Timing and Liveness in Continuous Petri nets

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Abstract

Fluidification constitutes a relaxation technique for studying discrete event systems through fluidified approximated models, thus avoiding the state explosion problem. Moreover, the class of continuous models thus obtained may be interesting in itself. In Petri nets, fluidification leads to the so called continuous Petri nets, that are technically hybrid models. Under infinite server semantics, timing a continuous Petri net model preserves the liveness property, but the converse is not necessarily true, and if the autonomous net model is not live, the timing may transform it into live. In this paper we investigate the conditions on the firing rates of timed continuous models that makes live a given continuous system.

Key words: Petri-nets; Continuous Systems; Deadlock; Safety analysis; Concurrent systems.

1 Introduction

In the literature, a lot of results related to the analysis and verification of liveness properties of untimed Discrete Event Systems (DES) can be found. Applications involve a wide range of systems including manufacturing, telecommunication, traffic and logistic systems. Nevertheless, the analysis and synthesis problems for such systems sometimes becomes untractable, due the computational complexity involved.

The fluidification constitutes a relaxation technique for studying discrete event systems through continuous approximated models, avoiding thus the state explosion problem and reducing the computational complexity of the analysis. Furthermore, by using fluid models, more analytical techniques can be used in order to study interesting properties. In Petri nets (PN), fluidification has been introduced from different perspectives ([2], [9]). In this work, timed continuous Petri net ($TCPN$) models under *infinite server semantics* are considered, since it has been found that such systems best approximate interesting classes of DES [6].

Regarding to *discrete* Petri nets, it is a well known fact that the addition of timing constraints to the firing of transitions (i.e., T-timing) may not preserve liveness or non-liveness. It is also in the folklore of the field that, for stochastic discrete models, these properties are preserved when the support of the stochastic functions associated to the firing of transitions is infinite. Let us study this a little more deeply by means of a couple of examples. The (discrete) net system in fig. 1(a), seen as autonomous (i.e., with no timing), is obviously live. Nevertheless, if we associate deterministic timings θ_1 and θ_3 to transitions t_1 and t_3, respectively (i.e., t_1 (t_3) fires after θ_1 (θ_3) time units of being enabled), with θ_3 smaller (thus faster) than θ_1, t_2 will never be enabled, thus cannot be fired, and non liveness follows. Considering now the net system in fig. 1(b), it is also immediate to asses that it is non-live as autonomous; nevertheless, if t_1 and t_2 are deterministically timed with $\theta_1 = \theta_2$, the system becomes live. Therefore, liveness of the discrete autonomous model is neither necessary, nor sufficient for that of the (at least partially) deterministically-timed interpreted model. Regarding deadlock-freeness, things are a bit simpler: if a (discrete) system is deadlock-free as autonomous it will be deadlock-free if it is T-timed.

If we consider now a classical Markovian timed interpretation (here all transitions have associated an exponential probability distribution function), then the Markov chain and the reachability graph are isomorphous [7]. Thus, any autonomous discrete net, and the result of timing it with arbitrary positive rates, are both simultaneously live or both equally non-live. Therefore, even if the net system in fig. 1(a) is non live for the mentioned deterministic timing, it is live for any positive exponential timing; moreover, the net system in fig. 1(b) is live

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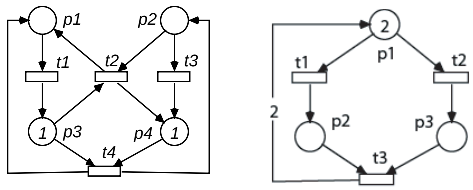


(a) (b)

Fig. 1. (a) Live as autonomous *discrete* net system but non-live under certain deterministic timing: $\theta_1 > \theta_3$. (b) Non live as autonomous, but live as timed if $\theta_1 = \theta_2$.

for the deterministic timing, but non live for any Markovian case, even if the rates of t_1 and t_2 are equal.

In this paper liveness is studied for the *timed continuous* model ($TCPN$) under infinite server semantics. The results in this context are of two types: If the *continuous* Petri net (*contPN*) is already live, it will remain live for any infinite server semantics interpretation (already advanced for particular net subclasses in [3]), and the new contribution: if the autonomous *contPN* is non-live, particular numerical timings of the *continuous* model can eventually transform it into live. The results hold even for deadlock-free marking non-monotonic systems (i.e., systems that being deadlock-free, run into a deadlock if the initial marking is increased). Non-liveness is associated to net *siphons* that are emptied. Intuitively speaking, the creation of some token conservations laws around *siphons* avoids them to become empty.

The analysis achieved here improves the understanding of the relation between the structure, the timing and the dynamical behavior of *contPN*s, useful in future studies about performance and control. From a practical point of view, the knowledge of a timing that makes a system live is interesting during its design, since this property is frequently desired. This kind of analysis is classical in discrete Petri nets, where the performance evaluation of stochastically timed models is frequently used, for example, in parametric optimization in order to decide the best rates for the transitions, define conflict policies, or allocate resources in different kind of systems. Furthermore, a timing that induces liveness in a continuous PN can be interpreted as a control action applied to the net system with a different nominal timing. This is already advanced in the last section of this paper.

This work is an extension of a very preliminary one [12], in which deadlock-freeness was studied for the $TCPN$ model. Here, those results are extended to liveness. After introducing some basic concepts in section 2, the problem considered in this work is presented in section 3, i.e., the study of the relation between the timing and liveness in $TCPN$ systems. Later, the existence of non-live potential steady states is studied from an algebraic perspective in section 4, while a sufficient condition for avoiding such states is introduced in section 5. In section 6, some classical stability results from the linear systems theory are applied to $TCPN$ systems in order to decide if a non-live potential steady state can be reached. Some examples are shown in Section 7. Finally, a timing that makes the system to avoid a non-live potential steady state is interpreted as a control action in section 8.

2 Basic Concepts and Notation

We assume that the reader is familiar with Petri nets (for notation see [10]). The set of the input (output) nodes of v is denoted as ${}^\bullet v$ ($v^\bullet$). The structure $\mathcal{N} = \langle P, T, \mathbf{Pre}, \mathbf{Post} \rangle$ of *continuous Petri nets* (*contPN*) is the same as the structure of *discrete PN*s. That is, P is a finite set of places, T is a finite set of transitions with $P \cap T = \emptyset$, $\mathbf{Pre}$ and $\mathbf{Post}$ are $|P| \times |T|$ sized, natural valued, *pre- and post- incidence matrices.* We assume that $\mathcal{N}$ is connected and that every place has a successor, i.e., $|p^\bullet| \geq 1$. The usual PN system, $\langle \mathcal{N}, \mathbf{m}_0 \rangle$, will be said to be *discrete* so as to distinguish it from a *continuous* PN system, in which $\mathbf{m} \in \mathbb{R}_{\geq 0}^{|P|}$. The main difference between both formalisms is in the evolution rule, since in *contPN*s firings are not restricted to be done in integer amounts [11]. As a consequence the marking is not forced to be integer.

A transition t is *enabled* at $\mathbf{m}$ iff for every $p \in {}^\bullet t$, $\mathbf{m}(p) > 0$, and its *enabling degree* is $\text{enab}(t, \mathbf{m}) = \min_{p \in {}^\bullet t}\{\mathbf{m}(p)/\mathbf{Pre}(p,t)\}$. The firing of t in a certain amount $\alpha \leq \text{enab}(t, \mathbf{m})$ leads to a new marking $\mathbf{m}' = \mathbf{m} + \alpha \cdot \mathbf{C}(t)$, where $\mathbf{C} = \mathbf{Post} - \mathbf{Pre}$ is the token-flow matrix and $\mathbf{C}(t)$ represents the column of $\mathbf{C}$ related to transition t.

As in discrete systems, a column vector $\mathbf{y}$ s.t. $\mathbf{y}^T \cdot \mathbf{C} = \mathbf{0}$ ($\mathbf{x}$ s.t. $\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$) is called *P-flow* (*T-flow*). When they are nonnegative, they are called *P-* and *T-semiflows.* Here, we always consider net systems whose initial marking $\mathbf{m}_0$ marks all P-semiflows. Matrix $\mathbf{B}_y$ denotes a basis of $P - flows$. If $\exists \mathbf{y} > \mathbf{0}$ s.t. $\mathbf{y}^T \cdot \mathbf{C} = \mathbf{0}$, the net is said to be *conservative*, and if $\exists \mathbf{x} > \mathbf{0}$ s.t. $\mathbf{C} \cdot \mathbf{x} = \mathbf{0}$ the net is said to be *consistent.* A set of places Σ is a *siphon* iff ${}^\bullet\Sigma \subseteq \Sigma^\bullet$ (the set of input transitions is included in the corresponding output one), and it is *minimal* if it does not contain another *siphon.* For example, in the net in fig. 2(a), $\{p_4, p_5, p_6\}$ defines a *minimal siphon*, while $\{p_3, p_4, p_5, p_6\}$ is also a *siphon*, but non *minimal.*

A continuous PN system is called deadlock-free if for every reachable marking $\exists t \in T$ s.t. $\text{enab}(t, \mathbf{m}) > 0$ (a marking reached in the limit of an infinitely long firing sequence is considered reachable [3]).

2.1 Timed Continuous Petri Nets

A *Timed Continuous Petri Net* ($TCPN$) is a *continuous PN* together with a vector $\boldsymbol{\lambda} \in \mathbb{R}_{>0}^{|T|}$. Different se-

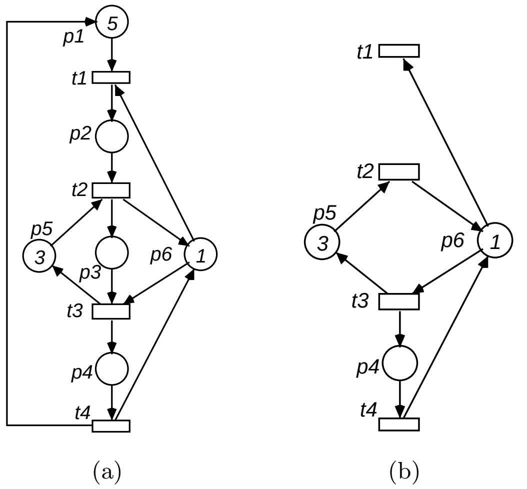


Fig. 2. (a) Deadlock system as autonomous, but deadlock--free as timed if $\lambda_3 > \lambda_1$. (b) Minimal siphon of the net.

mantics have been defined for *continuous* timed transitions, the two most important being *infinite server* or *variable speed*, and *finite server* or *constant speed* (see [9]). Here, infinite server semantics will be considered, since it usually provides a better approximation to the original model (already proved in [6] for an important subclass of nets). Like in purely Markovian *discrete* net models, under *infinite server semantics*, the flow through a timed transition t is the product of the rate, $\boldsymbol{\lambda}(t)$, and enab$(t, \mathbf{m})$, the instantaneous enabling of the transition, i.e., $\mathbf{f}(t, \mathbf{m}) = \boldsymbol{\lambda}(t) \cdot \text{enab}(t, \mathbf{m}) = \boldsymbol{\lambda}(t) \cdot \min_{p \in {}^\bullet t}\{\mathbf{m}(p)/\mathbf{Pre}(p, t)\}$. For the flow to be well defined, every transition must have at least one input place, hence in the following we will assume $\forall t \in T, |{}^\bullet t| \geq 1$.

The "min" in the above definition leads to the concept of *configurations*: a *configuration* is a set of pairs $\mathcal{C} = \{(t_1, p^1), (t_2, p^2), ..., (t_{|T|}, p^{|T|})\}$ where $\forall t_j \in T, p^j \in {}^\bullet t_j$ is a place that, for some markings, provides the minimum ratio $\mathbf{m}(p^j)/\mathbf{Pre}(p^j, t_j)$. In such case, it is said that p^j *constrains* t_j. For instance, in the net system of fig. 2(a), the configuration at the current marking is $\mathcal{C} = \{(t_1, p_6), (t_2, p_2), (t_3, p_3), (t_4, p_4)\}$, since $\mathbf{m}(p_6) < \mathbf{m}(p_1)$, $\mathbf{m}(p_2) < \mathbf{m}(p_5)$ and $\mathbf{m}(p_3) < \mathbf{m}(p_6)$. An upper bound for the number of configurations is $\prod_{t \in T} |{}^\bullet t|$. The set of all nonnegative markings that agree with the *P-flows* is denoted as $Class(\mathbf{m}_0) = \{\mathbf{m} \geq \mathbf{0} | \mathbf{B}_y^T \mathbf{m} = \mathbf{B}_y^T \mathbf{m}_0\}$, so, any reachable marking belongs to it.

The flow through the transitions can be written in a vectorial form as $\mathbf{f}(\mathbf{m}) = \boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbf{m}$, where $\boldsymbol{\Lambda}$ is a diagonal matrix whose elements are those of $\boldsymbol{\lambda}$, and $\boldsymbol{\Pi}(\mathbf{m})$ is the configuration operator matrix at $\mathbf{m}$, which is defined by elements as

$$\boldsymbol{\Pi}(\mathbf{m})_{i,j} = \begin{cases} \frac{1}{\mathbf{Pre}(p_j, t_i)} & \text{if } p_j \text{ is constraining } t_i \\ 0 & \text{otherwise} \end{cases}$$

If more than one place is constraining the flow of a transition at a given marking, any of them can be used, but only one is taken. Notice that the i-th entry of the vector $\boldsymbol{\Pi}(\mathbf{m})\mathbf{m}$ is equal to the enabling degree of transition t_i. For example, in the net of fig. 1(b) a marking $\mathbf{m} = [1, 1, 0]^T$ defines a configuration at which p_3 is constraining the flow of t_3 (p_1 always constrain t_1 and t_2). Then, the instantaneous transitions flow vector is

$$\mathbf{f}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}\lambda_1 & 0 & 0\\0 & \lambda_2 & 0\\0 & 0 & \lambda_3\end{bmatrix}\begin{bmatrix}1&0&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}\lambda_1\\\lambda_2\\0\end{bmatrix}$$

The set $Class(\mathbf{m}_0)$ can be divided into *marking regions* according to the *configurations*. A marking *region* is defined as the set $\Re_i = \{\mathbf{m} \in Class(\mathbf{m}_0) | \boldsymbol{\Pi}_i \mathbf{m} \leq \boldsymbol{\Pi}_j \mathbf{m}, \forall \boldsymbol{\Pi}_j\}$. Thus, for each configuration $\mathcal{C}_i$, it is associated a value $\boldsymbol{\Pi}_i$ that the configuration matrix can take, and a marking region $\Re_i$. These regions are polyhedrons, and are disjoint, except on the borders.

The dynamical behavior of a $TCPN$ system is described by its state equation:

$$\dot{\mathbf{m}} = \mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbf{m}$$

A $TCPN$ system $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$ that reaches a steady state (i.e., a marking $\mathbf{m}_{ss}$ s.t. $\mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m}_{ss})\mathbf{m}_{ss} = \mathbf{0}$) is called *live* iff the steady state flow is positive, which is expressed as $f_{ss}(\mathbf{m}_{ss}) > \mathbf{0}$ ([3]). In the same way, $\langle \mathcal{N}, \boldsymbol{\lambda} \rangle$ is *structurally live* iff there exists an initial marking $\mathbf{m}_0$ such that $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$ is live. Finally, a system $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$ that reaches a steady state deadlocks iff the steady state flow is null, i.e., $f_{ss}(\mathbf{m}_{ss}) = \mathbf{0}$.

A marking $\mathbf{m}_D$ is a deadlock if no transition is enabled. In such case, the set $\Sigma_D = \{p \in P | \mathbf{m}_D(p) = 0\}$ is a (usually non *minimal*) *siphon* whose outputs cover all transitions (i.e., $\Sigma_D^\bullet = T$). For instance, in fig. 2(a), the marking $\mathbf{m}_D = [1, 1, 3, 0, 0, 0]^T$ is the only deadlock related to the (minimal) *siphon* $\{p_4, p_5, p_6\}$ (i.e., it is empty at $\mathbf{m}_D$).

2.2 Eigenvalues of the state equation

Inside each region, the state equation is linear, since $\boldsymbol{\Pi}(\mathbf{m})$ is constant. Thus, the behavior of a $TCPN$ can be analyzed *by regions* through the knowledge of the eigenvalues and eigenvectors of the corresponding state matrices. In particular, given a configuration matrix $\boldsymbol{\Pi}_i$, a number $s \in \mathbb{C}$ (in general, complex) is called *eigenvalue* if there exists a column vector $\mathbf{v} \in \mathbb{C}^{|P|}$ s.t.:

$$(s \cdot \mathbf{I} - \mathbf{C}\boldsymbol{\Lambda}\boldsymbol{\Pi}_i) \cdot \mathbf{v} = \mathbf{0} \tag{1}$$

Vector $\mathbf{v}$ is called *column eigenvector* related to s. Furthermore, if there exists such eigenvalue, there also exist a row vector $\mathbf{w}$, called *row eigenvector* related to s s.t.:

$$\mathbf{w} \cdot (s \cdot \mathbf{I} - \mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}_i) = \mathbf{0} \qquad (2)$$

Remark 1 *Eigenvectors are related to P- and T-flows:*

1) *Given a P-flow* $\mathbf{y}$*, then* $\forall \mathbf{\Lambda}, \mathbf{\Pi}_D$ *it holds* $\mathbf{y}^T \cdot \mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}_D = \mathbf{0}$*, which is equivalent to (2) with* $s = 0$*. Therefore, P-flows are row eigenvectors related to a zero valued eigenvalue (i.e.,* $s = 0$*), for any timing and any configuration (already shown in [5]).*
2) *Given a column eigenvector* $\mathbf{v}$ *related to a zero valued eigenvalue (i.e.,* $\mathbf{v}$ *that fulfills (1) with* $s = 0$*), then the vector* $\mathbf{\Lambda}\mathbf{\Pi}_D\mathbf{v}$ *is a T-flow.*

We call *fixed eigenvalues* of $\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}_i$ those that do not depend on $\boldsymbol{\lambda}$, i.e., they are timing independent, while others are called *variable*. In particular, zero valued eigenvalues related to P-flows are fixed. Similarly, let us distinguish between *fixed* (timing independent) and *variable* poles.

Consider for instance the net system of fig. 1(b) and the configuration $\mathbf{\Pi}_1$ in which p_2 constraints t_3. The eigenvalues of $\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}_1$ are computed for two different timings $\boldsymbol{\lambda}^1 = [1, 2, 1]$ and $\boldsymbol{\lambda}^2 = [3, 2, 1]$, obtaining the values of $\{0, -0.26, -3.73\}$ and $\{0, 0.16, -6.16\}$, respectively. The last two eigenvalues for both timings are variable (they change for a different timing) while the first eigenvalue 0 is fixed, since it appears for any timing (it is associated to the P-flow $\mathbf{y} = [1, 1, 1]^T$).

3 Timing-dependent liveness in *contPN* systems: setting the problem.

Regarding autonomous (i.e., untimed) *continuous* net systems, it has been proved that deadlock-freeness and liveness are decidable [8]. If a system reaches a deadlock as timed, it also deadlocks as untimed (already stated for a subclass of nets in [3]). This clearly holds also for liveness since the evolution of a timed system just gives a particular trajectory of the untimed model.

Proposition 1 *If the contPN system* $\langle \mathcal{N}, \mathbf{m}_0 \rangle$ *is live, then for any* $\boldsymbol{\lambda} > \mathbf{0}$*,* $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$ *is live.*

On the contrary, a *contPN* that deadlocks as autonomous can be live as timed for particular timings. The next example shows a simple case.

Example 1 *The system of fig. 1(b) deadlocks as untimed, but the timed system is live if* $\lambda_1 = \lambda_2$*. It may seem that the set of rates for which this kind of things occurs has to be of null measure (i.e., a smaller dimension manifold), but it is not so. For example, the contPN system in fig. 2(a) deadlocks as autonomous, but is deadlock-free as timed if* $\lambda_3 > \lambda_1$*. Let us prove this by showing that under such timing the* siphon $\{p_4, p_5, p_6\}$ *will never empty. First, the deadlock belongs to a configuration in which* $\mathbf{m}(p_6) \leq \mathbf{m}(p_3)$*, because* $\mathbf{m}(p_6) = 0$ *at the deadlock. However, inside this configuration, the marking of the* siphon *is always increasing, since* $\mathbf{m}(p_4) + \mathbf{m}(p_5) + \mathbf{m}(p_6) = \mathbf{m}_0(p_4) + \mathbf{m}_0(p_5) + \mathbf{m}_0(p_6) + \int (\mathbf{f}(t_3) - \mathbf{f}(t_1)) d\tau$*, and* $\int (\mathbf{f}(t_3) - \mathbf{f}(t_1)) d\tau = \int (\lambda_3 - \lambda_1) \cdot \mathbf{m}(p_6) \cdot d\tau > 0$*. Clearly, if* $\lambda_3 \geq \lambda_1$ *the* siphon *never empties and the system is deadlock-free (sooner or later the deadlock configuration is left). In particular, if* $\lambda_1 = \lambda_3$ *the total marking of the siphon will remain constant. In any case, no deadlock occurs if the initial marking at* p_1 *is* 3 *instead of* 5*. That is, deadlock-freeness is non monotonic with respect to the marking: increasing the number of resources (*$\mathbf{m}_0(p_1) > 3$*) can kill the system!*

However, for some nets there do not exist rates that make the system deadlock-free.

Example 2 *Consider the system in fig. 1(b) but with a weight of* 1 *at arc* (t_3, p_1)*. Here the reasoning is purely structural: the net is structurally bounded but not consistent so it is non live for any timing.*

Through this paper, it will be investigated the existence of a timing that makes a non-live *contPN* system to be live as timed. If it is assumed that the timed system reaches a steady state marking, liveness can be studied from the flow at such marking.

From an algebraic perspective, steady states in *TCPN* systems are *equilibrium* markings, i.e., solutions of $\dot{\mathbf{m}} = \mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}(\mathbf{m})\mathbf{m} = \mathbf{0}$ with $\mathbf{m} \in Class(\mathbf{m}_0)$.

Definition 1 *An equilibrium marking* $\mathbf{m}_{ss}$ *can be classified according to its corresponding flow:*

1) *If* $\mathbf{f}(\mathbf{m}_{ss}) = \mathbf{0}$ *then* $\mathbf{m}_{ss}$ *is called* deadlock marking*. The configuration and region related to* $\mathbf{m}_{ss}$ *are called* deadlock configuration *and* deadlock region*, respectively.*
2) *If for some transition* t_j*,* $[\mathbf{f}(\mathbf{m}_{ss})]_j = 0$ *(where* $[\mathbf{f}(\mathbf{m}_{ss})]_j$ *denotes the j-th entry of* $\mathbf{f}(\mathbf{m}_{ss})$*) then* $\mathbf{m}_{ss}$ *is called* non-live marking*. The configuration and region related to* $\mathbf{m}_{ss}$ *are called* non-live configuration *and* non-live region*, respectively.*
3) *If* $\mathbf{f}(\mathbf{m}_{ss}) > \mathbf{0}$ *then* $\mathbf{m}_{ss}$ *is called* live marking.
If in a given region $\Re_i$ *there do not exist non-live equilibrium markings, then* $\Re_i$ *is called* live region*, and the associated configuration is said* live.

Remark 2 *According to the deadlock-freeness definition, a deadlock occurs when the system (asymptotically) reaches a deadlock marking. Similarly, the system becomes non-live when it (asymptotically) reaches a non-live marking. If the system reaches a live equilibrium marking, then it is live.*

Since deadlock markings are particular cases of non-live ones, only non-live markings will be studied in the sequel. Through this paper, liveness will be studied by using the following approach:

1) Given a $TCPN$ system $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$, compute non-live equilibrium markings in $Class(\mathbf{m}_0)$.
2) For a particular initial marking or a set of them, solve the reachability of the non-live markings.

The advantage of this approach is that the first problem can be solved by using an algebraic perspective, which will be done through the next section. The second problem can be studied by using results from Control Theory, which will be investigated in sections 5 and 6.

4 Live and non-live equilibrium markings

Non-live markings can be easily computed by using the following Linear Programming Problem (LPP).

Let $\mathbf{\Pi}_D$ be a given configuration matrix, and let T_D be a set of transitions. The following LPP computes an equilibrium marking $\mathbf{m}_D \in \Re_D$, if it exists, at which all the transitions in T_D are dead.

$$
\begin{aligned}
&max\ [1,...,1]\cdot \mathbf{m}_D \text{ subject to} \\
&\quad \mathbf{m}_D \geq \mathbf{0} \\
&\quad \mathbf{B}_y^T(\mathbf{m}_D - \mathbf{m}_0) = \mathbf{0} && \{\text{in } Class(\mathbf{m}_0)\} \\
&\quad \mathbf{C}\boldsymbol{\Lambda}\mathbf{\Pi}_D\mathbf{m}_D = \mathbf{0} && \{\text{equilibrium marking}\} \\
&\quad \mathbf{\Pi}_D\mathbf{m}_D \leq \mathbf{\Pi}_j\mathbf{m}_D \quad \forall \mathbf{\Pi}_j && \{\text{belongs to } \Re_D\} \\
&\quad [\mathbf{\Pi}_D]_i\, \mathbf{m}_D = 0 \quad \forall t_i \in T_D && \{\text{non-live}\}
\end{aligned}
$$

where $[\mathbf{\Pi}_D]_i$ denotes the $i-th$ row of $\mathbf{\Pi}_D$.

Proof: Since $\mathbf{m}_D \geq \mathbf{0}$ and $\mathbf{B}_y^T(\mathbf{m}_D - \mathbf{m}_0) = \mathbf{0}$ then, by definition, $\mathbf{m}_D \in Class(\mathbf{m}_0)$. Furthermore, $\mathbf{\Pi}_D\mathbf{m}_D \leq \mathbf{\Pi}_j\mathbf{m}_D\ \forall \mathbf{\Pi}_j$ implies that $\mathbf{m}_D \in \Re_D$. The marking $\mathbf{m}_D$ is an equilibrium one because $\mathbf{C}\boldsymbol{\Lambda}\mathbf{\Pi}_D\mathbf{m}_D = \mathbf{0}$. Finally, since $\boldsymbol{\Lambda}$ is a diagonal matrix, $[\mathbf{\Pi}_D]_i\mathbf{m}_D = 0$ implies $[\mathbf{f}(\mathbf{m}_D)]_i = [\boldsymbol{\Lambda}\mathbf{\Pi}_D\mathbf{m}_D]_i = 0$. Then, $\forall t_i \in T_D\ [\mathbf{f}(\mathbf{m}_D)]_i = 0$. Therefore, $\mathbf{m}_D$ is a non-live equilibrium marking in $\Re_D$ in which all the transitions in T_D are dead. □

Non-live markings are related to empty siphons. This is proved in the following proposition.

Proposition 2 *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$ be a $TCPN$ system. Consider an equilibrium marking $\mathbf{m}_D \in Class(\mathbf{m}_0)$. There are empty siphons at $\mathbf{m}_D$ iff it is non-live.*

Proof: Suppose that the system is at an equilibrium marking $\mathbf{m}_D$ at which there are empty siphons. Since empty siphons never gain marks, then the places belonging to supports of those empty siphons remain unmarked, so, their output transitions never become enabled and thus they are dead. For the other implication, suppose that the system is at a non-live equilibrium marking $\mathbf{m}_D$. Then, at least one input place of each dead transition is empty at $\mathbf{m}_D$, and they remain empty for future time (it is an equilibrium marking). If there exists a place p_i that always remains unmarked, then for each input transition t_j to this place, it must exist an input place p_k to t_j, which remains also unmarked for all time. Repeating this reasoning, it can be seen that p_i should belong to an unmarked siphon. □

Non-live markings can appear in different regions. In general, when more than one non-live marking appear, they can be isolated or connected in $Class(\mathbf{m}_0)$, but even in this case they may not describe a convex set. Nevertheless, if in a given region there exist two non-live equilibrium markings $\mathbf{m}_1$ and $\mathbf{m}_2$, at which transition t_i is dead, then all markings in the linear segment defined by $\mathbf{m}_1$ and $\mathbf{m}_2$ are also non-live equilibrium markings with t_i dead. This is proven in the following:

Proposition 3 *Given a $TCPN$ system, consider a marking region $\Re_D \subseteq Class(\mathbf{m}_0)$ with an equilibrium marking $\mathbf{m}_D \in \Re_D$. If there exists $\boldsymbol{\eta} \geq \mathbf{0}$ s.t.*

$$\begin{bmatrix} \mathbf{\Pi}_D \\ \mathbf{B}_y^T \end{bmatrix} \eta = \mathbf{0} \tag{3}$$

then all the markings in $S = \{\mathbf{m} \in \Re_D | (\mathbf{m} - \mathbf{m}_D) = \boldsymbol{\eta} \cdot \alpha, \alpha \in \mathbb{R}\}$ are also equilibrium markings having the same flow, i.e., $\forall \mathbf{m} \in S$ it holds $\mathbf{f}(\mathbf{m}) = \mathbf{f}(\mathbf{m}_D)$. If $\mathbf{m}_D$ is in the interior of $\Re_D$ then $\{S/\mathbf{m}_D\} \neq \emptyset$. In this way, if $\mathbf{m}_D$ is a non-live marking in which all the transitions in T_D are dead, then $\forall \mathbf{m} \in S$, $\mathbf{m}_D$ is non-live and all the transitions in T_D are dead at this marking.

Proof: Consider a marking $\mathbf{m} \in S$. Then, $\mathbf{f}(\mathbf{m}) - \mathbf{f}(\mathbf{m}_D) = \boldsymbol{\Lambda}\mathbf{\Pi}_D\mathbf{m} - \boldsymbol{\Lambda}\mathbf{\Pi}_D\mathbf{m}_D = \boldsymbol{\Lambda}\mathbf{\Pi}_D(\mathbf{m} - \mathbf{m}_D)$. Since $\mathbf{m} \in S$ then $\exists \alpha$ s.t. $(\mathbf{m} - \mathbf{m}_D) = \boldsymbol{\eta}\alpha$. According to (3), $\mathbf{\Pi}_D(\mathbf{m} - \mathbf{m}_D) = \mathbf{\Pi}_D\boldsymbol{\eta}\alpha = \mathbf{0}$. Therefore, $\mathbf{f}(\mathbf{m}) - \mathbf{f}(\mathbf{m}_D) = \boldsymbol{\Lambda}\mathbf{\Pi}_D(\mathbf{m} - \mathbf{m}_D) = \mathbf{0}$, meaning that $\mathbf{f}(\mathbf{m}) = \mathbf{f}(\mathbf{m}_D)$. In this way, since both markings have the same flow, if $\mathbf{m}_D$ is an equilibrium marking at which all the transitions in T_D are dead, then $\mathbf{m}$ is also an equilibrium marking in which all the transitions in T_D are dead.
Now, assume that $\mathbf{m}_D$ is in the interior of $\Re_D$. Then $\mathbf{m}_D > \mathbf{0}$, which implies that for a small enough $\alpha \in \mathbb{R}$ the marking $\mathbf{m}' = \mathbf{m}_D + \boldsymbol{\eta}\alpha > \mathbf{0}$. Furthermore, $\mathbf{B}_y^T\boldsymbol{\eta} = \mathbf{0}$ implies that $\mathbf{B}_y^T\mathbf{m}' = \mathbf{B}_y^T\mathbf{m}_D$, thus $\mathbf{m}' \in Class(\mathbf{m}_0)$. Moreover, since $\mathbf{m}_D$ is in the interior of $\Re_D$ then, for a small enough α, $\mathbf{m}' \in \Re_D$, thus $\{S/\mathbf{m}_D\} \neq \emptyset$. □

Live markings may exist in non-live regions. This is interesting since, if the timing is s.t. a live marking is attractive (i.e., it is asymptotically stable) then the system will avoid the non-live markings, and thus liveness follows. A sufficient condition for the existence of such live markings is introduced next:

Proposition 4 *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$ be a TCPN system. Let $\mathbf{m}_D$ be a non-live marking and let $\mathbf{\Pi}_D$ and $\Re_D$ be its associated configuration matrix and region, respectively.*

1) *If all the eigenvalues of $\mathbf{C\Lambda\Pi}_D$ non associated to* P-flows *are not null, then $\forall \mathbf{v}$ s.t. $\mathbf{C\Lambda\Pi}_D\mathbf{v} = \mathbf{0}$ and $\mathbf{B}_y^T\mathbf{v} = 0$ it fulfills that $\mathbf{v} = \mathbf{0}$. As a consequence, $\mathbf{m}_D$ is the only equilibrium marking in $\Re_D$.*
2) *If there exists an eigenvector $\mathbf{v}$, associated to a* variable *zero valued eigenvalue, s.t. $\mathbf{\Lambda\Pi}_D\mathbf{v}$ is a* T-semiflow*, $dim(\Re_D) = rank(\mathbf{C})$, and $\mathbf{m}_D$ is associated only to one configuration, then there exist infinite non deadlock equilibrium markings in $\Re_D$, at which the transitions related to positive values of $\mathbf{\Lambda\Pi}_D\mathbf{v}$ are live.*

Proof: First, in order to separate the null eigenvalues related to *P-flows* from the others, let us define a similarity transformation $[\mathbf{Z^T}, \mathbf{B}_y]^T$, where $\mathbf{B}_y^T$ is a basis for *P-flows* and $\mathbf{Z}$ is a suitable matrix for completing the rank. Denoting by $[\mathbf{A}, \mathbf{B}]$ the inverse transformation, the transformed state matrix is described by

$$\begin{bmatrix} \mathbf{Z} \\ \mathbf{B}_y^T \end{bmatrix} \mathbf{C\Lambda\Pi}_D \begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{ZC\Lambda\Pi}_D\mathbf{A} & \mathbf{ZC\Lambda\Pi}_D\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

since $\mathbf{B}_y^T\mathbf{C} = \mathbf{0}$. According to this transformation, the eigenvalues of $\mathbf{ZC\Lambda\Pi}_D\mathbf{A}$ are those of $\mathbf{C\Lambda\Pi}_D$ non associated to *P-flows*.
Statement 1). By hypothesis, all the eigenvalues, non associated to the *P-flows*, are not null. Then, all the eigenvalues of $\mathbf{ZC\Lambda\Pi}_D\mathbf{A}$ are not null, which implies that it has full rank. Let $\mathbf{v}$ be a vector s.t $\mathbf{C\Lambda\Pi}_D\mathbf{v} = \mathbf{0}$ and $\mathbf{B}_y^T\mathbf{v} = \mathbf{0}$. Applying the similarity transformation:

$$\begin{bmatrix} \mathbf{ZC\Lambda\Pi}_D\mathbf{A} & \mathbf{ZC\Lambda\Pi}_D\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{Zv} \\ \mathbf{B}_y^T\mathbf{v} \end{bmatrix} = \mathbf{0}$$

Now, since $\mathbf{B}_y^T\mathbf{v} = \mathbf{0}$ and $\mathbf{ZC\Lambda\Pi}_D\mathbf{A}$ has full rank then $\mathbf{v} = \mathbf{0}$. Finally, since every equilibrium marking $\mathbf{m}_1 \in \Re_D$ must satisfy $\mathbf{C\Lambda\Pi}_D(\mathbf{m}_1 - \mathbf{m}_D) = \mathbf{0}$ and $\mathbf{B}_y^T(\mathbf{m}_1 - \mathbf{m}_D) = \mathbf{0}$, then $(\mathbf{m}_1 - \mathbf{m}_D)$ is null, so $\mathbf{m}_1 = \mathbf{m}_D$.
Statement 2). By hypothesis, there exists $\mathbf{v} \neq \mathbf{0}$ such that $\mathbf{C\Lambda\Pi}_D\mathbf{v} = \mathbf{0}$, $\mathbf{B}_y^T\mathbf{v} = \mathbf{0}$ and $\mathbf{\Lambda\Pi}_D\mathbf{v}$ is a *T-semiflow.* Now, consider a vector $\mathbf{m}_1 = \mathbf{m}_D + \mathbf{v}\alpha$, notice that it is nonnegative for a small enough $\alpha \geq 0$ ($\forall p_j$ s.t. $\mathbf{m}_D(p_j) = 0$ it fulfills that $v_j \geq 0$, because p_j is constraining a transition and $\mathbf{\Lambda\Pi}_D\mathbf{v} \geq \mathbf{0}$). Furthermore, since $\mathbf{B}_y^T\mathbf{m}_1 = \mathbf{B}_y^T\mathbf{m}_D$, $dim(\Re_D) = rank(\mathbf{C})$ and $\mathbf{m}_D$ is related to only one configuration, there always exists a small enough $\alpha \geq 0$ s.t. $\mathbf{m}_1 \in \Re_D$. Moreover, $\mathbf{C\Lambda\Pi}_D\mathbf{m}_1 = \mathbf{0}$ and $\mathbf{\Lambda\Pi}_D\mathbf{v} \neq 0$ (which implies that $\mathbf{\Lambda\Pi}_D\mathbf{m}_1 \neq \mathbf{\Lambda\Pi}_D\mathbf{m}_D = \mathbf{0}$), i.e., $\mathbf{m}_1$ is a non deadlock marking in which the transitions related to positive entries of $\mathbf{\Lambda\Pi}_D\mathbf{v}$ are live. Finally, by linearity, every marking in the convex described by $\mathbf{m}_1$ and $\mathbf{m}_D$ is also a non deadlock equilibrium marking, and the flow at those markings is positive at those transitions related to positive entries of $\mathbf{\Lambda\Pi}_D\mathbf{v}$, i.e., they are live. □

A particular case of statement 2 of previous proposition occurs when the eigenvector $\mathbf{v}$ is s.t. $\mathbf{\Lambda\Pi}_D\mathbf{v} > \mathbf{0}$. In such case there exist infinite live equilibrium markings in $\Re_D$.

5 Timing to avoid non-live markings

In this section, a sufficient condition for avoiding non-live markings by suitably choosing the timing $\boldsymbol{\lambda}$, will be provided.

Recalling from subsection 2.2, *P-flows* are related to null eigenvalues that does not depend on the timing $\boldsymbol{\lambda}$ (equivalently, *P-flows* are related to fixed null poles [5]). However, not all the null poles are related to P-flows.

Remark 3 *Null poles can be:*

1) *Fixed and related to P-flows, i.e., $\exists \mathbf{y} \neq \mathbf{0}$ s.t. $\mathbf{y}^T\mathbf{C} = \mathbf{0}$, then $\forall \boldsymbol{\lambda}$, $\forall \mathbf{\Pi}_i$, $\mathbf{y}^T\mathbf{C\Lambda\Pi}_i = \mathbf{0}$.*
2) *Fixed but not related to P-flows, i.e., $\forall \boldsymbol{\lambda}\ \exists \mathbf{y} \neq \mathbf{0}$ s.t. $\mathbf{y}^T\mathbf{C\Lambda\Pi}_i = \mathbf{0}$ but $\mathbf{y}^T\mathbf{C} \neq \mathbf{0}$. These null poles appear in particular configurations, when $rank(\mathbf{\Pi}_i) < rank(\mathbf{C})$.*
3) *Variable, i.e., $\exists \boldsymbol{\lambda}\ \exists \mathbf{y} \neq \mathbf{0}$ s.t. $\mathbf{y}^T\mathbf{C\Lambda\Pi}_i = \mathbf{0}$ with $\mathbf{y}^T\mathbf{C} \neq \mathbf{0}$. These null poles appear even if $rank(\mathbf{\Pi}_i) = rank(\mathbf{C})$, but in this case, the span of $\mathbf{\Lambda\Pi}_i$ includes T-flows, i.e., $span(\Lambda\mathbf{\Pi}_i) \cap span(\mathbf{B}_x) \neq \emptyset$, where $\mathbf{B}_x$ is a basis of T-flows (right annuller of $\mathbf{C}$).*

Variable zero valued poles are interesting since they are related to marking conservation laws that are not *P-flows* (i.e., $\mathbf{y}$ s.t. $\mathbf{y}\dot{\mathbf{m}} = \mathbf{yC\Lambda\Pi}_i = \mathbf{0}$ but $\mathbf{yC} \neq \mathbf{0}$). This property can affect the reachability of non-live markings. In detail, if the marking conservation law involves the support of a siphon, this will never empty, thus the non-live markings at which this siphon is empty are not reachable. This is formally introduced in the following proposition.

Proposition 5 *Consider a non-live marking $\mathbf{m}_D \in \Re_D$, in which a siphon Σ_D is empty. If the timing $\boldsymbol{\lambda}$ is s.t. there exists a variable zero valued pole of $\mathbf{C\Lambda\Pi}_D$, related to an eigenvector $\mathbf{y} \geq \mathbf{0}$ whose support is equal to Σ_D (i.e., $\forall j$, $\mathbf{y}_j > 0$ iff $p_j \in \Sigma_D$), then the siphon Σ_D cannot be emptied (assuming it is initially marked) while the system evolves inside $\Re_D$. Consequently, $\mathbf{m}_D$*

is not reachable from any $\mathbf{m}_0 \in \Re_D$ s.t. $\mathbf{y}^T\mathbf{m}_0 > 0$ (i.e., that marks Σ_D), through a trajectory inside $\Re_D$.

Proof: First, since $\mathbf{y} \geq \mathbf{0}$ is s.t. $\mathbf{y}^T\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}_D = \mathbf{0}$, then, premultiplying the state equation, $\mathbf{y}^T\dot{\mathbf{m}} = \mathbf{y}^T\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}_D = \mathbf{0}$. Denote as $\mathbf{m}(\tau)$ the marking reached after τ time units (assuming that the system evolves inside $\Re_D$). By integrating previous equation, it is obtained $\int_0^\tau \mathbf{y}^T\dot{\mathbf{m}}d\tau = \mathbf{y}^T(\mathbf{m}(\tau) - \mathbf{m}_0) = \mathbf{0}$, equivalently $\mathbf{y}^T\mathbf{m}(\tau) = \mathbf{y}^T\mathbf{m}_0$. Next, assume that the siphon is initially marked, i.e., $\exists j$ s.t. $[\mathbf{m}_0]_j > 0$ and $p_j \in \Sigma_D$. Thus, according to the definition of $\mathbf{y}$, $p_j \in \Sigma_D$ implies $[\mathbf{y}]_j > 0$, then $\mathbf{y}^T\mathbf{m}_0 > 0$. In this way, $\mathbf{y}^T\mathbf{m}(\tau) > 0$. Furthermore, since $\mathbf{y} \geq \mathbf{0}$ and $\mathbf{m}(\tau) \geq \mathbf{0}$ then $\exists k$ s.t. $[\mathbf{y}]_k > 0$ and $[\mathbf{m}(\tau)]_k > 0$. Finally, $[\mathbf{y}]_k > 0$ implies that $p_k \in \Sigma_D$, thus, the siphon is not empty at $\mathbf{m}(\tau)$ (at least p_k is marked). Since this reasoning holds for any marking $\mathbf{m}(\tau)$ reached through a trajectory inside $\Re_D$, the proof is completed. □

Previous proposition represents the central idea of this section: it may be possible to make a siphon Σ_D to remain marked by suitably choosing $\boldsymbol{\lambda}$. This occurs, as already advance, in the system of fig. 2 with $\lambda_1 = \lambda_3$, where the timing is s.t. a marking conservation law $\mathbf{y}^T\mathbf{m} = \mathbf{y}^T\mathbf{m}_0$ (with $\mathbf{y}^T = [0, 0, 0, 1, 1, 1]$) appears involving the places in the siphon $\Sigma_D = \{p_4, p_5, p_6\}$, thus the corresponding non-live marking is avoided.

In case of several siphons that may become empty at a given non-live region $\Re_D$, it is not required to analyze each one. According to next proposition, the existence of a live marking in $\Re_D$ is enough for avoiding all the non-live markings in the same region.

Proposition 6 *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0\rangle$ be a TCPN system. Consider that $\mathbf{m}_0 > \mathbf{0}$ belongs to a non-live region $\Re_D$, and let $\mathbf{\Pi}_D$ be its associated configuration operator matrix. If there exists an equilibrium marking $\mathbf{m}_L \in \Re_D$ s.t. $\mathbf{\Lambda}\mathbf{\Pi}_D\mathbf{m}_L > \mathbf{0}$ then non-live markings in $\Re_D$ are not reachable through a trajectory in $\Re_D$.*

Proof: Let $\mathbf{m}_D$ be a non-live marking of $\Re_D$. Define $\mathbf{v} = (\mathbf{m}_L - \mathbf{m}_D)$, then $\mathbf{B}_y^T\mathbf{v} = \mathbf{0}$ and $\mathbf{\Lambda}\mathbf{\Pi}_D\mathbf{v}$ is a *T-flow.* Furthermore, for each j-th entry of $\mathbf{m}_D$ that is null, the j-th entry of $\mathbf{v}$ is positive. Now, every marking $\mathbf{m}_1$ reachable from $\mathbf{m}_0 > 0$, through a trajectory inside $\Re_D$, must fulfill the solution of the state equation (see, for instance, [1]): $\mathbf{m}_1 = e^{\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}_D\tau}\mathbf{m}_0$, for some time τ (while in $\Re_D$). In this way, considering an initial marking $\mathbf{m}'_0 = \mathbf{m}_0 + \mathbf{v}\alpha > \mathbf{0}$ (where α is a small enough scalar s.t. $\mathbf{m}'_0 \in \Re_D$), the marking reachable at time τ is given by: $\mathbf{m}'_1 = e^{\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}_D\tau}\mathbf{m}'_0$. Then, substituting $\mathbf{m}'_0$ and considering that $e^{\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}_D\tau}\mathbf{v} = \mathbf{v}$ (something easy to see by expanding the exponential matrix in Taylor's series), it follows that $\mathbf{m}'_1 = \mathbf{m}_1 + \mathbf{v}\alpha$.

Now, consider a positive initial marking $\mathbf{m}_0 \in \Re_D$. Let us reason by contradiction. Suppose that the system converges asymptotically from $\mathbf{m}_0 > \mathbf{0}$ towards the non-live marking $\mathbf{m}_D \in \Re_D$ through a trajectory in the interior of $\Re_D$. It implies that for a positive initial marking $\mathbf{m}'_0 = \mathbf{m}_0 + \mathbf{v}\alpha$, in which $\alpha < 0$, the system converges asymptotically towards $\mathbf{m}'_D = \mathbf{m}_D + \mathbf{v}\alpha$ through a trajectory in $\Re_D$ (at least for a small enough magnitude of α), but, since $\mathbf{m}_D$ has null entries whose corresponding elements in $\mathbf{v}\alpha$ are negative then $\mathbf{m}'_D$ has negative entries, which is a contradiction. Therefore, the corresponding non-live marking $\mathbf{m}_D$ is not reachable, through a trajectory in $\Re_D$, from any positive marking $\mathbf{m}_0 \in \Re_D$. □

The following proposition establishes that such live equilibrium marking $\mathbf{m}_L$ can be *induced* by timing properly the system (i.e., by choosing a suitable value for $\boldsymbol{\lambda}$) if the net is consistent.

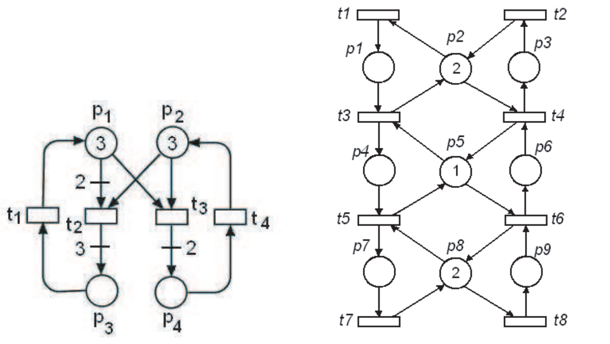
Proposition 7 *Let $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0\rangle$ be a TCPN system. Consider a non-live marking $\mathbf{m}_D$ that belongs to only one region $\Re_D$. There exists a timing $\boldsymbol{\lambda}$ that* induces *a live equilibrium marking $\mathbf{m}_L$ in $\Re_D$ iff $\mathcal{N}$ is consistent.*

Proof: If the net is not consistent then $\nexists \mathbf{x} > \mathbf{0}$ s.t. $\mathbf{C}\mathbf{x} = \mathbf{0}$, so, for any $\mathbf{\Lambda}$ and $\mathbf{m}_L > \mathbf{0}$ we cannot obtain $\mathbf{C}\mathbf{\Lambda}\mathbf{\Pi}_D\mathbf{m}_L = \mathbf{0}$, thus $\mathbf{m}_L$ is not an equilibrium marking.

For the other implication, by using the fact that all the *P-semiflows* are marked (assumed in Section 2) and the hypothesis that $\mathbf{m}_D$ belongs only to one region $\Re_D$, it can be proved that $\exists \mathbf{m}_L > \mathbf{0}$ that belongs to $\Re_D$. Now, if the net is consistent then there exists a vector $\mathbf{x} > \mathbf{0}$ s.t. $\mathbf{C}\mathbf{x} = \mathbf{0}$, so, a diagonal matrix $\mathbf{\Lambda} > \mathbf{0}$ s.t. $\mathbf{\Lambda}\mathbf{\Pi}_D\mathbf{m}_L = \mathbf{x}$ can be computed. Therefore, $\mathbf{m}_L \in \Re_D$ is an equilibrium marking and $\mathbf{\Lambda}\mathbf{\Pi}_D\mathbf{m}_L > \mathbf{0}$. □

Remark 4 *Consistence is no longer sufficient to guarantee the existence of a timing $\boldsymbol{\lambda}$ for avoiding non-live markings in different regions (but it is still necessary).*

Example 3 *Consider the net system of fig. 3(a) that has two deadlocks, $\mathbf{m}_D^1 = [0, 6, 0, 0]^T$ and $\mathbf{m}_D^2 = [6, 0, 0, 0]^T$. The corresponding deadlock configurations are $\mathcal{C}_D^1 = \{(t_1, p_1), (t_2, p_1), (t_3, p_3), (t_4, p_4)\}$ and $\mathcal{C}_D^2 = \{(t_1, p_2), (t_2, p_2), (t_3, p_3), (t_4, p_4)\}$, respectively. Since this net is consistent, according to proposition 7, for each configuration it is possible to find a timing for avoiding the corresponding deadlock marking. However, for this net it does not exist a timing that induces simultaneously live equilibrium markings in* both *regions. In order to prove this, notice that a basis for* T-flows *is given by $\mathbf{x} = [1, 1, 1, 1]^T$, then, if there exist live equilibrium markings $\mathbf{m}_1$ and $\mathbf{m}_2$ they must fulfill $\mathbf{\Lambda}\mathbf{\Pi}_D^1\mathbf{m}_1 = \beta\mathbf{\Lambda}\mathbf{\Pi}_D^2\mathbf{m}_2$ for some $\beta > 0$. This equality can be written by elements as $[0.5\lambda_1 m_{11}, \lambda_2 m_{11}, \lambda_3 m_{13}, \lambda_4 m_{14}]^T = \beta[\lambda_1 m_{22}, \lambda_2 m_{22}, \lambda_3 m_{23}, \lambda_4 m_{24}]^T$ (where m_{ij} means the $j-th$ entry of $\mathbf{m}_i$). Therefore $0.5m_{11} = \beta m_{22}$ and $m_{11} = \beta m_{22}$, but such equalities*



(a) (b)

Fig. 3. (a) *TCPN* system with deadlock markings in two different regions. (b) *TCPN* system with two independet T-semiflows.

do not have positive simultaneous solutions, so, there does not exist a timing $\boldsymbol{\lambda}$ *that induces live equilibrium markings in all deadlock regions. Nevertheless, it does not mean that the timed system is dead for all timing (e.g., with* $\boldsymbol{\lambda} = [1, 2, 1, 1]^T$ *the timed system converges to* $\mathbf{m}_1 = [0.75, 1.5, 2.25, 1.5]^T$*, which is live).*

6 Stability of non-live equilibrium markings

From a control theory perspective, non-live equilibrium markings are equilibrium points, so, the knowledge of the value of the poles (in each non-live configuration) is useful to decide if, given a particular timing, a non-live marking will be reached or will not. This idea is captured in the following propositions, whose proofs can be found in [12]:

Proposition 8 *Let* $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$ *be a TCPN system. Given a non-live marking* $\mathbf{m}_D$ *that belongs to a unique region* $\Re_D$*, then*

1) *If the real parts of the poles of* $\mathbf{C\Lambda\Pi}_D$*, non associated to the P-flows, are* negative*, then* $\mathbf{m}_D$ *is locally asymptotically stable, i.e., there exists a neighborhood of* $\mathbf{m}_D$*, named* $N(\mathbf{m}_D)$*, s.t. if* $\mathbf{m}_0 \in N(\mathbf{m}_D)$ *then the system* inevitably *reaches* $\mathbf{m}_D$*.*
2) *If* $\mathbf{C\Lambda\Pi}_D$ *has* a zero valued *pole, non associated to the P-flows, and the real parts of the others are* negative*, then* $\mathbf{m}_D$ *is stable (nevertheless* $\mathbf{m}_D$ *could be reached or not).*
3) *If there exists a* variable *null eigenvalue of* $\mathbf{C\Lambda\Pi}_D$*, with an associated eigenvector* $\mathbf{v}$ *s.t.* $\mathbf{B}_y^T\mathbf{v} = \mathbf{0}$ *and* $\mathbf{\Lambda\Pi}_D\mathbf{v}$ *is a* T-flow *with positive entries at those transitions that are dead at* $\mathbf{m}_D$*, then* $\mathbf{m}_D$ *is not reachable from a positive marking* $\mathbf{m}_0 \in \Re_D$*, through a trajectory in* $\Re_D$*.*
4) *If there exists a pole of* $\mathbf{C\Lambda\Pi}_D$ *having a positive real part, then* $\mathbf{m}_D$ *is unstable, so it is not reachable from another marking, through a trajectory in* $\Re_D$*.*

The stability analysis of a non-live marking $\mathbf{m}_D$, which is related to more than one configuration, is more complex, since it is a stability problem of a piecewise linear system. However, it is possible to know what could happen for particular cases.

Proposition 9 *Let* $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$ *be a TCPN system. Given a non-live marking* $\mathbf{m}_D$ *that belongs to different regions* $\Re_D^1$*,...,*$\Re_D^k$*, then*

1) *If for every region* $\Re_D^i$*, to which* $\mathbf{m}_D$ *belongs, the poles of* $\mathbf{C\Lambda\Pi}_D^i$ *are real and negative, then* $\mathbf{m}_D$ *is locally asymptotically stable, i.e., there exists a neighborhood of* $\mathbf{m}_D$*, named* $N(\mathbf{m}_D)$*, s.t. if* $\mathbf{m}_0 \in N(\mathbf{m}_D)$ *then the system inevitably reaches* $\mathbf{m}_D$*.*
2) *If for all regions* $\Re_D^i$*, there exists an eigenvector* $\mathbf{v}^i$*, associated to a* variable *zero eigenvalue of* $\mathbf{C\Lambda\Pi}_D^i$*, s.t.* $\mathbf{B}_y^T\mathbf{v}^i = \mathbf{0}$ *and* $\mathbf{\Lambda\Pi}_D^i\mathbf{v}^i > 0$*, then* $\mathbf{m}_D$ *is not reachable from* $\mathbf{m}_0 > 0$ *through a trajectory in* $\bigcup \Re_D^i$*.*

The first statement of previous proposition can be extended by using the classical Common Lyapunov Function (CLF) criterion (see, for instance, [4]), i.e., there exists a neighborhood of $\mathbf{m}_D$ from where the system inevitably reaches $\mathbf{m}_D$ if there exist symmetric positive definite matrices $\mathbf{P}$ and $\mathbf{Q}^i$ s.t. $(\mathbf{ZC\Lambda\Pi}_D^i\mathbf{A})^T\mathbf{P} + \mathbf{P}(\mathbf{ZC\Lambda\Pi}_D^i\mathbf{A}) = -\mathbf{Q}^i$ for each $\Re_D^i$ to which $\mathbf{m}_D$ belongs, where $\mathbf{Z}$ and $\mathbf{A}$ are the matrices introduced in the proof of proposition 4.

7 Examples: towards an interpretation at net level

Through this section, a few examples will be analyzed in order to illustrate the potential application of the results previously introduced.

In the sequel, given a non-live configuration $\mathcal{C}_D$, the possibility of choosing $\boldsymbol{\lambda}$ s.t. *variable* zero valued poles are induced will firstly be analyzed through the characteristic polynomial of $\mathbf{C\Lambda\Pi}_D$.

Remark 5 *Consider the characteristic polynomial of* $\mathbf{C\Lambda\Pi}_D$*, where* $\mathbf{\Lambda}$ *is in parametric form. The order of the lower order term is equal to the number of* fixed *zero valued poles, and a particular* $\mathbf{\Lambda}$ *that makes this lower order term be zero leads to a* variable *zero valued pole. In such case, statement* 3 *of proposition 8 may follow.*

Example 4 *Consider the TCPN system of fig. 2(a). In this case, there exists a unique deadlock* $\mathbf{m}_D = [1, 1, 3, 0, 0, 0]^T$*, belonging to a unique configuration* $\Re_D$*. The lower order term of the characteristic polynomial of* $\mathbf{C\Lambda\Pi}_D$ *is* $s^3(\lambda_4\lambda_1\lambda_2 - \lambda_4\lambda_2\lambda_3)$*. It means that there exist* 3 fixed *zero valued poles (related to* 3 *P-semiflows), and that a timing* $\boldsymbol{\lambda}$ *s.t* $\lambda_2\lambda_4(\lambda_1 - \lambda_3) = 0$

creates an additional zero valued pole. Then, a timing $\boldsymbol{\lambda}$ *s.t.* $\lambda_1 = \lambda_3$ *fulfills that condition, in which case, according to proposition 8,* $\mathbf{m}_D$ *is not reachable from any* $\mathbf{m}_0 > 0$ *through a trajectory in* $\Re_D$*. Moreover, since* $\mathbf{m}_D$ *belongs only to* $\Re_D$*, then the* $TCPN$ *system is deadlock-free. Such* $\boldsymbol{\lambda}$ *establishes an "equilibrium" between the flow going into the siphon* $\Sigma = \{p_4, p_5, p_6\}$ *and the flow going out of it, as it was proved in Section 3. Furthermore, if* $\lambda_3 > \lambda_1$*, then the coefficient of this term becomes negative and it can be demonstrated, through the Routh-Hurwitz criterion (see, for instance, [1]), that* $\mathbf{m}_D$ *is unstable (at leats one pole becomes positive), then the system is deadlock-free (proposition 8). Since this system has only one elementary T-semiflow, deadlock-freeness implies liveness (equivalently,* $\mathbf{m}_D$ *is the unique non-live equilibrium marking).*

Example 5 *Now, consider the system of fig. 3(b). It has 16 configurations but only three with deadlocks:*

$$\mathcal{C}_1 = \{(t_3, p_5), (t_4, p_6), (t_5, p_8), (t_6, p_5)\}$$
$$\mathcal{C}_2 = \{(t_3, p_5), (t_4, p_2), (t_5, p_8), (t_6, p_5)\}$$
$$\mathcal{C}_3 = \{(t_3, p_5), (t_4, p_2), (t_5, p_4), (t_6, p_5)\}$$

(the arcs that constrain transitions t_1, t_2, t_7 *and* t_8 *are not written because they are the same for all the configurations). Configuration* $\mathcal{C}_2$ *has infinite deadlocks (*$\exists \boldsymbol{\eta} \neq 0$ *satisfying (3)). All deadlocks in the system are connected. Computing the lower order terms of the characteristic polynomial for the three cases we obtain*

$$\mathcal{C}_1: \quad s^3\lambda_1\lambda_2\lambda_7\lambda_4(\lambda_3\lambda_8 - \lambda_5\lambda_6)$$
$$\mathcal{C}_2: \quad s^4\left[\lambda_1\lambda_2\lambda_7(\lambda_3\lambda_8 - \lambda_5\lambda_6) + \lambda_2\lambda_7\lambda_8(\lambda_1\lambda_6 - \lambda_3\lambda_4)\right]$$
$$\mathcal{C}_3: \quad s^3\lambda_2\lambda_7\lambda_8\lambda_5(\lambda_1\lambda_6 - \lambda_3\lambda_4)$$

For any timing $\boldsymbol{\lambda}$ *s.t.* $\lambda_3\lambda_8 = \lambda_5\lambda_6$ *and* $\lambda_1\lambda_6 = \lambda_3\lambda_4$*, a* variable *zero valued pole is added to every deadlock configuration. In this system, every possible non-live equilibrium marking* $\mathbf{m}_D$ *is actually a deadlock one (i.e.,* $\Sigma_D^\bullet = T$*). However, if a variable zero valued eigenvalue is added then there exists an eigenvector* $\mathbf{v} \neq \mathbf{0}$ *s.t.* $[\boldsymbol{\Lambda\Pi}_D\mathbf{v}]_i > 0$ *for some* t_i*, so, according to proposition 9, non live markings in which* t_i *is dead are not reachable from a positive initial marking. Finally, since every non-live marking is a deadlock one, then the system is live.*

Example 6 *The system of fig. 4 has two different* minimal T-semiflows*, whose supports are covered, independently, by* siphons $\Sigma_1 = \{p_4, p_5, p_6\}$ *and* $\Sigma_2 = \{p_9, p_{10}, p_{11}\}$*. That means that there exist non-live equilibrium markings that are not deadlocks. Now, if the timing* $\boldsymbol{\lambda}$ *is s.t.* $\lambda_1 = \lambda_3$*, the* siphon Σ_1 *conserves its total marking (as in the system of figure 2(a)). But, if* $\lambda_5 > \lambda_7$ *then the* siphon Σ_2 *will empty, so, the system does not reach a deadlock, but it becomes non live. Nevertheless, if* $\boldsymbol{\lambda}$ *is s.t.* $\lambda_1 = \lambda_3$ *and* $\lambda_5 = \lambda_7$*, both* siphons *remain marked, for all time, i.e., the timed system is live. In such case, we are inducing a live equilibrium marking in all non-live configurations.*

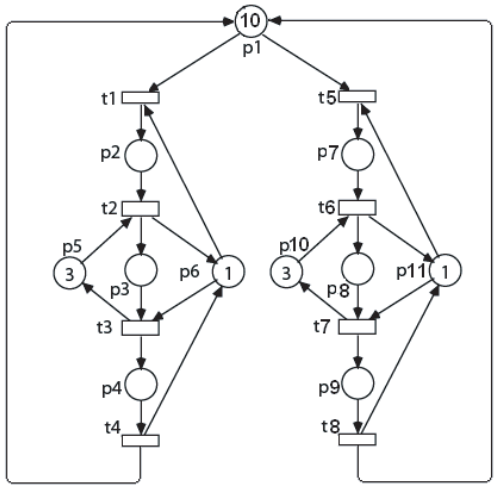


Fig. 4. $TCPN$ system with two independent *siphons*.

8 Control Interpretation

Through this paper, some tools have been provided in order to compute $\boldsymbol{\lambda}$ s.t. non-live markings are avoided. This can be interpreted as the synthesis of a controller for meeting *safety* specifications, i.e., for avoiding forbidden (non-live) states. Let us detail this through this section.

Definition 2 *Given a* $TCPN$ *system having fixed nominal rates* $\boldsymbol{\lambda}$*, a control action is defined as a reduction of the flow through the transitions [11]. Transitions in which a control action can be applied are called* controllable*. The effective flow through a controllable transition can be represented as:* $\mathbf{f}_i(\tau) = \lambda(t_i) \cdot enab(\tau, t_i) - u(\tau, t_i)$*, where* $u(\tau, t_i)$ *represents the control action on* t_i *and* $0 \leq u(\tau, t_i) \leq \lambda(t_i) \cdot enab(\tau, t_i)$*.*
The control vector $\mathbf{u} \in \mathbb{R}^{|T|}$ *is defined s.t.* $\mathbf{u}_i$ *represents the control action on* t_i*. If* t_i *is not controllable then* $\mathbf{u}_i = 0$*. The set of all controllable transitions is denoted by* T_c*, and the set of uncontrollable transitions is* $T_{nc} = T - T_c$*.*

The behavior of a $TCPN$ *forced* system is described by the state equation:

$$\dot{\mathbf{m}} = \mathbf{C\Lambda\Pi(m)m} - \mathbf{Cu} \tag{4}$$

with the input constraint $\mathbf{0} \leq \mathbf{u} \leq \mathbf{\Lambda\Pi(m)m}$.

The results presented through this paper can be interpreted as the solution of a liveness-enforcing control problem.

Proposition 10 *Consider a* $TCPN$ *system* $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$*, where* $\boldsymbol{\lambda}$ *are the nominal firing rates, and a non-live* *marking $\mathbf{m}_D \in Class(\mathbf{m}_0)$ reachable in the autonomous contPN. Let $\boldsymbol{\lambda}^l$ be a firing rate vector s.t. the timed system $\langle \mathcal{N}, \boldsymbol{\lambda}^l, \mathbf{m}_0 \rangle$ avoids the non-live marking $\mathbf{m}_D$. If all the transitions are controllable (i.e., $T = T_c$) then $\exists \beta > 0$ s.t. the control law $\mathbf{u}^l = \left[\boldsymbol{\Lambda} - \beta \boldsymbol{\Lambda}^l\right] \boldsymbol{\Pi}(\mathbf{m})\mathbf{m}$ makes the system $\langle \mathcal{N}, \boldsymbol{\lambda}, \mathbf{m}_0 \rangle$ to avoid $\mathbf{m}_D$ and $\mathbf{0} \leq \mathbf{u}^l \leq \boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbf{m}$.*

Proof: The flow of the system with the timing $\boldsymbol{\lambda}^l$ is equivalent to the flow of the system with the original nominal rates $\boldsymbol{\lambda}$ but under a control action $\mathbf{u}^l$, i.e.: $\boldsymbol{\Lambda}^l \boldsymbol{\Pi}(\mathbf{m})\mathbf{m} = \boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbf{m} - \mathbf{u}^l$, where $\mathbf{u}^l = \left[\boldsymbol{\Lambda} - \boldsymbol{\Lambda}^l\right] \boldsymbol{\Pi}(\mathbf{m})\mathbf{m}$. Nevertheless, since the control action must fulfill $\mathbf{0} \leq \mathbf{u}^l \leq \boldsymbol{\Lambda}\boldsymbol{\Pi}(\mathbf{m})\mathbf{m}$ and $\mathbf{u}^l_i = 0 \, \forall t_i \in T_{nc}$, then it is required that $\mathbf{0} \leq \boldsymbol{\lambda}^l \leq \boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^l = \boldsymbol{\lambda} \, \forall t_i \in T_{nc}$. If all the transitions are controllable (i.e., $T = T_c$) then it is always possible to define a small enough scalar $\beta > 0$ s.t. $\mathbf{0} \leq \beta \boldsymbol{\lambda}^l \leq \boldsymbol{\lambda}$. Furthermore, the timing $\beta \boldsymbol{\lambda}^l$ also makes the system to avoid $\mathbf{m}_D$, then the control law $\mathbf{u}^l = \left[\boldsymbol{\Lambda} - \beta \boldsymbol{\Lambda}^l\right] \boldsymbol{\Pi}(\mathbf{m})\mathbf{m}$ meets the required specification. □

Example 7 *For instance, consider the system of fig. 3(b) having nominal rates $\boldsymbol{\lambda} = [1, 5, 6, 8, 2, 4, 10, 2]^T$. For these rates the system reaches a deadlock $\mathbf{m}_D = [2, 0, 0, 1, 0, 0, 0, 0, 2]^T$. Now, according to the analysis done in the example 5, the timing $\boldsymbol{\lambda}^l = [2, 2, 2, 2, 2, 2, 2, 2]^T$ makes this net system live. Notice that $\boldsymbol{\lambda}^l \nleq \boldsymbol{\lambda}$, but defining $\beta = 0.5$ it fulfills $\mathbf{0} \leq \beta \boldsymbol{\lambda}^l \leq \boldsymbol{\lambda}$, so, by applying the control law $\mathbf{u}^l = \left[\boldsymbol{\Lambda} - \beta \boldsymbol{\Lambda}^l\right] \boldsymbol{\Pi}(\mathbf{m})\mathbf{m}$ the system avoids all the deadlock markings.*

This control interpretation can be extended to the case in which some uncontrollable transitions are considered. However, such case is much more complex, since it is required that $\forall t_i \in T_{nc}$, $\boldsymbol{\lambda}^l = \boldsymbol{\lambda}$. The analysis o such case is beyond the scope of this paper.

9 Conclusions

Through this paper, liveness of timed continuous Petri nets has been studied. First, the liveness problem has been divided into two different ones: the existence of non-live equilibrium markings, and the reachability of them. The first problem is an algebraic one that can be easily solved in polynomial time (LPP in Section 4). It was shown that non-live markings are strongly related to unmarked siphons (proposition 2).

By using some classical results on stability in linear systems, a couple of sufficient conditions for avoiding non-live equilibrium markings, and sufficient conditions for reaching them, have been introduced (propositions 8 and 9). The main contribution consists in providing a sufficient condition for avoiding non-live markings that belong to a unique region (proposition 6). The existence of such condition is related to consistency (proposition 7). Some illustrative examples were presented in order to interpret the obtained results at the net level.

Finally it was shown that, given a $TCPN$ system with particular nominal rates, a control law that enforce liveness can be computed by using a timing that makes the system to avoid non-live markings.

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