Citation: D. Paesa, C. Franco, S. Llorente, G. López-Nicolás, and C. Sagüés. On robust PI Adaptive Observers for Nonlinear Uncertain Systems with Bounded Disturbances. Mediterranean Conference on Control and Automation, pp. 1031-1036. Marrakech-Morocco, June 23-25, 2010

On Robust PI Adaptive Observers for Nonlinear Uncertain Systems with Bounded Disturbances

D. Paesa, C. Franco, S. Llorente, G. Lopez-Nicolas and C. Sagues

***Abstract*— Robust adaptive observers for uncertain systems corrupted by bounded disturbances have to overcome two main problems. They have to guarantee a bounded parameter estimation, because it is well known that an arbitrarily small disturbance might drift to infinity the parameter estimation error even if the state estimate error remains bounded, and they have to ensure a robust state and parameter estimation independently of the uncertainty of the system. The contribution of this paper is a solution of both issues. We present a robust adaptive observer based on a normalized dead zone which guarantees a bounded estimate dealing with noise corrupted systems and on an adaptive gain which increases the robustness against uncertain systems. Tuning parameters are computed minimizing the effect of disturbances on the estimation error. The performance and the stability of the proposed adaptive observer is analyzed and demonstrated through simulation examples.**

I. Introduction

An adaptive observer is an algorithm used to estimate the state as well as the unknown parameters of a system from the information available (e.g. system input and output measurements). Therefore, this sort of algorithms represents a useful tool in order to cope with problems that may appear in any industrial application. For instance, they can be used to deal with systems whose parameters are initially unknown due to modeling uncertainties and also to handle systems whose parameters are time variant. Additionally, they have important applications not only in adaptive control but also in fault detection and isolation.

Adaptive observers were applied to linear time invariant systems at first [1], [2], and afterwards to nonlinear time variant systems [3], [4], [5]. All those works were characterized by having only a proportional feedback term of the output observation error in both state observer and parameter adaptation law. This proportional approach ensures a bounded estimation of the state and the unknown parameter, assuming a persistent excitation condition as well as the lack of disturbances.

Some authors have proposed adding an additional integral term in the state adaptive law in order to improve the steady state accuracy and increase the robustness against modeling errors and disturbances. Such adaptive observers are known as Proportional Integral Adaptive Observers. They were at first applied to linear systems [6], [7], [8] and recently to a class of nonlinear systems [9].

This work was supported in part by DGA project PI065/09 and BSH Home Appliances Group.

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Nevertheless, all these adaptive observers can be unstable in the presence of disturbances. As it is shown in [10] a bounded disturbance might drive to infinity the parameter estimation of an adaptive observer even if the state estimation error remains bounded. Several modifications to the adaptive law for updating unknown parameters have been proposed in order to overcome this drawback. All these new methods are based on including an operator that constrains the parameter estimation inside a bounded set [11], [10], [12], [9]. The definition of this bounded set is based on the maximum difference between the parameter estimation and the nominal value of the parameter. Therefore, prior knowledge of the system is required in order to define a proper nominal parameter value. However, although the parameter estimation error is bounded, it does not mean that the parameter estimation performance is reasonable. In [9] it is pointed out that adaptive laws with projection operators may estimate parameters completely different than the real values.

This paper proposes a robust adaptive observer based on a dead zone rather than on the addition of a projection operator and therefore it does not need any prior knowledge of the variation range of the unknown parameter. It guarantees a bounded parameter estimate if the system is corrupted by bounded disturbances. Additionally, our proposal introduces an adaptive gain in order to increase the robustness of the estimate independently of the system uncertainty.

This paper is organized as follows. In Section II, the non linear system formulation is used as starting point to develop the proposed robust adaptive observer. Section III shows the convergence and stability analysis of our proposal even if the system is corrupted by bounded disturbances. Additionally, a method to compute the optimal parameters of the adaptive observer minimizing the $\mathcal{L}_2$ gain is described. Some simulation examples are presented in Section IV in order to test the robustness of our proposed adaptive scheme. Finally, concluding remarks are outlined in Section V.

II. Problem Statement

In this paper we address the problem of designing a robust adaptive observer for uncertain nonlinear systems which can be described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \Delta\phi(t, y, u)\theta + B_w w(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^l$ is the input vector, $y(t) \in \mathbb{R}^m$ is the output vector, $\theta \in \mathbb{R}^p$

is the unknown constant parameter vector which can be used to represent modeling uncertainties, $w(t) \in \mathbb{R}^n$ is the disturbance vector, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{m\times n}$, $\Delta \in \mathbb{R}^{n\times m}$, and $B_w \in \mathbb{R}^{n\times n}$ are known constant matrices. The nonlinearity $\phi(t,y,u) \in \mathbb{R}^{m\times p}$ is a time-varying matrix which depends on the input $u(t)$ and/or the output $y(t)$. In addition, $u(t)$ and $\phi(t,y,u)$ are assumed persistently exciting, and the pair (A,C) is assumed completely observable. As it is shown in [13], [14], general nonlinear systems can be formulated as a system described by (1) through a change of coordinates.

Let us consider the following robust adaptive observer for systems described by (1)

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + \Delta\phi(t,y,u)\hat{\theta} \\ &\quad + K_P(y(t)-\hat{y}) + K_I z(t) \\ \hat{y}(t) &= C\hat{x}(t) \\ \dot{z}(t) &= A_z z(t) + y(t) - \hat{y}(t) \end{aligned} \tag{2}$$

where, $K_P \in \mathbb{R}^{n\times m}$ and $K_I \in \mathbb{R}^{n\times m}$ can be regarded as the proportional observer gain and the integral observer gain respectively. Both have to be tunned depending on the performance specifications. The vector $z(t)$ represents the integral of the output estimation error whose time response can be modified using $A_z \in \mathbb{R}^{m\times m}$ which is a diagonal exponentially stable matrix.

We propose a parameter adaptive law of the robust adaptive observer (2) which is based on the addition of a dead zone and an adaptive gain. On one hand, the dead zone ensures that the parameter estimate is adjusted only when the normalized estimation error is larger than a predetermined value. Therefore it guarantees a bounded estimate dealing with noise corrupted systems. On the other hand the adaptive gain increases the robustness of the estimate coping with uncertain systems. Unlike other proposals, like projection operators [9], [10], our adaptive law does not need any prior knowledge of the system. The proposed parameter adaptive law dynamic is described by

$$\begin{aligned} \dot{\hat{\theta}}(t) &= \Gamma(t)\phi^T(t)C^T(y(t)-\hat{y}(t)) \quad & \epsilon > \delta \\ \dot{\hat{\theta}}(t) &= 0 & \epsilon \le \delta \end{aligned} \tag{3}$$

where $\delta \in \mathbb{R}$ denotes the threshold of the dead zone, $\epsilon = (|y-\hat{y}|)/(|y|+b)$ is the normalized estimation error, $\Gamma \in \mathbb{R}^{p\times p}$ is an adaptive gain and b is the normalization bias, which is typically chosen to be a small positive constant that prevents potential division by zero.

Adaptive gain dynamics [15], [16](page 169), are generally described by

$$\dot{\Gamma}(t) = A_\Gamma(t)\Gamma(t) - \Gamma(t)\Omega^T(t)\Omega(t)\Gamma(t), \tag{4}$$

where A_Γ is a matrix that regulates the transient response of $\Gamma(t)$ whereas $\Omega(t)$ is a diagonal positive matrix which depends on $\phi(t)$ and consequently, it depends also on $u(t)$ and/or $y(t)$. Specifically, we use an adaptive gain with exponential forgetting factor proposed by [15] whose dynamics are

$$\begin{aligned} \dot{\Gamma}(t) &= \lambda\Gamma(t) - \Gamma(t)y_\Omega^T(t)C^TCy_\Omega(t)\Gamma(t) \\ (\dot{\Gamma^{-1}})(t) &= -\lambda\Gamma^{-1}(t) + y_\Omega^T(t)C^TCy_\Omega(t) \end{aligned} \tag{5}$$

where the forgetting factor $\lambda > 0$ is a scalar real number, y_Ω is a dynamic transformation obtained as $\dot{y}_\Omega = (A-K_P)y_\Omega + \Delta\phi$ and the initial value $\Gamma(t_0)$ of the adaptive gain is positive definite.

III. Stability and Convergence Analysis

Let us begin analyzing the error system dynamics which can be obtained subtracting (2) from (1). Then, the state error dynamic $\tilde{x} = x - \hat{x}$ is defined by

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + \Delta\phi(t)\tilde{\theta} - K_PC\tilde{x}(t) \\ &\quad - K_I z(t) + B_w w(t) \\ \tilde{y}(t) &= C\tilde{x}(t) \end{aligned} \tag{6}$$

while the parameter error $\tilde{\theta} = \theta - \hat{\theta}$ is described by

$$\begin{aligned} \dot{\tilde{\theta}}(t) &= -\Gamma(t)\phi^T(t)(y(t)-\hat{y}(t)) \quad & \epsilon > \delta \\ \dot{\tilde{\theta}}(t) &= 0 & \epsilon \le \delta \end{aligned} \tag{7}$$

A. Stability properties

Now, we present computable sufficient conditions for $\mathcal{L}_2$ stability via quadratic Lyapunov functions based on a linear matrix inequality approach.

Theorem 1. *For given A_z, K_P, K_I and by introducing the variable $A_x = A - K_PC$, the error dynamics shown in (6) and (7) are quadratically stable and have a $\mathcal{L}_2$ gain from w to $\tilde{y}$ which is smaller than γ, if there exist $P = P^T > 0$, $Q = Q^T > 0$, and $\gamma > 0$ subject to*

$$\begin{bmatrix} A_x^TP + PA_x + C^TC + \beta I & C^TQ - PK_I & P \\ QC - K_I^TP & A_z^TQ + QA_z & 0 \\ P & 0 & -\gamma^2 I \end{bmatrix} < 0, \tag{8}$$

$P\Delta = C^T$, and to $\lambda\Gamma^{-1} > y_\Omega^TC^TCy_\Omega$.

Proof. Let us consider the following quadratic Lyapunov function for the error dynamics described by (6) and (7):

$$V(\tilde{x},z,\tilde{\theta}) = \tilde{x}^TP\tilde{x} + z^TQz + \tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \tag{9}$$

where $P = P^T > 0$, $Q = Q^T > 0$ and $\Gamma = \Gamma^T > 0$. The next step is to prove the stability of the error dynamics (6)-(7) and to ensure that the $\mathcal{L}_2$ gain of our proposed adaptive observer is no more than γ, which guarantees that all estimates are bounded. Let us define the $\mathcal{L}_2$ gain of the adaptive observer as:

$$\|\tilde{y}\|_2 < \gamma\|w\|_2 \tag{10}$$

Rearranging terms, (10) can be written as:

$$\tilde{y}^T\tilde{y} - \gamma^2 w^Tw < 0 \tag{11}$$

Consequently, to check the quadratically stability condition of our robust adaptive observer and to bound the $\mathcal{L}_2$

gain of the error dynamics (6) and (7), we have to ensure that the inequality

$$\dot{V} + \tilde{y}^T\tilde{y} - \gamma^2 w^T w < 0 \tag{12}$$

is satisfied inside and outside of the dead zone. Let us begin analyzing the behavior of (12) outside the dead zone. Let us define $V_{\tilde{x}} = \tilde{x}^T P\tilde{x}$, $V_z = z^T Qz$ and $V_{\tilde{\theta}} = \tilde{\theta}^T\Gamma^{-1}\tilde{\theta}$. Then, let us take derivative of (9) to obtain

$$\dot{V} = \dot{V_{\tilde{x}}} + \dot{V_z} + \dot{V_{\tilde{\theta}}} \tag{13}$$

where $\dot{V_{\tilde{x}}}$ is defined as

$$\begin{aligned}\dot{V_{\tilde{x}}} &= \tilde{x}^T((A-K_PC)^TP + P(A-K_PC))\tilde{x} \\ &+\tilde{x}^TPw + w^TP\tilde{x} - \tilde{x}^TPK_Iz - z^TK_I^TP\tilde{x} \\ &+\tilde{x}^TP\Delta\phi\tilde{\theta} + \tilde{\theta}^T\phi^T\Delta^TP\tilde{x}\end{aligned} \tag{14}$$

while $\dot{V_z}$ is defined as

$$\begin{aligned}\dot{V_z} &= \tilde{x}^TC^TQz + z^TQC\tilde{x} \\ &+z^TA_z^TQz + z^TQA_zz\end{aligned} \tag{15}$$

and finally, $\dot{V_{\tilde{\theta}}}$ is defined as

$$\begin{aligned}\dot{V_{\tilde{\theta}}} &= -\tilde{x}^TC^T\phi\Gamma\Gamma^{-1}\tilde{\theta} - \tilde{\theta}^T\Gamma^{-1}\Gamma\phi^TC\tilde{x} \\ &+\tilde{\theta}^T(\dot{\Gamma^{-1}})\tilde{\theta} \\ &= -\tilde{x}^TC^T\phi\tilde{\theta} - \tilde{\theta}^T\phi^TC\tilde{x} \\ &+\tilde{\theta}^T(-\lambda\Gamma^{-1} + y_\Omega^TC^TCy_\Omega)\tilde{\theta}\end{aligned} \tag{16}$$

Rearranging terms and using $P\Delta = C^T$ and $\lambda\Gamma^{-1} > y_\Omega^TC^TCy_\Omega$, the time derivative of the Lyapunov function candidate (9) outside the dead zone is negative semi-definite if the following inequality holds.

$$\begin{aligned}&\tilde{x}^T((A-K_PC)^TP + P(A-K_PC))\tilde{x} + \tilde{x}^TPw \\ +\ & w^TP\tilde{x} - \tilde{x}^TPK_Iz - z^TK_I^TP\tilde{x} + \tilde{x}^TC^TQz \\ +\ & z^TQC\tilde{x} + z^TA_z^TQz + z^TQA_zz \\ +\ & \tilde{y}^T\tilde{y} - \gamma^2w^Tw < 0\end{aligned} \tag{17}$$

By introducing $A_x = A - K_PC$ and $\tilde{y}^T\tilde{y} = \tilde{x}^TC^TC\tilde{x}$, the above inequality can be rewritten as

$$\begin{bmatrix} \tilde{x} \\ z \\ w \end{bmatrix}^T \cdot \begin{bmatrix} A_x^TP + PA_x + C^TC & C^TQ - PK_I & P \\ QC - K_I^TP & A_z^TQ + QA_z & 0 \\ P & 0 & -\gamma^2I \end{bmatrix} \cdot \begin{bmatrix} \tilde{x} \\ z \\ w \end{bmatrix} < 0 \tag{18}$$

and consequently, it can be also rearranged as an equivalent LMI problem over the variables P and Q as follows

$$\begin{bmatrix} A_x^TP + PA_x + C^TC & C^TQ - PK_I & P \\ QC - K_I^TP & A_z^TQ + QA_z & 0 \\ P & 0 & -\gamma^2I \end{bmatrix} < 0. \tag{19}$$

Then (9) is a Lyapunov function when (19) holds. Consequently, the error dynamics (6) and (7) are stable outside the dead zone.

Let us continue analyzing the behavior of our Lyapunov function (9) inside the dead zone. In this case $\dot{V_{\tilde{x}}}$ is defined as

$$\begin{aligned}\dot{V_{\tilde{x}}} &= \tilde{x}^T((A-K_PC)^TP + P(A-K_PC))\tilde{x} \\ &+\tilde{x}^TPw + w^TP\tilde{x} - \tilde{x}^TPK_Iz - z^TK_I^TP\tilde{x} \\ &+\tilde{x}^TP\Delta\phi\tilde{\theta} + \tilde{\theta}^T\phi^T\Delta^TP\tilde{x}\end{aligned} \tag{20}$$

while $\dot{V_z}$ is defined as

$$\begin{aligned}\dot{V_z} &= \tilde{x}^TC^TQz + z^TQC\tilde{x} \\ &+z^TA_z^TQz + z^TQA_zz\end{aligned} \tag{21}$$

and finally, $\dot{V_{\tilde{\theta}}}$ is defined as

$$\dot{V_{\tilde{\theta}}} = \tilde{\theta}^T(-\lambda\Gamma^{-1} + y_\Omega^TC^TCy_\Omega)\tilde{\theta} \tag{22}$$

Rearranging terms and using $P\Delta = C^T$, $\left\|\phi\tilde{\theta}\right\| < \beta\left\|\tilde{x}\right\|$ and $\lambda\Gamma^{-1} > y_\Omega^TC^TCy_\Omega$, the time derivative of the Lyapunov function candidate (9) inside the dead zone is negative semi-definite if the following inequality holds.

$$\begin{aligned}&\tilde{x}^T((A-K_PC)^TP + P(A-K_PC))\tilde{x} + \tilde{x}^TPw \\ +\ & w^TP\tilde{x} - \tilde{x}^TPK_Iz - z^TK_I^TP\tilde{x} + \tilde{x}^TC^TQz \\ +\ & z^TQC\tilde{x} + z^TA_z^TQz + z^TQA_zz \\ +\ & \tilde{y}^T\tilde{y} - \gamma^2w^Tw + \beta\tilde{x}^T\tilde{x} < 0\end{aligned} \tag{23}$$

By introducing $A_x = A - K_PC$ and $\tilde{y}^T\tilde{y} = \tilde{x}^TC^TC\tilde{x}$, the above inequality can be also rearranged as an equivalent LMI problem over the variables P and Q as follows

$$\begin{bmatrix} A_x^TP + PA_x + C^TC + \beta I & C^TQ - PK_I & P \\ QC - K_I^TP & A_z^TQ + QA_z & 0 \\ P & 0 & -\gamma^2I \end{bmatrix} < 0. \tag{24}$$

Then (9) is a Lyapunov function when (24) is satisfied. Consequently, the error dynamics (6) and (7) are stable inside the dead zone.

Since (19) is always satisfied if (24) holds. We can ensure the stability of our proposed robust adaptive observer inside and outside the dead zone by solving the LMI problem (24). □

B. Optimal gain design

Since our proposed adaptive observers involves several tuning matrices, we give a method to compute the optimal parameters by minimizing the effect of disturbances on the estimation error. That is, minimizing the $\mathcal{L}_2$ gain in (8).

Theorem 2. *There exist optimal parameters K_P, K_I and A_z that minimize the $\mathcal{L}_2$ gain from w to $\tilde{y}$ if there exist the matrices $P = P^T > 0$, $Q = Q^T > 0$, R, S, $U = U^T < 0$ and the scalar $\gamma > 0$ subject to*

$$\begin{bmatrix} \Xi + \beta I - R^TC - C^TR & C^TQ - S & P \\ QC - S^T & 2U & 0 \\ P & 0 & -\gamma^2I \end{bmatrix} < 0, \tag{25}$$

$P\Delta = C^T$, $\lambda\Gamma^{-1} > y_\Omega^T C^T C y_\Omega$, *and to* $\Xi = A^T P + PA + C^T C$.

Proof. By introducing the following new variables $R = K_P^T\, P$, $S = P\, K_I$, $U = A_z\, Q$ and substituting them into (23), the following inequality is obtained:

$$\begin{aligned} & \tilde{x}^T(A^T P + PA - R^T C - C^T R)\tilde{x} + \tilde{x}^T P w \\ + \quad & w^T P \tilde{x} - \tilde{x}^T S z - z^T S^T \tilde{x} + \tilde{x}^T C^T Q z \\ + \quad & z^T Q C \tilde{x} + z^T U z + z^T U^T z \\ + \quad & \tilde{y}^T \tilde{y} - \gamma^2 w^T w + \beta \tilde{x}^T \tilde{x} < 0 \end{aligned} \tag{26}$$

which can be straightforwardly rearranged as the minimization LMI problem shown in (25) completing the proof. Notice that once the solution of the minimization problem has been obtained, the resulting optimal parameters can be computed as $K_P = P^{-1}\, R^T$, $K_I = P^{-1}\, S$, $A_z = V\, Q^{-1}$. □

IV. Simulation Results

This section aims to show the potential benefit of combining a dead zone with an adaptive gain in the same parameter adaptive law. The effectiveness of our proposed robust adaptive observer is shown through two simulation examples. The first example was proposed by Jung *et al.* [9], there, the authors pointed out that some adaptive observer may estimate a non bounded parameter error $\tilde{\theta}(t)$ although the system disturbance $w(t)$, the system input $u(t)$ and the state error $\tilde{x}(t)$ are bounded. The second one is a fourth order nonlinear system with disturbances whose uncertain parameter vector is time varying. In this case, our adaptive observers has to achieve a good performance for any value of the uncertain parameter.

A. Jung Example [9]

Let us consider the scalar system

$$\begin{aligned} \dot{x}_1(t) &= -x_2(t) + y^3(t)\theta + 0.1 w(t) \\ \dot{x}_2(t) &= -x_1(t) - 2x_2(t) + 0.1 w(t) \\ y(t) &= x_1(t) \end{aligned} \tag{27}$$

with $x(t=0) = [-0.5, 0.2]$, $\theta = 1.1$, and $w(t) = sin(0.5t)$. Solving the $\mathcal{L}_2$ minimization problem shown in (26) we obtain the optimal matrices of our observer:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0.1256 \end{bmatrix}$$

$$S = \begin{bmatrix} 0.6392 \\ 0.0019 \end{bmatrix}$$

$$R = \begin{bmatrix} 9.4211 & 0.8719 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.8459 \end{bmatrix}$$

$$U = \begin{bmatrix} -1.6513 \end{bmatrix}$$

and by using the relations $K_P = P^{-1}\, R^T$, $K_I = P^{-1}\, S$, $A_z = V\, Q^{-1}$ we can obtain the optimal tunning parameters of our robust adaptive observers which are:

$$K_P = \begin{bmatrix} 9.4211 \\ 6.9415 \end{bmatrix}$$

$$K_I = \begin{bmatrix} 0.6392 \\ 0.0148 \end{bmatrix}$$

$$A_z = \begin{bmatrix} -1.6513 \end{bmatrix}$$

Finally, we chose the adaptive gain with an initial value $\Gamma(t=0) = 50$ and a forgetting factor $\lambda = 0.4$.

Fig. 1 shows the estimation state error $\tilde{x}(t)$ while Fig. 2 shows the real parameter and the estimated parameter. As long as we compare the results obtained by our observer with the results obtained in [9], we can conclude that both have a similar behavior since both guarantee a bounded state and parameter estimation error. Nevertheless, since our proposal is based on a normalized dead zone rather than in a projection operator, our robust adaptive observer needs neither any knowledge about the nominal value of the unknown parameter nor the bound of the unknown parameter. This is one of the main contributions of our proposal compared with other robust adaptive observer [9], [10].

It is worth nothing that the parameter estimation error is caused due to the fact that the system (27) has $u(t) = 0$. Our proposed robust adaptive observer can achieve an accurate parameter estimation for persistent excited systems as it shown in the following example.

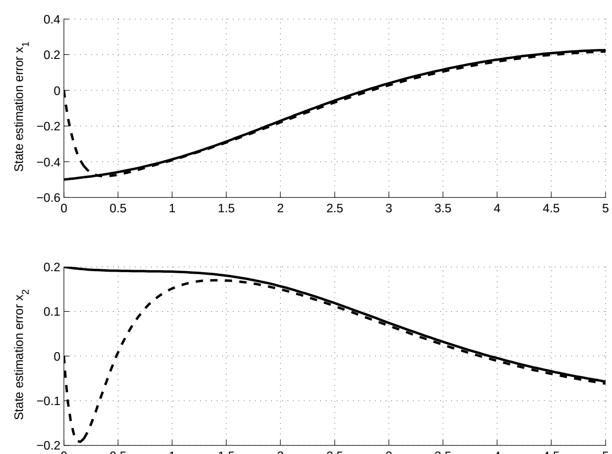

Fig. 1. Estimation results of the state variable. From top to bottom, state estimation error $\tilde{x}_1(t)$, $\tilde{x}_2(t)$.

B. High order system

Let us now consider the following uncertain nonlinear system

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 & 2 & 1 \\ -0 & -3 & 1 & 2 \\ 3 & 0 & -1 & 5 \\ -1 & 0 & -3 & -4 \end{bmatrix} x(t)$$

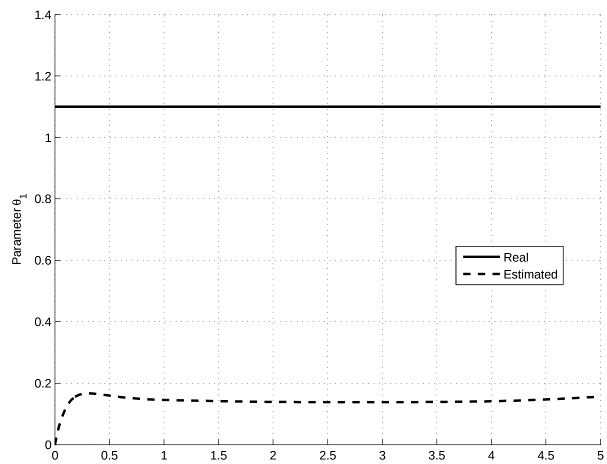

Fig. 2. Estimation results of the parameter. Solid lines is the real parameter. Dashed lines is the estimated parameter.

$$
\begin{aligned}
&+ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \end{bmatrix} u(t) + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} w(t) \\
&+ \begin{bmatrix} 0.8 & 0 \\ 0 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_3^3 & 0 \\ 0 & y_1^3 \end{bmatrix} \theta(t) \\
y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t) \qquad (28)
\end{aligned}
$$

with $x(t=0) = [1.4, 2, 3, -1]$, $u(t) = 5cos(5t) + 4$ and $w(t) = sin(50t)$. Moreover, to check the robustness of our proposal dealing with uncertain systems the unknown parameter θ will change during the simulation. At first, it is defined as $\theta = [0.5, -0.5]$ and it switches to $\theta = [1.5, -2.5]$ at $t = 10$ seconds.

Our proposed robust adaptive observer is designed solving the $\mathcal{L}_2$ minimization problem shown in (26). Specifically, the resulting optimal solutions are:

$$
P = \begin{bmatrix} 1.2500 & 0 & 0 & 0 \\ 0 & 0.0691 & 0 & 0 \\ 0 & 0 & 0.2500 & 0 \\ 0 & 0 & 0 & 0.1346 \end{bmatrix}
$$

$$
S = \begin{bmatrix} 1.9383 & 0.0386 \\ -0.0019 & -0.0058 \\ 0.0838 & 0.8995 \\ 0.0054 & 0.0146 \end{bmatrix}
$$

$$
R = \begin{bmatrix} 22.8158 & 1.2534 & 1.4454 & 1.1030 \\ 1.4454 & 0.0974 & 4.3292 & 0.7822 \end{bmatrix}
$$

$$
Q = \begin{bmatrix} 2.5896 & 0 \\ 0 & 2.9949 \end{bmatrix}
$$

$$
U = \begin{bmatrix} -4.8535 & 0 \\ 0 & -4.8878 \end{bmatrix}
$$

and by using the relations $K_P = P^{-1}\,R^T$, $K_I = P^{-1}\,S$, $A_z = V\,Q^{-1}$ we can obtain the optimal tunning parameters of our robust adaptive observers which are:

$$
K_P = \begin{bmatrix} 18.2526 & 1.1563 \\ 18.1382 & 1.4088 \\ 5.7816 & 17.3169 \\ 8.1923 & 5.8098 \end{bmatrix}
$$

$$
K_I = \begin{bmatrix} 1.5506 & 0.0309 \\ -0.0274 & -0.0845 \\ 0.3354 & 3.5978 \\ 0.0405 & 0.1084 \end{bmatrix}
$$

$$
A_z = \begin{bmatrix} -4.8535 & 0 \\ 0 & -4.8878 \end{bmatrix}
$$

Finally, we chose the adaptive gain with an initial value $\Gamma(t=0) = diag[0.5, 0.5]$ and a forgetting factor $\lambda = 0.4$.

Simulation results are shown in Fig. 3-4. Namely, Fig. 3 shows the estimation state error $\tilde{x}(t)$ while Fig. 4 shows the real parameter and the estimated parameter. It is evident that our proposed adaptive observer guarantees a robust behavior since state estimation error tends to zero independently of the uncertain parameter $\theta(t)$. Additionally, it can be seen that, for each parameter change, the convergence of the parameter estimation error is reestablished after a brief transient time.

V. Conclusion

Classical proportional adaptive observers might suffer large estimation errors in the presence of disturbances. Consequently, different techniques have been introduced to prevent this situation and to ensure a bounded parameter estimation error. All of these techniques modify the parameter adaptive law using an operator defined in function of the prior knowledge of the system. Therefore, information on the nominal parameter values are required to guarantee a bounded parameter estimation error.

To overcome this limitation, we have proposed a robust adaptive observer based on a new formulation of the parameter adaptive law. It contains a dead zone which guarantees a bounded estimate dealing with noise corrupted systems and an adaptive gain which increases the robustness against uncertain systems. Moreover, it also contains an additional integral term which increases the converge speed of the estimation error. Unlike other proposals, this new design does not need any initial knowledge of the nominal values of the system. Since our proposed adaptive observers involves several tuning matrices, we have given a method to compute the optimal parameters by minimizing the effect of disturbances on the estimation error. The effectiveness of the proposed robust adaptive observer has been verified through simulation examples, which confirm the good performance of our algorithm.

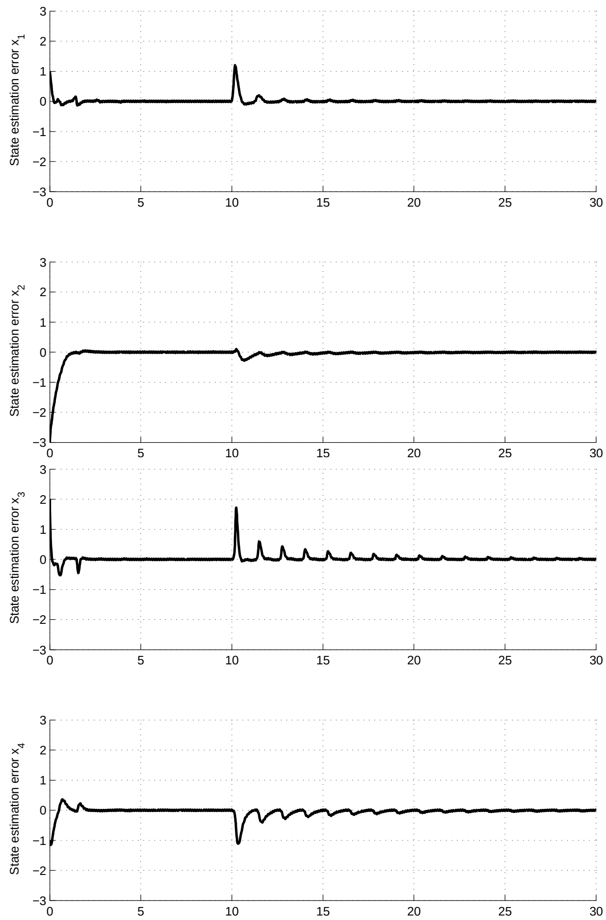

Fig. 3. Estimation results of the state variable. From top to bottom, state estimation error $\tilde{x}_1(t)$, $\tilde{x}_2(t)$, $\tilde{x}_3(t)$ and $\tilde{x}_4(t)$.

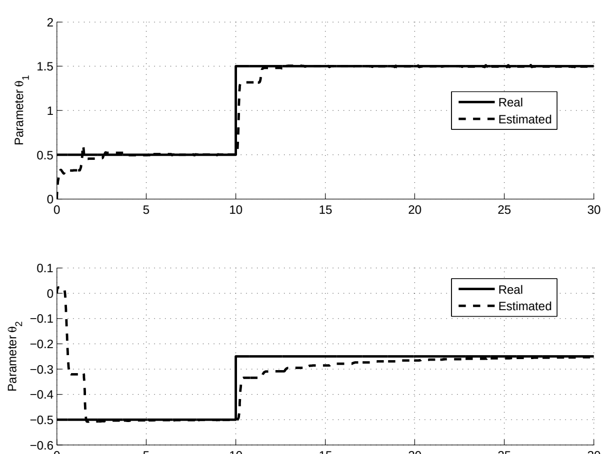

Fig. 4. Estimation results of the parameter. Solid lines are the real parameter. Dashed lines are the estimated parameter.

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