\ \mathbf{m_{k+1}} &\simeq \mathbf{m_k} +^c \mathbf{C}^c \cdot \boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m}_k)\mathbf{m}_k\Delta\tau +^c \mathbf{C}^d \cdot \Delta\boldsymbol{\sigma}(\Delta\mathbf{F}_k) \end{aligned} \tag{6}$$

Notice that the flow of the fluid transitions only depends on the marking at the fluid places. On the contrary, the firing of discrete transitions depends on the marking of both discrete and fluid places, because the parameters of the Poisson random variables are $\Delta\mathbf{F}_k = \boldsymbol{\Lambda}^d \cdot Enab(\mathbf{M}_k) \cdot \Delta\tau$ (i.e., is a function of the whole marking $\mathbf{M}_k$).

In the system given by (6) discrete transitions fire with random delays, while the continuous ones are deterministic w.r.t. the fluid marking. However, it is possible to add uncorrelated gaussian noise to the continuous transitions in order to improve the approximation of the flow at these (as done in $TCPN$ model), obtaining the following system:



(a)

(b)

Fig. 1. a) A PN system with $\lambda = [1 \quad 3 \quad 1 \quad 2]^T$. b) Average marking trajectories of 1000 simulations. As a hybrid model, nodes t_1, t_2, p_4 and p_5 are discrete. $E\{\mathbf{M}\}$ corresponds to the original system MPN, while others represents the corresponding relaxations.

$$\begin{array}{ll} \boldsymbol{\mu}_{\mathbf{k+1}} & \simeq \boldsymbol{\mu}_{\mathbf{k}} +^d \mathbf{C}^d \cdot \Delta\boldsymbol{\sigma}(\Delta\mathbf{F}_k) \\ \mathbf{m}_{\mathbf{k+1}} & \simeq \mathbf{m}_{\mathbf{k}} +^c \mathbf{C}^c \cdot \boldsymbol{\Lambda\Pi}(\mathbf{m}_k)\mathbf{m}_k\Delta\tau +^c \mathbf{C}^c \cdot \mathbf{v}_k \\ & +^c\mathbf{C}^d \cdot \Delta\boldsymbol{\sigma}(\Delta\mathbf{F}_k) \end{array} \quad (7)$$

where $\mathbf{v}_k$ is defined for the fluid transitions as in (4). In the sequel, model (7) will be denoted as $MHPN + \mathbf{v}_k$.

5. APPROXIMATION OF THE MPN MODEL BY THE CORRESPONDING $MHPN$

The $MHPN$ model is defined as a partial relaxation of the MPN, so, one could think that the approximation provided by the *hybrid* system to the original *discrete* one should be better than that provided by the totally relaxed *continuous* model. However, that is not always the case.

Consider for instance the PN system of fig. 1(a) with rates $\lambda = [1 \quad 3 \quad 1 \quad 2]^T$. This PN was simulated 1000 times as a discrete, fluid and hybrid system, in order to obtain mean trajectories of the marking at p_1. As a hybrid model, nodes t_1, t_2, p_4, p_5 are discrete, while others are continuous. Fig. 1(b) shows the resulting mean trajectories. It can be seen that fluid models $TCPN$ (3) and $MCPN$ (5) provide a better approximation to the Markovian PN (denoted as $E\{\mathbf{M}\}$) than hybrid models $MHPN$ (6) and $MHPN + \mathbf{v}_k$ (7), i.e., *a partial relaxation can be worse than a full relaxation!* Let us analyze this in the following subsection.

5.1 Approximation analysis

The dynamical behavior of the MPN is achieved by the firing of its transitions, which is characterized by the firing count vector $\Delta\sigma(\Delta\mathbf{F}_k)$ in (1). In this way, if at some time step k, the average marking of the MPN is well approximated by the average marking of a given relaxed model (either fluid or hybrid) and their transitions fire in the same amount in both (the MPN and the relaxed model), then the marking approximation will hold for the next time step $k+1$. Therefore, following an inductive reasoning, if the initial condition of both systems coincide and the firing count vector of the relaxed model approximates that of the MPN system through the time, then the marking approximation is achieved (errors are not accumulated because, roughly speaking, the ergodicity of the MPN implies asymptotic stability in the relaxed model, i.e., early errors will not affect the long term behavior). However, it is important to remember that the firing count vector of the MPN is a random variable, then the corresponding firing count vector of the relaxed model should approximate the moments of the original one, i.e, mean value and covariance. Let us focus first in the mean value approximation through this subsection.

The mean value of the firing count vector of the MPN is approximated by the flow (but multiplied by $\Delta\tau$) at the continuous transitions in the relaxed fluid model. In (Vázquez, Recalde & Silva (2008)) it was found that such approximation is effective if Conditions 1 and 2 hold. A similar reasoning holds for the fluid transitions in the hybrid models, so no more analysis in these is required.

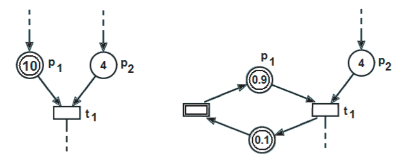
On the other hand, in the hybrid models, discrete transitions can have as input places either discrete or continuous ones. If discrete transitions have only discrete input places no problem occurs (there is no relaxation, then the approximation is perfect at these transitions). However, if a discrete transition has input continuous places then it can lead to a bad approximation, as in the case of the system of fig. 1(a). Now, consider the synchronization of fig. 2(a). Transition t_1 is a discrete transition having as input places $p_1 \in P^c$ and $p_2 \in P^d$. The expected number of firings of t_1 during a time interval $\Delta\tau$ is proportional to the expected value of its enabling degree, i.e., $E\{\Delta\sigma(\Delta\mathbf{F}(t_1))\} = \lambda_1 \cdot \Delta\tau \cdot E\{Enab(t_1)\}$, which can be computed by using the total probability theorem for the MPN as:

$$\begin{aligned} E\ \{Enab(t_1)\} &= \\ &= E\{min(\lfloor \mathbf{M}(p_1)/Pre(p_1,t_1)\rfloor, \lfloor \mathbf{M}(p_2)/Pre(p_2,t_1)\rfloor)\} \\ &= \sum_{S_{\mathbf{M}_2}} \sum_{S_{\mathbf{M}_1}} min(M_1,M_2) \cdot Prob(M_1|M_2)Prob(M_2) \end{aligned} \quad (8)$$

and for the hybrid Petri net as:

$$E\{Enab(t_1)\} = \sum_{S_{\mathbf{M}_2}} \int min(\lfloor x \rfloor, M_2) f_{1|2}(x)dx \cdot Prob(M_2) \quad (9)$$

where $S_{\mathbf{M}_1}$ ($S_{\mathbf{M}_2}$) denotes all the possible values for the marking at p_1 (p_2), and $f_{1|2}(\cdot)$ is the probability density function of the marking at fluid place p_1 given $\mathbf{M}(p_2) = M_2$ (M_1, M_2 denote fixed values for $\mathbf{M}(p_1), \mathbf{M}(p_2)$). Then, the approximation of the firing count vector is achieved if the fluid marking at p_1 is representative of the marking (the value of the marking, not the mean value of this) of the *original* MPN w.r.t. the enabling degree function, i.e., if the value of $\sum_{S_{\mathbf{M}_1}} min(M_1,M_2)Prob(M_1|M_2)$ in (8) is close to the value of $\int min(\lfloor x \rfloor, M_2)f_{1|2}(x)dx$ in (9). For instance, in the synchronization of fig. 2(a), markings at places p_1 and p_2 are random variables, but given the current marking, for the most probable values of p_1 and p_2, p_2 will constraint t_1, i.e., in (9) $\int min(\lfloor x \rfloor, M_2)f_{1|2}(x)dx \simeq M_2$. Following a similar reasoning, in the original MPN



(a) (b)

Fig. 2. a) Discrete transition in a $MHPN$ with a continuous and a discrete input places. b) t_1 only is enabled only at infinite time as a hybrid model.

the probability that p_1 constraints t_1 is negligible, i.e., in (8) $\sum_{S_{\mathbf{M}_1}} min(M_1, M_2) \cdot Prob(M_1|M_2) \simeq M_2$, so the enabling degree of t_1, and thus the firing count, is well approximated in the relaxed model. This case can be generalized as a sufficient condition for obtaining a good approximation, during a time interval $(\tau_0, \tau_0 + n\Delta\tau)$:
Condition 3) The probability that the discrete transitions be constrained by discrete places at $\mathbf{M}_k$, for any time step k in the interval $(\tau_0, \tau_0 + n\Delta\tau)$, is near to 1.

Condition 3 implies that the arcs between fluid places and discrete transitions are temporarily implicit (i.e., the discrete subnet evolves independently of the marking at fluid places). Furthermore, this condition can be generalized. Consider again the discrete transition of fig. 2(a). If at the current time step k the distribution of the marking at p_1 in the MPN were well approximated by the distribution of the fluid marking for the same place but in the $MHPN$, then the expected enabling degree of t_1 in both models would be similar. Such condition can be stated as:

$$Prob(M_1) \simeq \int_{M_1}^{M_1+1} f_1(x)dx \qquad \forall M_1 \in S_{\mathbf{M}_1} \quad (10)$$

In order to obtain such approximation, it is required that the marking at p_1 be large enough, i.e., that $S_{\mathbf{M}_1}$ consists in several probable values for the marking at p_1, so the marking approximation errors will be small w.r.t. their mean values. Furthermore, it is very important that p_1 at least enables t_1 with probability near to 1, otherwise a minimum error in the marking can lead to a big one in the firing count. For instance, consider the PN of fig. 2(b). Notice that as a hybrid model t_1 is enabled only at infinite time (only in infinite time the marking at p_1 is 1), but in the discrete MPN t_1 is enabled with a significant probability (since the mean value of the marking in the MPN is close to 0.9, the probability that $\mathbf{M}(p_1) = 1$ is significant). Then, Condition 3 is generalized as:

Condition 4) Discrete transitions can be constrained by either discrete or fluid places, but fluid places constraining discrete transitions enable them with probability near to 1 at $\mathbf{M}_k$, for any time step k in the interval $(\tau_0, \tau_0 + n\Delta\tau)$, i.e., such output discrete transitions are always enabled. The larger the marking at such fluid places, the better the approximation.

For instance, consider again the system of figure 1(a) with the same firing rates. The MPN and $MHPN$ systems have been simulated 1000 times for different initial markings at p_1, while the initial markings for the other places remain as in fig. 1(a). Table 1 resumes the results thus obtained. The first column represents the initial marking. Columns 2 and 3 are the expected values at the steady state of p_1 for the MPN and $MHPN$, respectively. Next column deals with to the approximation error, while the last column is the probability that p_5 constraints t_1 (i.e., Condition 3 for t_1). As expected, the error is lower when p_1 does not constraint t_1 (i.e., when $P.C3 \to 1$). On the other hand, in the last experiment an initial marking of $\mathbf{M}_0 = [10, 0, 0, 10, 0]^T$ was used. It can be seen that the approximation by the *hybrid* model is good, even if the probability that p_5, the discrete place, constraints t_1 is almost 0, i.e., p_1 constraints t_1 so Condition 3 is not fulfilled. However, in this last case the value of p_1 is large enough, which means that Condition 4 holds.

Table 1. Initial and steady state markings at p_1 for the MPN and $MHPN$ of fig. 1(a)

$\mathbf{M}_0(p_1)$	MPN	MHPN	error	P. C3
2	1.95	2.40	23.1%	0.42
3	2.54	2.74	7.87%	0.52
4	3.27	3.36	2.8%	0.65
5	4.04	4.02	0.5%	0.78
10	8.52	8.58	0.7%	0.98
$\mathbf{M}_0(p_1) = \mathbf{M}_0(p_4) = 10$	9.50	9.8	3.2%	0.01

5.2 Improvement of MHPN by adding noise

As recalled in subsection 3.3, the addition of noise $\mathbf{v}_k$ (4) to the fluid transitions in the $TCPN$ model improves the approximation of the firing count of the discrete transitions in the MPN, in particular when they represent synchronizations (Vázquez, Recalde & Silva (2008)). Moreover, the approximation is achieved not only at the mean value but also at the covariance. Following a similar reasoning, the addition of noise to the continuous transitions in the $MHPN$ model may improve the approximation to the original MPN, obtaining thus the $MHPN+\mathbf{v}_k$ model (7). Nevertheless, the difference between the approximation provided by the $MHPN$ model (6) and the hybrid model with noise (7) is not so important as in the case of fully continuous systems.

For instance, consider the system of fig. 3. As hybrid, transitions t_6, t_7 and places p_4, p_5, p_6, p_7, p_8 and p_9 are continuous, while other transitions and places are discrete. Consider rates as $\lambda = [30, 2, 30, 30, 30, 3.5, 10]^T$. The system was simulated 400 times as discrete (MPN), continuous ($TCPN$ and $MCPN$) and hybrid ($MHPN$ and $MHPN + \mathbf{v}_k$). The mean trajectories for the marking at place p_6 are shown in fig. 4(a). As it can be seen, the addition of the noise in the $MCPN$ model represents an important improvement to the approximation of the MPN (denoted as $E\{\mathbf{M}\}$) with respect to the $TCPN$ system. However, the improvement in the hybrid models (compare $MHPN$ against $MHPN + \mathbf{v}_k$) is not so important. The reason for that is the stochastic behavior of the discrete transitions in the hybrid model, i.e., the stochastic behavior of the firing of discrete transitions in the $MHPN$ makes the marking at continuous places be also stochastic, so it approximates not only the mean value but also the covariance (in certain degree) of the marking in the MPN.

However, the addition of noise in the hybrid model becomes important when discrete transitions are constrained by continuous places (i.e., Condition 3 does not hold) and the marking at those places is low (i.e., Condition 4 is

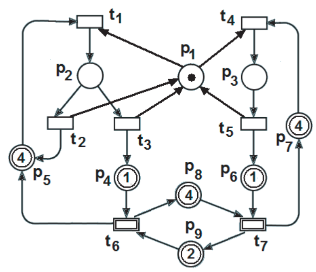
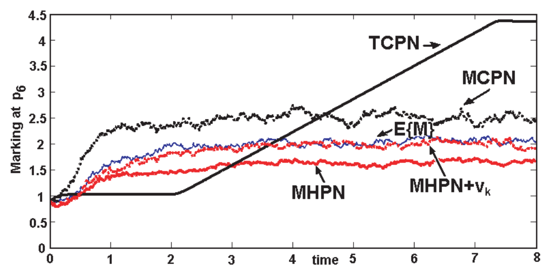
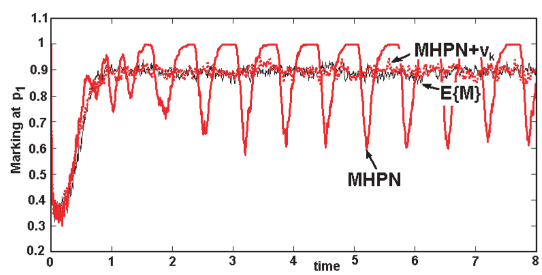


Fig. 3. As a hybrid PN, nodes $t_6, t_7, p_4, p_5, p_6, p_7, p_8, p_9$ are continuous while others are discrete.



(a)



(b)

Fig. 4. a) Expected marking at p_6 of the PN of fig. 3, with $\lambda = [30, 2, 30, 30, 30, 3.5, 10]^T$. b) Expected marking at p_1 with $\lambda = [30, 2, 30, 30, 30, 0.5, 0.5]^T$. In both figures, the average trajectories were obtained after 400 simulations. $E\{\mathbf{M}\}$ corresponds to the MPN, while other curves represent the corresponding relaxations.

barely fulfilled). For example, consider again the system of fig. 3 with $\lambda = [30, 2, 30, 30, 30, 0.5, 0.5]^T$. Fig. 4(b) shows the mean trajectory of 400 simulations, of the marking at p_1 for the discrete (MPN) and hybrid ($MHPN$ and $MHPN + \mathbf{v}_k$) systems (the periodic behavior of the $MHPN$ is explained by the low rates of t_6 and t_7 that increase the marking at p_5 and p_7 during a long deterministic period until enabling t_1 and t_4, which fire almost instantaneously w.r.t. the fluid dynamics). It can be seen that $MHPN + \mathbf{v}_k$ provides a better approximation to the MPN than $MHPN$. In this case, the noise added to the continuous transitions makes the p.d.f. of the continuous marking at p_5 and p_7 approximate better the p.d.f. of the corresponding markings in the MPN, because in this not only the mean value of the marking is being approximated but also the covariance, so (10) is closer to be fulfilled.

6. CONCLUSIONS

In this paper, a hybrid Petri net model $MHPN$ is introduced as a partial relaxation of a MPN. Such hybrid model is enriched by adding gaussian noise to the continuous transitions, in order to improve the approximation, obtaining thus another hybrid system $MHPN + \mathbf{v}_k$. It was found that in order to approximate a MPN by a hybrid relaxation, next conditions should be taken into account:

(1) All the fluid transitions should be enabled with probability near to 1 (Condition 1).
(2) Discrete transitions should be constrained by discrete places with probability near to 1, i.e., Condition 3. Otherwise, continuous places that constraint discrete transitions should enable such transitions with probability near to 1; the larger the marking at those fluid places, the better the approximation (Condition 4).
(3) If more than one place constraint fluid transitions that represent synchronizations, then gaussian noise (4) should be added to them for improving the approximation, obtaining thus a $MHPN + \mathbf{v}_k$ model.
(4) If Condition 3 does not hold for some discrete transitions, and Condition 4 is barely fulfilled, then the addition of the gaussian noise (4) becomes important for the approximation, i.e., the $MHPN + \mathbf{v}_k$ model should be considered.

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