Structural Performance Analysis of Stochastic Petri Nets*

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Abstract

Structure performance analysis theory and techniques is an essay to avoid the computational complexity problem associated to Markovian and discrete event simulation techniques. Even if a finished conceptual and technical framework is not yet available, important benefits have been obtained not only from performance but also from correctness analysis point of view. In this survey we overview some of the achievements developed by the authors and collaborators towards a structure theory for performance evaluation of net based models. Concepts and techniques for the computation of performance bounds and approximate and exact evaluation are described in a semi-formal/illustrative way through a selected collection of examples.

1 Introduction

Petri nets consist of a few simple objects, relations, and rules to model discrete event dynamic systems with potentially very complex behaviours. Therefore qualitative/logical analysis of net models has been traditionally a problem to which much attention has been devoted. Roughly speaking these analysis techniques fall into the following categories [20, 23]: (1) *enumeration*, in which the state space is (partially) explored; (2) *transformation*, where the basic idea is to transform the original system into another under some "equivalence" notion and easier to analyse; (3) *structural*, where the key point is to obtain the maximum of information about the behaviour reasoning on the net structure (i.e., the static) and the initial marking (essentially considered as a parameter); and (4) *simulation*, where one or several behaviours are explored.

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Petri nets have been provided with several timed stochastic interpretations leading to performance models, that can be viewed as *queueing networks with choices and synchronizations* [5]. Performance analysis techniques of timed stochastic interpretation can be also classified into analogous groups. Enumeration techniques are strongly related to techniques in which certain *Markov chains* are solved. Simulation is now essentially a chapter of *discrete event dynamic systems simulation*. Transformation techniques are not too much developed and the existing methods are deeply based on structural concepts and objects. This invited survey overviews and illustrates some of the achievements towards a structure theory for performance computation of net based models. It is heavily based on the work carried out by our group.

As general remarks, we want to point out two important facts: (a) today we have *not* a full-fledged/finished theory for structural computation of performance figures, but several important concepts and techniques are already available; nevertheless several new algorithms improving the more classical approaches are computationally much less complex (even with polynomial time complexity in several cases!); and (b) the interleaving of correctness and performance analysis concepts and techniques has already produced several benefits, that we can summarize as follows [22]:

(1) *Introduction of new concepts for the analysis of autonomous (non-timed, in our context) net systems*, as the liveness bound of a transition (a quantitative generalization of the classical liveness concept, strongly related to the notion of number of servers in a station in steady-state).

(2) *Bridges between logical and performance properties*, as that existing between the existence of home spaces (often required for correctness) and ergodicity of the stochastic model. In section 4, the analysis technique taken as example introduces an elaborated bridge for a class of nets be-

tween synchronic distances (a correctness notion) and potential ergodicity (a performance notion).

(3) *Performance analysis suggests new results for the correctness analysis of autonomous net models.* This is the case, for example, of the so called rank theorems, that allow to characterize, in polynomial time, structural liveness and boundedness of certain net classes (e.g., equal conflict [27, 26]; deterministic systems of sequential processes [21]).

(4) *Untimed net systems analysis techniques allow the development of new performance analysis techniques* in which full state exploration and/or simulation are not needed.

In other words, the interleaving of correctness and performance analysis theories produces a synergic situation, in which each one contributes to the development of the other.

The rest of the paper, through a selected collection of examples, shows concepts and techniques for the structural analysis of performance net models. We assume the reader is familiar with the general concepts and notation on (stochastic) Petri net systems [20, 5, 23, 25].

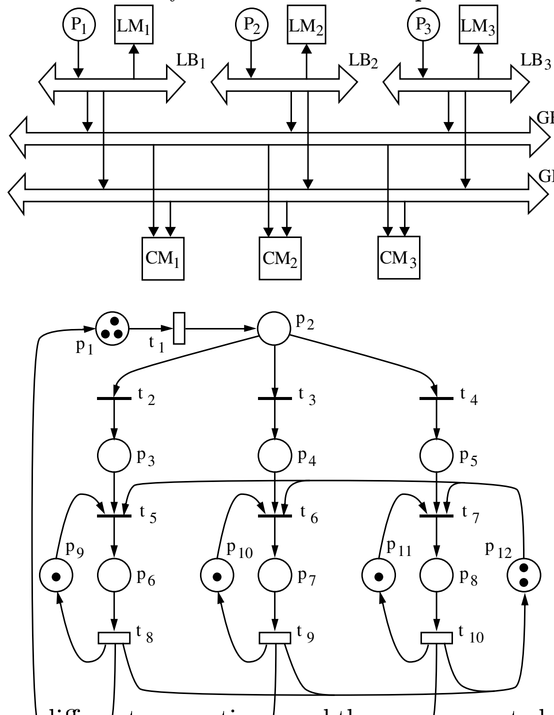
2 Performance bounds

In this section, we consider the computation of performance bounds for the steady-state behaviour of stochastic Petri net systems under *weak ergodicity* assumption for the firing and the marking processes [4] and with infinite-server semantics. We denote by $\overline{M}$ and $\vec{\sigma}^*$ the *limit average marking* and the *limit throughput* vector, respectively. For each transition t_i, the inverse of its throughput, $\Gamma^{(i)} = 1/\sigma^*(t_i)$, is called *mean interfiring time* of t_i.

If no assumption is made on the probability distribution of transition service times, *insensitive* bounds for the limit throughput can be derived based on some structural characteristics of the model. Such results are presented in section 2.2. Some improvements are briefly presented in section 2.3. The structural concepts needed to derive the performance results are introduced in section 2.1.

As an example of application of the results included in this section, we consider the Petri net model of a multiple-bus multiprocessor with external common memory depicted in Fig. 1 (three processors, three external common memories, and two buses). We assume that the common memory request rates of all processors are equal, and that the three common memories

Figure 1: A multiple-bus multiprocessor with external common memory and its Petri net representation.



have different access times and they are requested with different probabilities.

2.1 Structural basis

If C denotes the incidence matrix of the net, vectors $Y \geq 0$ such that $Y^T \cdot C = 0$ ($X \geq 0$ such that $C \cdot X = 0$) are called *P-semiflows* or conservative components (*T-semiflows* or consistent components). The *support* of a P-semiflow (T-semiflow) is defined as $||Y|| = \{p \in P \mid Y(p) \neq 0\}$ ($||X|| = \{t \in T \mid X(t) \neq 0\}$). A ($P$- or T-) semiflow is called *minimal* if it has minimal support (i.e., non strictly included in another).

The *visit ratio* of transition t_j with respect to t_i, $v_j^{(i)}$, is the average number of times t_j is visited (fired) for each visit to (firing of) the reference transition t_i. The vector of visit ratios of a bounded system must be a T-semiflow (the input weighted flow to each place must be equal to the output weighted flow). The visit ratios of transitions in *equal conflict*, i.e., such that their preconditions are the same ($PRE[t_j] = PRE[t_k]$) must be fixed by the corresponding routing rates. For instance, for the net in Fig. 1, the following equations hold: $r_3 v_2^{(i)} - r_2 v_3^{(i)} = 0$; $r_4 v_2^{(i)} - r_2 v_4^{(i)} = 0$, where r_2, r_3, and r_4 are the routing rates of t_2, t_3, and t_4. In summary, the next result can be stated:

Theorem 2.1 [10] *The vector of visit ratios with re-*

spect to transition t_i of a live and bounded net system must be a solution of:

$$C \cdot \vec{v}^{(i)} = 0; \;\; R \cdot \vec{v}^{(i)} = 0; \;\; v_i^{(i)} = 1$$

where C is the incidence matrix and R is a matrix that relates the visit ratios of transitions in equal conflict ($PRE[t_j] = PRE[t_k]$) according to the corresponding routing rates.

Equations in the above theorem have been shown to characterize a unique vector $\vec{v}^{(i)}$ for important net subclasses [5, 10] (a condition over the rank of C and the number of equal conflicts underlies these cases).

The *average service demand* from transition t_j with respect to t_i is defined as $D_j^{(i)} = s_j \, v_j^{(i)}$.

The performance of a net with infinite-server semantics of transitions depends on the maximum degree of enabling of the transitions, the *enabling bound.* Interpreting the net transitions as queueing stations, the *liveness bound*, $L(t)$, is a lower bound of the number of servers in steady state. The liveness bound is just a numerical refinement of the classical liveness concept in Petri nets ($L(t) > 0$ iff t is live) required for performance computations.

Definition 2.1 *Let $\langle \mathcal{N}, M_0 \rangle$ be a net system.*

The enabling bound of a transition t of $\mathcal{N}$ is $E(t) \stackrel{\text{def}}{=} \max\{k \mid \exists M \in R(\mathcal{N}, M_0) : M \geq kPRE[t]\}$.

The liveness bound of a transition t is $L(t) \stackrel{\text{def}}{=} \max\{k \mid \forall M' \in R(\mathcal{N}, M_0), \exists M \in R(\mathcal{N}, M') : M \geq kPRE[t]\}$.

The structural enabling bound of a transition t is: $SE(t) \stackrel{\text{def}}{=} \max\{k \mid M_0 + C \cdot \vec{\sigma} \geq k\, PRE[t];\ \vec{\sigma} \geq 0\}$.

Note that the definition of structural enabling bound reduces to the formulation of a linear programming problem. The following result relates the above three concepts:

Theorem 2.2 [4] *Let $\langle \mathcal{N}, M_0 \rangle$ be a net system, then for every transition t of $\mathcal{N}$, $SE(t) \geq E(t) \geq L(t)$.*

The interest of the above property lies in the fact that for those net systems whose exact liveness bounds of transitions cannot be efficiently computed, upper bounds (i.e., optimistic values) can be always obtained by solving some linear programming problems, i.e., by computing the structural enabling bounds.

2.2 Computation of insensitive bounds

If, in addition of weak ergodicity, we assume that the *residence time* of a token at each place (time spent by the token within the place) is bounded[1], the following result can be derived from Little's law and the basic properties of P-semiflows:

Theorem 2.3 [10] *For any live and bounded system, a lower bound for the mean interfiring time $\Gamma^{(i)}$ of transition t_i can be computed by the following linear programming problem:*

$$\Gamma^{(i)} \geq \begin{array}{ll} \text{maximum} & Y^T \cdot PRE \cdot \vec{D}^{(i)} \\ \text{subject to} & Y^T \cdot C = 0 \\ & Y^T \cdot M_0 = 1 \\ & Y \geq 0 \end{array} \qquad \text{(LPP1)}$$

Let us consider, for instance, the Petri net model in Fig. 1. The application of (LPP1) gives:

$$\Gamma^{(1)} \geq \max \left\{ \frac{s_1 v_1^{(1)} + s_8 v_8^{(1)} + s_9 v_9^{(1)} + s_{10} v_{10}^{(1)}}{3}, \frac{s_8 v_8^{(1)} + s_9 v_9^{(1)} + s_{10} v_{10}^{(1)}}{2}, \frac{s_8 v_8^{(1)}}{1}, \frac{s_9 v_9^{(1)}}{1}, \frac{s_{10} v_{10}^{(1)}}{1} \right\} \quad (1)$$

The quantities under the max operator above represent, for this particular case, the mean interfiring time, normalized for transition t_1, computed at each of the isolated subnets generated by the minimal P-semiflows of the net, assuming that all the nodes are delay stations (infinite-server semantics). Therefore, the lower bound for the mean interfiring time of t_1 in the original net system given by (1) is computed looking at the "slowest subsystem" generated by the P-semiflows, considered in isolation (with delay nodes).

For instance, if the common memory request mean time of each processor is $s_1 = 1$ modelled by transition t_1 with infinite-server semantics, the three memories have access times with mean $s_8 = 1$, $s_9 = 2$, and $s_{10} = 5$, and they are requested with routing rates $r_2 = 6$, $r_3 = 3$, and $r_4 = 1$, respectively, the bound given by (1) is $\Gamma^{(1)} \geq 0.9$, i.e., the throughput of t_1 is less than or equal to 1.1111. If exponentially distributed service times are assumed for timed transitions, the exact throughput of transitions t_1 can be computed by solving the embedded CTMC, and is equal to 0.8471 (i.e., the bound is 31% above).

With respect, to the lower bound on throughput, the following result can be stated:

Theorem 2.4 [10] *For any live and bounded system, an upper bound for the mean interfiring time $\Gamma^{(i)}$ of*

[1] This condition is assured for live and bounded net systems if a *locally fair* consumption of tokens at each place is assumed (for instance, FIFO discipline or random order for the selection of the token).

transition t_i is given by: $\Gamma^{(i)} \leq \sum_{j=1}^{m} D_j^{(i)}$. If the system is free choice, then the bound can be improved: $\Gamma^{(i)} \leq \sum_{j=1}^{m} \frac{D_j^{(i)}}{SE(t_j)}$.

In the general case, the bound corresponds to a complete sequentialization of all the transition services. In the free-choice case, internal self-concurrency at transitions is considered by dividing each average service demand by the corresponding structural enabling bound.

In the case of the net in Fig. 1, which is not free choice, the lower bound on throughput of t_1 given by the inverse of $\sum_{j=1}^{m} D_j^{(1)}$ is 0.3704. This bound is 56% below the exact value for exponential service times (this pessimistic bound is close to the exact value for special probability distributions with very large coefficient of variation; see [3]).

2.3 Some improvements of the bounds

In previous section, only P-semiflows and enabling bounds from the net structure and mean service times from stochastic timing have been used for the computation of bounds. The improvement of such bounds can be addressed in two ways: (1) by introducing a greater amount of structural information or/and (2) by taking into account the probability distribution function of service times.

In [7], some improvements derived just from the first approach (the structural one) were derived. For instance, other structural linear marking relations than those derived from P-semiflows (mainly those derived from *traps*) were used as additional constraints for the linear programming bound computation. The addition of *implicit places* was shown to be another interesting tool for the improvement of structure-based performance bounds.

With respect to the second approach, i.e., that based on the use of the distribution function of service times, the reader is referred to [8] for an example. There, an improvement of throughput lower bounds for NBUE service distributions (thus, in particular, for exponential) is presented by using a kit of *transformation rules* to produce local pessimistic temporal behaviour. In general, the rules lead to an approximation for throughput, and in particular cases to some improvements of the lower bound.

We have selected to present here an improvement that can be considered a mixture of structural and stochastic interpretation approaches. The throughput upper bound presented in the previous section (theorem 2.3) is based on the computation of the mean interfiring time of transitions of subnets generated by P-semiflows considered in isolation. Since infinite-server semantics is considered for the isolated subnet, the real (unknown) residence time at places is lower bounded by the service time of transitions, but waiting time due to synchronizations is not considered at all.

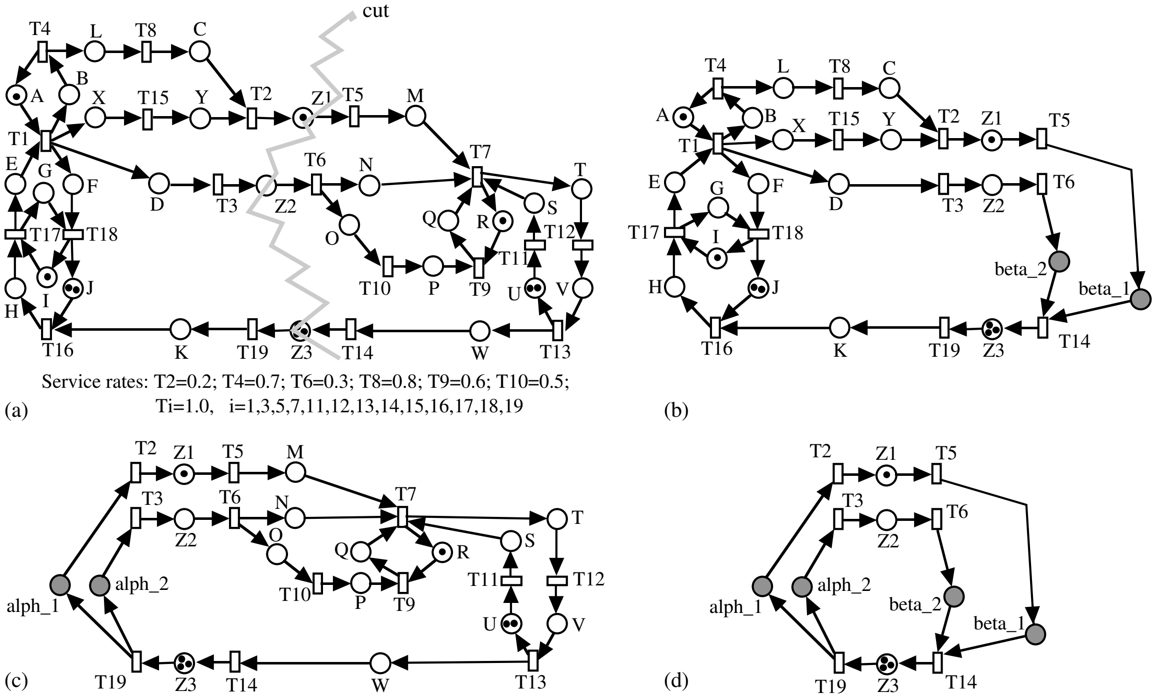
In the paragraphs below, we briefly present (the interested reader is referred to [11] for a detailed exposition) how the bound for the residence time at places can be improved taking into account not only the service time but also a part of the queueing (in places) time due to synchronizations: the time in queue when $L(t)$ servers is the maximum available at each transition t. In other words, we consider also the isolated subnets but with finite-server semantics for transitions (with $L(t)$ servers).

Since we are looking for very efficient computation, we restrict ourselves to analyse some subnets (that we call *RP-components*) isomorphous to product-form queueing networks. RP-components are subnets generated by P-semiflows with the structure of a strongly connected *state machine* (state machines are ordinary Petri nets whose transitions have only one input and only one output place) with the additional constraint that for any pair of transitions in conflict in the subnet, these transitions must be in equal conflict in the whole net.

The idea is that, if $\Gamma^{(i)}$ is the exact mean interfiring time of t_i in the whole net system, $\Gamma_{Y_L}^{(i)}$ is the exact mean interfiring time of t_i in the isolated RP-component generated by a minimal P-semiflow Y, with $L(t)$–server semantics for each involved transition t, and $\Gamma_{Y_\infty}^{(i)}$ is the value of the objective function of (LPP1) corresponding to Y, then, $\Gamma^{(i)} \geq \Gamma_{Y_L}^{(i)} \geq \Gamma_{Y_\infty}^{(i)}$ (provided some additional technical points that are presented in detail in [11]). In other words, the knowledge of the liveness bound of transitions for a given net can allow to improve the throughput upper bound computed in theorem 2.3 by investing an additional computational effort. For the example in Fig. 1, the optimal value of (LPP1) was obtained for the minimal P-semiflow Y_1 that covers the subset of places $\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$. If we consider the RP-component generated by Y_1 with single-server semantics for transitions t_8, t_9, and t_{10} (note that $L(t_8) = L(t_9) = L(t_{10}) = 1$), we obtain (by using, for instance, the *mean value analysis* algorithm) that the throughput of t_1 is less than or equal to 0.8648 (remember that the previous bound was 1.1111). Now, the bound is only 2% above the exact value (0.8471).

All the techniques presented in this section for the computation of bounds are based on the knowledge of

Figure 2: An MG, its decomposition in aggregated systems, and the basic skeleton.



the vector of visit ratios, but the structural computation of that vector is only possible for some net subclasses [10]. A general, but less intuitive, statement for the computation of bounds (without the explicit use of visit ratios) can be found in [12].

3 Approximate analysis

In this section, we present a general iterative technique for approximate throughput computation. In order to simplify the presentation we restrict to the consideration of stochastic strongly connected *marked graphs* (MG's). MG's are ordinary Petri nets whose places have only one input and only one output transition. Similar results can be found in the literature for other net subclasses [19, 16]. Hierarchical aggregation approximation techniques have been developed in [17, 2]. The approach presented here has two basic foundations. First, a deep understanding of the qualitative behaviour of MG's leads to a general (structural) decomposition technique. Second, after the decomposition phase, an iterative response time approximation method is applied for the computation of the throughput.

3.1 Structural decomposition

The basic idea is the following: a strongly connected and live MG (see Fig. 2.a) is split into two *aggregable* subsystems $\mathcal{N}_{A_1}$ and $\mathcal{N}_{A_2}$ (those in Figs. 2.b and 2.c without the shaded places) by a *cut* Q defined through some places ($Q = \{Z1, Z2, Z3\}$, in Fig. 2.a). Input and output transitions to places of the cut are called *interface transitions* ($T2$, $T5$, $T3$, $T6$, $T19$, and $T14$ in Fig. 2.a). The goal is to add *something* (the shaded places in Figs. 2.b and 2.c) to each aggregable subsystem summarizing the behaviour of the other aggregable subsystem, leading to what we call the *aggregated subsystems*, $\mathcal{AS}_1$ and $\mathcal{AS}_2$ (Figs. 2.b and 2.c). More precisely, the objective is to obtain aggregated subsystems whose reachability sets are projections of the whole system reachability set on the corresponding preserved places.

The method to derive the aggregated subsystem from the aggregable one consists of two steps: (1) add a place to the aggregable subsystem from each interface transition t to each interface transition t' such that there exists a path from t to t' in the original net (see the shaded places in Fig. 2); and (2) compute the minimum initial marking of the places added in the previous step such that the reachability set of the aggregated subsystem is the projection of the whole system reachability set on the corresponding preserved places.

For the computation of the initial marking of the shaded places, let us consider the aggregable subsystem $\mathcal{N}_{A_i}$. We derive from this system a directed graph $G_{A_i} = (V, E)$ as follows. Each vertex corresponds to a transition of the net. There exists a directed arc between two vertices if and only if there exists a place in the net connecting the two transitions that represent the two vertices. The sense of the arc is the sense of the tokens' flow between the transitions through the place. Each arc has a non-negative cost equal to the initial marking of the place that represents. Moreover, we add an arc $t \to t$ for each vertex t with a cost equal to ∞.

If we apply the algorithm of R.W. Floyd to solve the *all-pairs shortest paths* problem [1] to the directed graph G_{A_i}, we obtain for each ordered pair of interface transitions (t, t') the smallest length of any path from t to t', denoted $\mathrm{length}(t, t')$ (if this value is equal to ∞, there is no path from t to t'). Observe, that $\mathrm{length}(t, t') = \min\{\sum_{p_j \in \pi} M_0[p_j] \mid \pi$ is a path from t to $t'\}$. The computational complexity of this algorithm is $O(m^3)$, where m is the number of transitions of the considered net. We add a place p from t to t' if $\mathrm{length}(t, t') \neq \infty$, with initial marking $m_0[p] = \mathrm{length}(t, t')$.

Additionally, a *basic skeleton* system $\mathcal{BS}$ can be defined including only the cut, the interface transitions, and the shaded places (Fig. 2.d).

For the previously explained decomposition, the following result can be proven:

Theorem 3.1 [6] *Let $\langle \mathcal{N}, M_0 \rangle$ be a strongly connected and live MG, $Q \subseteq P$ a cut of $\mathcal{N}$, and $\mathcal{AS}_i$ be the aggregated subsystems. Then, the reachability set of $\mathcal{AS}_i$ is the projection of that of $\langle \mathcal{N}, M_0 \rangle$ on the preserved places.*

Very informally, the above theorem states that the structural decomposition technique leads to "exact qualitative aggregation": firing sequences and reachable markings on aggregated subsystems are the projections of the original one (the same can be stated as a corollary for the basic skeleton).

3.2 Iterative approximation algorithm

The technique for an approximate computation of the throughput that we comment here is, basically, a *response time approximation* method (other techniques as subrogate delays or Marie's method could also be used). The interface transitions of $\mathcal{N}_j$ in $\mathcal{AS}_i$ approximate the response time of all the subsystem $\mathcal{N}_j$ $(i = 1, 2; j \neq i)$. A direct (non-iterative) method to compute the constant service rates of such interface transitions in order to represent the aggregation of the subnet gives, in general, low accuracy. Therefore, we are forced to define a *fixed-point search iterative process*, with the possible drawback of the presence of convergence and efficiency problems. Even if we have not a convergence formal proof, experimentally the method converged for all considered cases, and typically in a maximum of three to five iterations (i.e., very efficiently).

In the iterative algorithm, the CTMC's underlying both aggregated subsystems are solved alternatively until throughput convergence is achieved. Each time that an aggregated subsystem is solved, only its throughput and the ratios among the service rates of some interface transitions are estimated (see the details in [6]). After that, a scale factor for these service rates is computed, by using the basic skeleton system (in such a way that the throughput of the basic skeleton and the throughput of the aggregated subsystem, computed before, are the same). A linear search of the scale factor must be implemented, but in a net system with very few states (the basic skeleton). In each iteration of this linear search, the basic skeleton may be solved by deriving the underlying CTMC.

Let us consider again the MG depicted in Fig. 2.a. The exact value of the throughput is equal to 0.138341 (if single-server semantics is assumed). The approximated value, obtained after five iteration steps, is 0.138343 (the error is 0.064333%). The following additional fact must be remarked: the underlying CTMC's of $\mathcal{AS}_1$ (Figs. 2.b), $\mathcal{AS}_2$ (Figs. 2.c), and the basic skeleton (Fig. 2.d) have 8288, 3440, and 231 states, respectively, while the original MG has 89358 states.

4 Exact analysis

Product-form solutions for the exact steady-state distribution of very restricted classes of stochastic Petri nets have been developed in similar terms to the classical results existing for queueing networks. In some cases, the conditions for the existence of a product-form solution can be checked at the structural level. However, in order to build that solution, a state space level computation must be done in many cases (see in [14] a comparison of several approaches).

An alternative approach, based on the rewriting of the infinitesimal generator of the embedded Markov chain from a Kronecker expression in terms of the infinitesimal generators of smaller components of the system, has been presented in [13]. Even if the solution is based on the recognition of net components that interact through transition synchronization, the method cannot be considered as a fully structural technique.

In this section, we present an efficient exact solution for a particular Petri net subclass mainly based on structural concepts. The presentation is deliberately more technical than before just to present a more detailed flash of concepts and their interleaving leading to new techniques. We consider stochastic Petri net systems with exponential service times constructed from cooperating sequential components (state machines) communicating with buffers (places). Some interesting qualitative results that can be derived from structure of these nets are presented in section 4.1. In particular, necessary and sufficient conditions for "potential ergodicity" (existence of a stochastic interpretation that can lead to ergodic systems) are studied. In section 4.2, necessary and sufficient ergodicity conditions and the steady-state performance measures are computed.

First, we recall the formal definition of the subclass we are going to deal with.

Definition 4.1 *A Petri net system* $\langle \mathcal{N}, M_0 \rangle = \langle \cup_{i=1}^{q} P_i \cup B, \cup_{i=1}^{q} T_i, Pre, Post, M_0 \rangle$ *is a deterministic system of sequential processes (deterministic system, DS for short) iff*

i) $P_i \cap P_j = \emptyset, T_i \cap T_j = \emptyset, P_i \cap B = \emptyset, i,j = 1,\ldots,q; i \neq j$;

ii) $\langle \mathcal{SM}_i, M_{0i} \rangle = \langle P_i, T_i, Pre_i, Post_i, M_{0i} \rangle, i = 1,\ldots,q$, *are strongly connected and 1–bounded state machines, where* Pre_i,$Post_i$ *and* M_{0i} *are the restrictions of* Pre, $Post$ *and* M_0 *to* P_i *and* T_i*; and*

iii) the set B *of buffers is such that* $\forall b \in B$:

  a) $^\bullet b \neq \emptyset$, $b^\bullet \neq \emptyset$;

  b) $\exists i,j \in \{1,\ldots,q\}, i \neq j$ *such that* $^\bullet b \subset T_i$ *and* $b^\bullet \subset T_j$*; and*

  c) $\forall p \in \cup_{i=1}^{q} P_i : t,t' \in p^\bullet \Rightarrow Pre(b,t) = Pre(b,t')$.

A totally open deterministic system of sequential processes (totally open system, TOS for short) is a DS without circuits containing buffers.

The first two items of the previous definition state that a DS is composed by a set of state machines ($\mathcal{SM}_i, i = 1,\ldots,q$) and a set of buffers ($B$). By item *iii.a*, buffers are neither source nor sink places. An input and output-private condition for buffers is expressed by *iii.b*. Due to requirement *iii.c*, the marking of buffers does not disturb the decisions taken by a state machine (this fact justifies the word "deterministic" in the name of the class).

4.1 Structural concepts

Some interesting qualitative results can be derived from the structure of these nets. In theorem 4.1, consistency (necessary condition for ergodicity) is shown to collapse with existence of home state for this subclass of nets. The rest of this section is devoted to the study of necessary and sufficient conditions for the "potential ergodicity": existence of a stochastic interpretation that can lead to ergodic systems.

The following theorem relates, for TOS's, a behavioural property (existence of home state) with a structural one (consistency).

Theorem 4.1 [9] *Let* $\langle \mathcal{N}, M_0 \rangle$ *be a TOS. Then* $\mathcal{N}$ *is consistent iff* M_0 *is a home state.*

Consistency is known to be a necessary condition for the marking ergodicity of a live stochastic Petri net system. Since there exist non consistent TOS's, it is convenient to check consistency (a polynomial time computation) of the TOS before computing ergodicity conditions. Taking into account the above remark, the following result with practical interest can be stated:

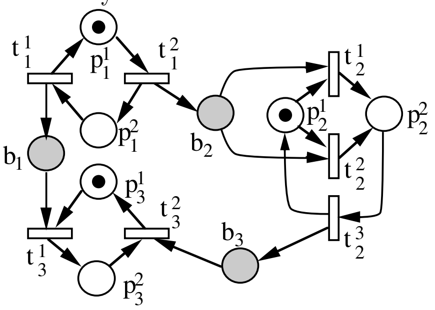
Theorem 4.2 [9] *There exist TOS's that are marking non ergodic for all timing interpretation. In particular, non-consistent systems are always marking non-ergodic.*

Unfortunately, it cannot be stated that for all consistent TOS, there exists an stochastic interpretation such that the resulting system is ergodic.

Let us now recall the concept of *global synchronic distance relation.* If two subsets of transitions are in global synchronic distance relation then it is not possible to fire an infinite number of times some transition of the first subset without firing any transition of the second subset, and vice versa. Even more, if two subsets of transitions are in global synchronic distance relation they behave like if they were included in a *regulation circuit* [18, 24]. This equivalence relation is used below for finding necessary and sufficient conditions for the existence of a stochastic interpretation that makes ergodic a TOS.

Definition 4.2 *Let* $\langle \mathcal{N}, M_0 \rangle$ *be a Petri net and* T_1, T_2 *subsets of transitions.* T_1, T_2 *are in global synchronic*

Figure 3: A TOS that admits stochastic timings making ergodic the system.



distance relation, $(T_1, T_2) \in SDR$*, iff* $\exists W_1, W_2 \in \mathbb{N}^m$ *vectors which express the weights associated with the transitions of the subsets* T_1 *and* T_2 *(i.e.,* $||W_1|| = T_1$ *and* $||W_2|| = T_2$*) and* $\exists k \in \mathbb{N}$ *such that*

$$\sup_{\substack{\sigma \in L(\mathcal{N}, M) \\ M \in R(\mathcal{N}, M_0)}} |(W_1 - W_2)^T \cdot \vec{\sigma}| \leq k$$

For characterizing the possible existence of a timing interpretation making a given TOS ergodic, let us give local rules that will be composed step by step for a large system. The first one gives necessary and sufficient conditions for the existence of a stochastic timing interpretation that makes ergodic a system composed by two state machines.

Theorem 4.3 [9] *Let* $\langle \mathcal{N}, M_0 \rangle = \langle P_1 \cup P_2 \cup B, T_1 \cup T_2, Pre, Post, M_0 \rangle$ *be a TOS composed by two state machines and a set of buffers* B *such that* $\forall b \in B$: ${}^\bullet b \subseteq T_1, b^\bullet \subseteq T_2$. *Then, there exists a stochastic interpretation making marking ergodic the system if and only if: (i)* $\mathcal{N}$ *is consistent and (ii)* $\forall b_i, b_j \in B$: $({}^\bullet b_i, {}^\bullet b_j) \in SDR$ *and* $(b_i{}^\bullet, b_j{}^\bullet) \in SDR$.

In case of TOS's composed by two state machines, if (i) and (ii) of the above theorem hold then the marking of all the buffers can be always computed from the marking of one buffer and the marking of the state machines. With the object of computing ergodicity conditions for a larger system including $\mathcal{N}$ as a subsystem, if (i) and (ii) hold, from the performance point of view, we can suppose without loss of generality that the two state machines are communicating with at most one buffer.

Let us now give the "transitivity rule" for three state machines communicating with buffers like in Fig. 3. This rule completes the stating of necessary and sufficient conditions for the existence of a stochastic timing that makes ergodic a given TOS.

Theorem 4.4 [9] *Let* $\langle \mathcal{N}, M_0 \rangle = \langle \cup_{i=1}^3 P_i \cup \cup_{i=1}^3 \{b_i\}, \cup_{i=1}^3 T_i, Pre, Post, M_0 \rangle$ *be a TOS composed by three state machines and three buffers such that* ${}^\bullet b_1 \subseteq T_1, b_1{}^\bullet \subseteq T_3, {}^\bullet b_2 \subseteq T_1, b_2{}^\bullet \subseteq T_2, {}^\bullet b_3 \subseteq T_2$*, and* $b_3{}^\bullet \subseteq T_3$. *Then, there exists a stochastic interpretation making ergodic the system if and only if: (i)* $\mathcal{N}$ *is consistent and (ii)* ${}^\bullet b_1, {}^\bullet b_2) \in SDR$*,* $(b_2{}^\bullet, {}^\bullet b_3) \in SDR$*,* $(b_1{}^\bullet, b_3{}^\bullet) \in SDR$.

If (i) and (ii) of the above theorem hold, then the marking of b_3 can be always computed from the marking of b_1, b_2 and the marking of the state machines. With the object of computing conditions for a larger system including $\mathcal{N}$ as a subsystem, if (i) and (ii) hold, the state machine $\mathcal{SM}_2$ and the buffers b_2, b_3 can be substituted by a unique buffer.

Theorems 4.3 and 4.4 provide rules for an iterative reduction of buffers of a TOS. These rules preserve the possibility of existence of a timing that makes the system ergodic if the necessary and sufficient conditions (stated in the mentioned theorems) hold.

Therefore the existence of a stochastic timing that makes a TOS ergodic is characterized in terms of pure structural conditions: consistency and some synchronic distance relations.

4.2 Ergodicity characterization and steady-state solution

Let us now consider a different problem: given a TOS, once that the previous conditions have been checked, we want to know if ergodicity conditions hold for a given timing. We study these conditions and the steady-state throughput of the system under ergodicity assumption.

In [15], an ergodicity theorem is proved for a particular class of open synchronized queueing networks. Let us recall now the concept of *saturated net* and the adaptation of the above-mentioned theorem for TOS's.

Definition 4.3 *Let* $\langle \mathcal{N}, M_0 \rangle$ *be an and* b *one of its buffers. The net obtained from* $\langle \mathcal{N}, M_0 \rangle$ *by deleting the buffer* b *and its adjacent arcs is called the saturated system according to* b.

Note that the saturated system according to b behaves like $\langle \mathcal{N}, M_0 \rangle$ in the case in which the buffer b is always marked.

Theorem 4.5 [15] *Let* $\langle \mathcal{N}, M_0 \rangle$ *be a TOS, and* $\vec{\sigma}^*_{(b)}$ *be the limit vector of transition throughputs of the saturated net according to* b.

1. *Let $B' \subseteq B$ be the subset of buffers the marking of which can vary independently. If $POST[b]\cdot\vec{\sigma}^*_{(b)} < PRE[b]\cdot\vec{\sigma}^*_{(b)}$, $\forall b \in B'$, then the associated Markov process is positive recurrent.*

2. *If there exists a buffer b such that $POST[b]\cdot\vec{\sigma}^*_{(b)} > PRE[b]\cdot\vec{\sigma}^*_{(b)}$, then the associated Markov process is transient.*

Part 1 of theorem 4.5 means that for each buffer the input flow must be less than the service rate of the output state machine.

Let us illustrate the numerical computation of the ergodicity criterion with the net in Fig. 3. The left and right hand side expressions of theorem 4.5.1 for buffer b_2 can be computed considering the state machines $\mathcal{SM}_1$ and $\mathcal{SM}_2$ in isolation:

$$POST[b_2]\cdot\vec{\sigma}^*_{(b_2)}=\frac{\lambda_1^1\lambda_1^2}{\lambda_1^1+\lambda_1^2};\;\; PRE[b_2]\cdot\vec{\sigma}^*_{(b_2)}=\frac{(\lambda_2^1+\lambda_2^2)\lambda_2^3}{\lambda_2^1+\lambda_2^2+\lambda_2^3}$$

where λ_j^i is the rate of the exponentially distributed random variable associated with transition t_j^i. We assume that the conflict at place p_2^1 is solved in favour of transition t_2^1 with probability $\lambda_2^1/(\lambda_2^1+\lambda_2^2)$ and in favour of t_2^2 with probability $\lambda_2^2/(\lambda_2^1+\lambda_2^2)$.

The same computation for the buffer b_1 leads to the expressions:

$$POST[b_1]T\cdot\vec{\sigma}^*_{(b_1)}=\frac{\lambda_1^1\lambda_1^2}{\lambda_1^1+\lambda_1^2};\;\; PRE[b_1]T\cdot\vec{\sigma}^*_{(b_1)}=\frac{\lambda_3^1\lambda_3^2}{\lambda_3^1+\lambda_3^2}$$

The marking of buffer b_3 linearly depends on the marking of the other buffers, so it must not be considered. Then, the system is ergodic if and only if:

$$\frac{\lambda_1^1\lambda_1^2}{\lambda_1^1+\lambda_1^2}<\min\left\{\frac{(\lambda_2^1+\lambda_2^2)\lambda_2^3}{\lambda_2^1+\lambda_2^2+\lambda_2^3},\frac{\lambda_3^1\lambda_3^2}{\lambda_3^1+\lambda_3^2}\right\}$$

Let us suppose now that the system ergodicity conditions given in theorem 4.5 are satisfied. The following theorem gives a method for computing efficiently the steady-state behaviour of the connected machines of a TOS. The idea is as follows. A partial order relation can be introduced on the set of strongly connected components of the system (set of sequential processes) as follows: a sequential process $\mathcal{SM}_i$ is "greater" than other $\mathcal{SM}_j$ iff there exists a directed path from nodes of $\mathcal{SM}_i$ to nodes of $\mathcal{SM}_j$. Maximal elements of this partial order are sequential processes that have not any input buffer.

Theorem 4.6 [9] *Let $\langle\mathcal{N}, M_0\rangle$ be a TOS. If its marking process is ergodic then:*

i) If $\mathcal{SM}_i$ has not any input buffer then the limit average marking of each place and the limit throughput of transitions can be computed solving the following marking invariant and flow equations:

$$\begin{array}{l}\sum_{p\in P_i}\overline{M}(p)=1;\\ \sigma^*(t)=\lambda_t\overline{M}(p),\;\; \mathit{if}\; Pre(p,t)=1,\forall t\in T_i\\ \sum_{t\in{}^\bullet p}\sigma^*(t)=\sum_{t\in p^\bullet}\sigma^*(t),\forall p\in P_i\end{array}$$

where $\sigma^*(t)$ is the limit throughput of transition t, λ_t is the rate of the exponentially distributed random variable associated with t, and $\overline{M}(p)$ is the limit average marking of place p.

ii) If $\mathcal{SM}_j$ has input buffers that are output buffers of the state machines $\mathcal{SM}_{i_1},\ldots,\mathcal{SM}_{i_r}$ then the limit average marking of each place and the limit throughput of transitions can be computed solving the equations:

$$\begin{array}{l}\sum_{p\in P_j}\overline{M}(p)=1;\\ \sigma^*(t)=\lambda_t\overline{M}(p),\;\; \mathit{if}\; Pre(p,t)=1,\forall t\in T_j:{}^\bullet t\cap B=\emptyset;\\ \sum_{t\in p^\bullet}\sigma^*(t)=\sum_{t'\in{}^\bullet p}\sigma^*(t'),\forall b\in B:b^\bullet\subset T_j\wedge{}^\bullet b\subset T_{i_1}\cup\ldots\cup T_{i_r};\\ \sum_{t\in{}^\bullet p}\sigma^*(t)=\sum_{t\in p^\bullet}\sigma^*(t),\forall p\in P_j\end{array}$$

We remark that the application of the method described in the above theorem has polynomial time complexity.

As an example, let us consider once more the net in Fig. 3. In this case, there exists one state machine without input buffers: $\mathcal{SM}_1$. Marking invariant and flow equations for this machine have the form:

$$\begin{array}{l}\overline{M}(p_1^1)+\overline{M}(p_1^2)=1;\\ \sigma^*(t_1^1)=\lambda_1^1\overline{M}(p_1^2);\; \sigma^*(t_1^2)=\lambda_1^2\overline{M}(p_1^1);\; \sigma^*(t_1^1)=\sigma^*(t_1^2)\end{array}$$

This system can be solved, obtaining:

$$\begin{array}{c}\overline{M}(p_1^1)=\frac{\lambda_1^1}{\lambda_1^1+\lambda_1^2};\; \overline{M}(p_1^2)=\frac{\lambda_1^2}{\lambda_1^1+\lambda_1^2};\\ \sigma^*(t_1^1)=\sigma^*(t_1^2)=\frac{\lambda_1^1\lambda_1^2}{\lambda_1^1+\lambda_1^2}\end{array}$$

Now, for computing the steady-state measures of the other state machines under the assumption of ergodicity, it is necessary to take into account that

$$\sigma^*(t_1^2)=\sigma^*(t_2^1)+\sigma^*(t_2^2)\quad\text{and}\quad\sigma^*(t_3^1)=\sigma^*(t_1^1)$$

that is, the input flow of tokens to each buffer in steady-state must be equal to the output flow, and:

$$\begin{array}{l}\overline{M}(p_2^1)=1-\overline{M}(p_2^2);\quad \overline{M}(p_2^2)=\frac{\lambda_1^1\lambda_1^2}{(\lambda_1^1+\lambda_1^2)\lambda_2^3};\\ \sigma^*(t_2^1)=\frac{\lambda_1^1\lambda_1^2\lambda_2^1}{(\lambda_1^1+\lambda_1^2)(\lambda_2^1+\lambda_2^2)};\quad \sigma^*(t_2^2)=\frac{\lambda_1^1\lambda_1^2\lambda_2^2}{(\lambda_1^1+\lambda_1^2)(\lambda_2^1+\lambda_2^2)};\\ \sigma^*(t_2^3)=\sigma^*(t_3^1)=\sigma^*(t_3^2)=\frac{\lambda_1^1\lambda_1^2}{\lambda_1^1+\lambda_1^2}\end{array}$$

Thus, the throughput of transitions and the average marking of places can be computed in polynomial time on the net size.

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