Citation: D. Paesa, C. Franco, S. Llorente, G. López-Nicolás and C. Sagüés. Reset observers applied to MIMO systems. Journal of Process Control, vol. 21, no. 4, pp. 613-619, 2011

Reset Observers Applied to MIMO Systems

D. Paesa[*,a], C. Franco[a], S. Llorente[b], G. Lopez-Nicolas[a], C. Sagues[a]

[a] *Department of Computer Science and Systems Engineering, Aragon, Institute of Engineering Research, University of Zaragoza, E-50018, Zaragoza Spain.*
[b] *Research and Development Department, Induction Technology, Product Division Cookers, BSH Home Appliances Group, E-50016, Zaragoza, Spain.*

Abstract

Reset observers (ReOs) are a novel sort of observer consisting of an integrator, and a reset law that resets the output of the integrator depending on a predefined switching condition. For SISO systems, the switching condition is defined in such a manner that the ReO is reset when the output estimation error and the reset term have different sign. However, the way to define the reset condition to deal with MIMO systems has not been analyzed previously. The contributions of this paper are a formulation to handle ReO for MIMO systems, and an algorithm to compute the $\mathcal{L}_2$ gain of the MIMO ReO for performance purposes. Additionally, the effectiveness of our proposed MIMO ReO is analyzed by simulations.

Key words: Reset Elements, State Observers, MIMO Systems, Hybrid Systems.

1. Introduction

State observers for linear time invariant systems (LTIs) have been widely studied since 1970s (see, for instance, [1], [2] and references therein). Those precursory works were characterized by having only a proportional feedback term in the estimation laws and were known as proportional observers (POs). After that, proportional integral observers (PIOs) were introduced to overcome the performance limitation of traditional proportional observers. Specifically, PIOs include an additional integral feedback loop, which increases robustness of the estimation process against disturbances, and modeling errors. They were initially introduced by Shafai and his co-workers for loop transfer recovery and robustness improvement in few publications (see [3], [4] and the references therein). The adaptive version of this observer for linear systems was reported in [5] and [6], and generalized for certain class of nonlinear systems in [7].

Although PIOs outperform traditional POs, they are still affected by the inherent limitations of linear feedback control. That is, they cannot decrease the overshooting and settling time of the estimation process simultaneously. To overcome this drawback,

[*]Corresponding author. Tel: +34 976578575; Fax: +34 976578584.
Email address: dpaesa@unizar.es (D. Paesa)

a novel sort of observer named reset observer (ReO), was proposed in [8]. ReOs are observers consisting of an integrator and a reset law that resets the output of the integrator depending on a predefined switching condition. Similar to traditional proportional observers and PIOs, ReO can be regarded as a recursive algorithm for state estimation in dynamic systems, and therefore, it can play a key role in many applications such as monitoring, maintenance and fault tolerant control [9], [10]. The main advantage of ReOs compared with traditional observers is that their estimation laws are no longer linear. Thus, the ReO overcome the inherent limitations of linear feedback laws. The introduction of the reset element, which is essentially nonlinear, in the estimation laws can improve the performance of the observer.

Reset elements for control purposes were firstly introduced by Clegg in 1958 [11], who proposed an integrator which was reset to zero when its input is zero. In 1974, Horowitz generalized that initial work substituting the Clegg integrator by a more general structure called the first order reset element (FORE) [12]. Nevertheless, the stability analysis of those early works were mainly based on simulations, and it would take two decades to find stability analysis demonstrations [13]. The main contribution of those works was a stability test applicable to reset control systems, called the H_β condition. However, the H_β condition is rather conservative, and it requires the plant dynamics to be stable [14]. Recently, reset elements based on the classical formulation have also been generalized to time-delay systems in [15], and [16], wherein the authors include the plant time delay in the stability analysis.

Additionally, some authors have been focused on improving the steady state performance of reset compensators. Since the state of the reset element is eventually reset, it does not have the characteristic of eliminating the steady state error in response to step disturbances by itself, thus, a steady state error is expected for all systems without an integrator. To overcome this drawback, [17] proposed reset elements with variable reset that improve the closed-loop performance of the system.

Other authors aim at reducing the conservatism of the H_β condition, and extending the stability analysis to unstable plants. A relevant survey on this line is [18]. There, the authors modified the reset condition in such a manner that, the system is no longer reset when the input of the FORE is zero, but when the input and the output of the FORE have different sign. Indeed, this novel reset criterion results in a much smaller flow region, and allows a significant relaxation of the stability and performance conditions. Nonetheless, this modification is more complicated to implement in real applications.

Another important issue to solve within the reset system framework, is how to deal with MIMO systems [19], since a proper formulation of the reset conditions in the MIMO case is far from trivial. In the SISO case there is only one reset condition whereas in the MIMO case there are many possible reset conditions, and it is not clear which choices are the best ones. For this reason, we present in this paper a ReO formulation to handle MIMO systems. According to the authors knowledge, this contribution can be regarded as the first result about how to cope with MIMO systems with multiple reset conditions within the reset system framework.

This paper is organized as follows. In Section 2, we present definitions in which our stability results are based. In Section 3, the ReO formulation for LTI MIMO systems is presented. Stability and convergence analysis of the ReO by using quadratic Lyapunov functions is given in Section 4. Besides, an algorithm to compute the $\mathcal{L}_2$ gain of the

MIMO ReO for performance purposes is also presented. A simulation example is shown in Section 5 in order to show the effectiveness of the proposed ReO. Finally, concluding remarks are given in Section 6.

Notation: In the following, we use the notation $(x, y) = \left[x^T \ y^T\right]^T$. Given a state variable x of a hybrid system with switches, we will denote its time derivative with respect to the time by $\dot{x}$. Furthermore, the value of the state variable after the switch will be denoted by x^+. Finally, we omit its time argument and we write $x(t)$ as x.

2. Preliminaries

For future reference, we present here the following definitions for asymptotic stability, and $\mathcal{L}_2$ gain stability of hybrid systems [20], [21], [22].

Definition 1. Consider the reset system $\mathcal{R}$ with state $x \in \mathbb{R}^n$, and the origin as an equilibrium point. Let two closed sets $\mathcal{F}_0$, and $\mathcal{J}_0$ be given, such that $\mathcal{F}_0 \cup \mathcal{J}_0 = \mathbb{R}^n$, wherein $\mathcal{F}_0$ is the flow set, and $\mathcal{J}_0$ is the jump set. If there exists a Lyapunov-function candidate $V(x) : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$ and

$$\begin{array}{ll} \dot{V}(x) < 0 & x \in \mathcal{F}_0 \\ V(x+) \leq V(x) & x \in \mathcal{J}_0 \end{array}$$

then the origin is asymptotically stable.

Definition 2. The set $\mathcal{L}_2$ consists of all measurable functions $f(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ such that $\int_0^\infty |f(t)|^2 \, dt < \infty$.

Definition 3. Consider the reset system $\mathcal{R}$ with input $d \in \mathcal{L}_2$, and output $y \in \mathcal{L}_2$. Let us define the $\mathcal{L}_2$ gain of the reset system $\mathcal{R}$ as the following quantity

$$\mathcal{L}_2 \, gain = \sup_{||d||_2 \neq 0} \frac{||y||_2}{||d||_2}$$

where $||y||_2$ denotes the 2-norm of y, defined as the square root of $||y||_2^2 = \int_0^\infty y^T y \, dt$, and $||d||_2$ is defined in a similar manner.

Additionally, we present the following lemma that will be used in the sequel [23].

Lemma 1. *The $\mathcal{L}_2$ gain of a LTI system with an input signal d, and an output signal y is less than γ, if there exists a quadratic function $V(x) = x^T P x$, $P > 0$ and $\gamma > 0$ such that*

$$\dot{V}(x) < \gamma^2 d^T d - y^T y$$

Definition 4. Consider the reset system $\mathcal{R}$ with state $x \in \mathbb{R}^n$, input $d \in \mathcal{L}_2$, output $y \in \mathcal{L}_2$, and the origin as an equilibrium point. Let two closed sets $\mathcal{F}_0$, and $\mathcal{J}_0$ be given, such that $\mathcal{F}_0 \cup \mathcal{J}_0 = \mathbb{R}^n$, wherein $\mathcal{F}_0$ is the flow set, and $\mathcal{J}_0$ is the jump set. If there exists a Lyapunov-function candidate $V(x) : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$ and

$$\begin{array}{ll} \dot{V}(x) < \gamma^2 d^T d - y^T y & x \in \mathcal{F}_0 \\ V(x+) \leq V(x) & x \in \mathcal{J}_0 \end{array}$$

then the origin is asymptotically stable and the reset system $\mathcal{R}$ has a $\mathcal{L}_2$ gain lower or equal to γ.

3. Problem Statement

In this paper, we address the problem of the state estimation of linear time invariant systems which are described by

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu + B_w w \\ y & = & Cx \\ y_L & = & C_L x \end{array} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^l$ is the input vector, $w \in \mathbb{R}$ is the disturbance vector, $y \in \mathbb{R}^m$ is the output vector. A, B, B_w, C, and C_L are constant matrices with appropriate dimensions. In addition, $y_L \in \mathbb{R}^{L_2}$ is a performance evaluation output which will be used to estimate the $\mathcal{L}_2$ gain of the MIMO ReO in a similar way than [24].

The structure of our proposed ReO applied to a LTI system (1) is given in Fig. 1. The ReO dynamics are described as follows:

$$\begin{array}{rcl} \dot{\hat{x}} & = & A\hat{x} + Bu + K_I\zeta + K_P\tilde{y} \\ \hat{y} & = & C\hat{x} \\ \hat{y}_L & = & C_L\hat{x} \end{array} \tag{2}$$

where $\hat{x}$ is the estimated state, K_I and K_P represent the integral and proportional gain respectively, $\tilde{y} = C\tilde{x} = C(x - \hat{x})$ is the output estimation error. We define $\tilde{y}_L = C_L\tilde{x}$ as the output estimation error for performance evaluation purposes and ζ is the reset integral term.

In the SISO case [8], ζ is computed as

$$\begin{array}{ll} \dot{\zeta} = A_\zeta\zeta + B_\zeta\tilde{y} & \tilde{y} \cdot \zeta \geq 0 \\ \zeta^+ = A_r\zeta & \tilde{y} \cdot \zeta \leq 0 \end{array} \tag{3}$$

where $A_\zeta \in \mathbb{R}$ and $B_\zeta \in \mathbb{R}$ are two tunning scalars which regulate the transient response of ζ, and A_r is the reset matrix. Specifically, we define $A_r = 0$, since the reset integral term ζ is reset to zero when $\tilde{y} \cdot \zeta \leq 0$. Notice that, A_ζ is not directly set to zero in order to fit the FORE formulation (see, for instance, [14] and [25]), and the PIAO formulation proposed in [7].

Moreover, it is worth mentioning that the definition of the reset law based on a sector condition for flowing ($\tilde{y} \cdot \zeta \geq 0$) and resetting ($\tilde{y} \cdot \zeta \leq 0$) is preferred throughout

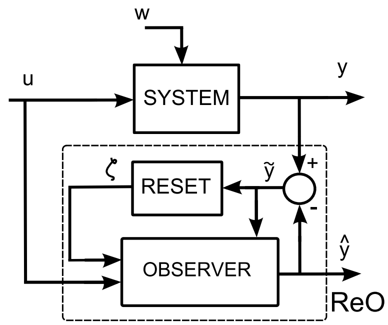

Figure 1: Reset observer applied to a LTI system.

this paper rather than classical condition ($\tilde{y} = 0$). Both definitions are equivalent for reset dynamics that correspond to first order reset elements like (3), and although the classical condition is clearly easier to implement, we have chosen the sector condition because it has advantages for stability and convergence analysis.

In the SISO case, the reset observer can be regarded as a hybrid system with one *flow set* $\mathcal{F}$ and with one *jump or reset set* $\mathcal{J}$. Regarding (3), the two conditions at the right side are the *flow* and the *jump* condition respectively. On one hand, as long as $(\tilde{y}, \zeta) \in \mathcal{F}$ the observer behaves as a proportional integral observer. On the other hand, if the pair $(\tilde{y}, \zeta)$ satisfies the *jump* condition, the integral term is reseted according to the reset map A_r. Thus, the observer flows whenever $\tilde{y} \cdot \zeta \geq 0$, that is, if $\tilde{y}$ and ζ have the same sign, whereas the observer jumps whenever $\tilde{y} \cdot \zeta \leq 0$, that is, if $\tilde{y}$ and ζ have different sign. Introducing the augmented state $\eta = [\tilde{x} \;\; \zeta]^T$ and by using $\tilde{y} = C\tilde{x}$ we can formalize the definition of both sets by using the following representation:

$$\mathcal{F} := \left\{\eta^T M \eta \geq 0\right\}, \; \mathcal{J} := \left\{\eta^T M \eta \leq 0\right\}, \tag{4}$$

where M is defined as

$$M = \begin{bmatrix} 0 & C^T \\ C & 0 \end{bmatrix}. \tag{5}$$

Nevertheless, this formulation is no longer valid when we are dealing with MIMO systems. In this case the reset observer can be regarded as a hybrid system with one *flow set* $\mathcal{F}$ and with several *jump or reset sets* $\mathcal{J}_i$ each one defined by a different reset condition. As before, a MIMO reset condition will depend on whether each output estimation error $\tilde{y}_k$ and its associated reset term ζ_k has or not different sign. However, since the reset observer has now several reset sets, the reset conditions have to be designed in such a manner that only one reset condition can be satisfied at the same time and, as a consequence, the ReO only can jump to one of the different jump sets. Indeed, the number of reset conditions can be obtained by computing the different combinations of reset terms. Undoubtedly, the number of reset conditions and, as a consequence, the number of reset regions depends on the number of outputs of the system. Using straightforward combinatorial mathematics, the number of different reset conditions β for a system with m outputs is

$$\beta = \sum_{i=1}^{m} \frac{m!}{i!(m-i)!}. \tag{6}$$

Additionally, let us define the following auxiliary sets for future reference. Let us assume without loss of generality that for each output $j = 1 \dots m$ there exists a matrix $M_j = M_j^T \in \mathbb{R}^{(n+m)\times(n+m)}$ which defines the following auxiliary sets:

$$\begin{aligned} \mathcal{S}_j &:= \eta^T\, M_j\, \eta \leq 0 \Rightarrow \tilde{y}_j \zeta_j \leq 0 \\ \mathcal{S}_j^{'} &:= \eta^T\, M_j\, \eta \geq 0 \Rightarrow \tilde{y}_j \zeta_j \geq 0 \end{aligned} \tag{7}$$

Using these auxiliary sets $\mathcal{S}_j$, $\mathcal{S}_j^{'}$ the different reset sets can be formalized. Let us define $\mathcal{A}$ as the set of all natural numbers from 1 to m. Additionally, for each different reset combination $i = 1 \dots \beta$ let us define $\mathcal{B}_i = \{k : \tilde{y}_k \zeta_k \leq 0, \mathcal{B}_k \subseteq \mathcal{A}\}$, which represents the indexes of the pairs $(\tilde{y}_k, \zeta_k)$ which have different sign, and $C_i = \{k : \tilde{y}_k \zeta_k \geq 0, C_k \subseteq \mathcal{A}\}$, which represents the indexes of the pairs $(\tilde{y}_k, \zeta_k)$ which have the same sign, in such a manner that $\mathcal{B}_i \cap C_i = \emptyset$ and $\mathcal{B}_i \cup C_i = \mathcal{A}, \forall i = 1 \dots \beta$. Therefore, the possible $i = 1 \dots \beta$ combinations of reset sets can be defined as:

$$\mathcal{J}_i := \bigcup_{j \in \mathcal{B}_i} \mathcal{S}_j \bigcup_{j \in C_i} \mathcal{S}_j^{'} \tag{8}$$

whereas the flow set would be defined as:

$$\mathcal{F} := \bigcup_{j \in \mathcal{A}} \mathcal{S}_j^{'} \tag{9}$$

Once the different reset conditions and reset sets have been properly defined, we can define the dynamic of the reset term within the flow region as well as after each different reset condition. Analogously to (3), the evolution of ζ for MIMO systems is defined as follows:

$$\begin{aligned} \dot{\zeta} &= A_\zeta \zeta + B_\zeta \tilde{y} && \eta \in \mathcal{F} \\ \zeta^+ &= A_{Ri} \zeta && \eta \in \mathcal{J}_i, \quad i = 1 \dots \beta \end{aligned} \tag{10}$$

where $A_{Ri} \in \mathbb{R}^{(n+m)\times(n+m)}$ are diagonal matrices defined as

$$A_{Ri} = \begin{bmatrix} I_n & 0 \\ 0 & A_{ri} \end{bmatrix}, \quad i = 1 \dots \beta \tag{11}$$

and $A_{ri} \in \mathbb{R}^{m\times m}$ are diagonal matrices whose diagonal terms $a_{r,r}^i$ are defined as follows

$$a_{r,r}^i = \left\{ \begin{array}{ll} 0 & \text{if } r \in \mathcal{B}_i \\ 1 & \text{if } r \notin \mathcal{B}_i \end{array} \right\} \quad i = 1 \dots \beta \tag{12}$$

Finally, we assume that the proposed reset observer for MIMO systems (2),(10) holds the following assumptions:

Table 1: $\beta = 7$ reset combinations for a systems with $m = 3$ outputs

i Combination	Strucure	Reset terms	Non reset terms
1	$\tilde{y}_1\zeta_1 \leq 0 \ \wedge \ \tilde{y}_2\zeta_2 \leq 0 \wedge \ \tilde{y}_3\zeta_3 \leq 0$	1^{st},2^{nd},3^{th}	None
2	$\tilde{y}_1\zeta_1 \leq 0 \ \wedge \ \tilde{y}_2\zeta_2 \leq 0 \wedge \ \tilde{y}_3\zeta_3 \geq 0$	1^{st},2^{nd}	3^{th}
3	$\tilde{y}_1\zeta_1 \leq 0 \ \wedge \ \tilde{y}_2\zeta_2 \geq 0 \wedge \ \tilde{y}_3\zeta_3 \leq 0$	1^{st},3^{th}	2^{nd}
4	$\tilde{y}_1\zeta_1 \geq 0 \ \wedge \ \tilde{y}_2\zeta_2 \leq 0 \wedge \ \tilde{y}_3\zeta_3 \leq 0$	2^{nd},3^{th}	1^{st}
5	$\tilde{y}_1\zeta_1 \leq 0 \ \wedge \ \tilde{y}_2\zeta_2 \geq 0 \wedge \ \tilde{y}_3\zeta_3 \geq 0$	1^{st}	2^{nd},3^{th}
6	$\tilde{y}_1\zeta_1 \geq 0 \ \wedge \ \tilde{y}_2\zeta_2 \leq 0 \wedge \ \tilde{y}_3\zeta_3 \geq 0$	2^{nd}	1^{st},3^{th}
7	$\tilde{y}_1\zeta_1 \geq 0 \ \wedge \ \tilde{y}_2\zeta_2 \geq 0 \wedge \ \tilde{y}_3\zeta_3 \leq 0$	3^{th}	1^{st},2^{nd}

Assumption 1. *The reset observer described by (2),(10) is such that* $\eta \in \mathcal{J}_i \ \Rightarrow \ A_{Ri}\eta \in \mathcal{F}$.

This condition guarantees that after each reset, the solution will be mapped to the flow set $\mathcal{F}$ and, as a consequence, it is possible to flow after resets.

Assumption 2. *The reset observer described by (2),(10) is such that the reset times* $t_{j+1} \ - t_j \geq \ \rho \ \forall j \in \mathbb{N}, \rho \in \mathbb{R} > 0$.

This assumption ensures that the reset observer uses time regularization to avoid Zeno solutions. It guarantees that the time interval between any two consecutive resets is not smaller than ρ which is a positive constant.

It is important to note that both assumptions are quite natural to assume for hybrid system, and consequently, these conditions are commonly used in most of current reset system formulations available in literature [14], [22].

To give insight into the proposed reset formulation for MIMO systems, we apply the previously explained design to the following example. Let us consider a system with $m = 3$ outputs $y = [y_1, y_2, y_3]$, their corresponding output estimation errors $\tilde{y} = [\tilde{y}_1, \tilde{y}_2, \tilde{y}_3]$ and their associated reset terms as $\zeta = [\zeta_1, \zeta_2, \zeta_3]$. According to (6), a ReO for a system with $m = 3$ outputs has $\beta = 7$ different combinations of reset conditions. These reset conditions are defined depending on the different combinations of how many reset terms can be reset at the same time. Specifically, these $\beta = 7$ reset combinations are outlined in Table 1.

Then by using (7), we could define $m = 3$ auxiliary sets $\mathcal{S}_1 := \eta\, M_1\, \eta \leq 0 \Rightarrow \tilde{y}_1\zeta_1 \leq 0$, $\mathcal{S}_2 := \eta\, M_2\, \eta \leq 0 \Rightarrow \tilde{y}_2\zeta_2 \leq 0$, $\mathcal{S}_3 := \eta\, M_3\, \eta \leq 0 \Rightarrow \tilde{y}_3\zeta_3 \leq 0$ and their corresponding complement sets $\mathcal{S}_1^{'}$, $\mathcal{S}_2^{'}$ and $\mathcal{S}_3^{'}$. Moreover, by using these auxiliary sets, we can state the different index sets and their associated reset regions which are defined according to (8), and additionally, the different A_{ri} reset matrices which are defined according to (12) as it shown in Table 2. Finally, since $m = 3$ then $\mathcal{A} = [1, 2, 3]$, and we can define the flow region as $\mathcal{F} := \mathcal{S}_1^{'} \cup \mathcal{S}_2^{'} \cup \mathcal{S}_3^{'}$ according to (9).

It is worth mentioning that the reset conditions (6), and the auxiliary sets (7), can lead to complex reset regions for large systems (e.g. $m \geq 10$). Therefore, regarding real-time implementation, it would be ill-advised to use this proposal for large systems, unless they were controlled with a powerful enough processor. Nevertheless, notice that the complexity is linear with the number of outputs m when considering all the

Table 2: β index sets $\mathcal{B}_i$, and C_i and their associated reset regions $\mathcal{J}_i$ for a systems with $m = 3$ outputs

i Combination	$\mathcal{B}_i$	C_i	$\mathcal{J}_i$	A_{ri}
1	$[1, 2, 3]$	$\emptyset$	$\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$	diag(0, 0, 0)
2	$[1, 2]$	$[3]$	$\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3^{'}$	diag(0, 0, 1)
3	$[1, 3]$	$[2]$	$\mathcal{S}_1 \cup \mathcal{S}_3 \cup \mathcal{S}_2^{'}$	diag(0, 1, 0)
4	$[2, 3]$	$[1]$	$\mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_1^{'}$	diag(1, 0, 0)
5	$[1]$	$[2, 3]$	$\mathcal{S}_1 \cup \mathcal{S}_2^{'} \cup \mathcal{S}_3^{'}$	diag(0, 1, 1)
6	$[2]$	$[1, 3]$	$\mathcal{S}_2 \cup \mathcal{S}_1^{'} \cup \mathcal{S}_3^{'}$	diag(1, 0, 1)
7	$[3]$	$[1, 2]$	$\mathcal{S}_3 \cup \mathcal{S}_1^{'} \cup \mathcal{S}_2^{'}$	diag(1, 1, 0)

reset conditions defined following (12), since the reset regions are explicitly defined just by checking the product of each output error $\tilde{y}_i$, and its corresponding reset term ζ_i.

4. Stability and Convergence Analysis

In this section we state computable sufficient conditions for $\mathcal{L}_2$ stability via quadratic Lyapunov functions for reset observers defined by (2) and (10) applied to LTI systems described by (1).

4.1. State stability analysis

Let us begin analyzing the augmented error system dynamics, which can be obtained subtracting (2) from (1) and by using the previously defined augmented state $\eta = [\tilde{x} \;\; \zeta]^T$,

$$\begin{aligned} \dot{\eta} \quad &= A_\eta\, \eta + B_\eta\, w \quad && \eta \in \mathcal{F} \\ \eta^+ \quad &= A_{Ri}\, \eta && \eta \in \mathcal{J}_i, \quad i = 1 \dots \beta \\ \xi \quad &= C_\eta\, \eta \\ \xi_L \quad &= C_{\eta_L}\, \eta \end{aligned} \tag{13}$$

where

$$A_\eta = \begin{bmatrix} A - K_P C & -K_I \\ B_\zeta C & A_\zeta \end{bmatrix},\; B_\eta = \begin{bmatrix} B_w \\ 0 \end{bmatrix},\; C_\eta = \begin{bmatrix} C \\ 0 \end{bmatrix},\; C_{\eta_L} = \begin{bmatrix} C_L \\ 0 \end{bmatrix}. \tag{14}$$

Let us now state a sufficient condition for the existence of a quadratically stable ReO based on a LMI approach.

Theorem 1. *For given A_η, B_η and A_{Ri} the augmented error dynamics shown in (13) with $B_w = 0$, m outputs, $i = 1 \dots \beta$ reset conditions computed according to (6), the collections of index sets $\mathcal{A}$, $\mathcal{B}_i$ and C_i, and $j = 1 \dots m$ matrices M_j defined according to (7), are quadratically stable if there exist a matrix $P = P^T > 0$ and scalars $\tau_{Fj} \geq$*

$0\ \forall j \in \mathcal{A}$, $\tau_{Jj_i} \geq 0\ \forall j \in \mathcal{B}_i$ and $\tau_{Jj_i} \geq 0\ \forall j \in C_i$ subject to

$$A_\eta^T P + PA_\eta + \sum_{j\in\mathcal{A}} \tau_{Fj} M_j \quad < \quad 0, \tag{15}$$

$$A_{Ri}^T PA_{Ri} - P - \sum_{j\in\mathcal{B}_i} \tau_{Jj_i} M_j + \sum_{j\in C_i} \tau_{Jj_i} M_j \quad \leq \quad 0,\ \ i = 1\ldots\beta, \tag{16}$$

which is a linear matrix inequality problem in the variables P, τ_{Fj} and τ_{Jj_i}.

Proof. Let us begin considering the following quadratic Lyapunov function for the augmented error dynamics described by (13):

$$V(\eta) = \eta^T\ P\ \eta \tag{17}$$

According to Definition 1, to prove the quadratically stability of our proposed reset observer, we have to check that:

$$\begin{array}{ll} \dot{V}(\eta) < 0 & \eta \in \mathcal{F} \\ V(\eta_i^+) \leq V(\eta) & \eta \in \mathcal{J}_i,\ \ i = 1\ldots\beta \end{array} \tag{18}$$

According to (4), since $\mathcal{F} := \bigcup_{j\in\mathcal{A}} \mathcal{S}_j' = \eta^T\ M_1\ \eta \geq 0 \wedge \ldots \wedge \eta^T\ M_m\ \eta \geq 0$ and employing the S-procedure [23], the first term of (18) is equivalent to the existence of $\tau_{Fj} \geq 0\ \forall j \in \mathcal{A}$ such that

$$\dot{V}(\eta) < -\sum_{j\in\mathcal{A}} \eta^T \tau_{Fj} M_j \eta \tag{19}$$

Then, let us take derivative of (17) to obtain

$$\begin{aligned} \dot{V}(\eta) &= \dot{\eta}^T P\eta + \eta^T P\dot{\eta} \\ &= \eta^T (A_\eta^T P + PA_\eta)\eta \end{aligned} \tag{20}$$

Rearranging terms of equations (19) and (20), the first term of (18) holds if the following inequality is satisfied

$$\eta^T (A_\eta^T P + PA_\eta)\eta + \sum_{j\in\mathcal{A}} \eta^T \tau_{Fj} M_j \eta < 0 \tag{21}$$

which can be rearranged as an equivalent LMI problem in the variables $P > 0$ and $\tau_{Fj} \geq 0$

$$A_\eta^T P + PA_\eta + \sum_{j\in\mathcal{A}} \tau_{Fj} M_j < 0, \tag{22}$$

which is analogous to (15) and consequently, proves the first equation of (18).

Similarly, employing again the S-procedure, the second term of (18) holds if for all $i = 1\ldots\beta$ there exist $\tau_{Jj_i} \geq 0\ \forall j \in \mathcal{B}_i$ and $\tau_{Jj_i} \geq 0\ \forall j \in C_i$ such that

$$V(\eta_i^+) \leq V(\eta) + \sum_{j\in\mathcal{B}_i} \eta^T \tau_{Jj_i} M_j \eta - \sum_{j\in C_i} \eta^T \tau_{Jj_i} M_j \eta,\ \ i = 1\ldots\beta \tag{23}$$

which is equivalent to

$$\eta^T A_{Ri}^T P A_{Ri}\eta - \eta^T P\eta - \sum_{j\in\mathcal{B}_i} \eta^T \tau_{Jj_i} M_j \eta + \sum_{j\in C_i} \eta^T \tau_{Jj_i} M_j \eta \le 0, \;\; i = 1\ldots\beta \tag{24}$$

Rearranging terms, (24) can be also rewritten as an equivalent LMI problem in the variables $P > 0$ and $\tau_{Jj_i} \ge 0$ as follows

$$A_{Ri}^T P A_{Ri} - P - \sum_{j\in\mathcal{B}_i} \tau_{Jj_i} M_j + \sum_{j\in C_i} \tau_{Jj_i} M_j \le 0, \;\; i = 1\ldots\beta \tag{25}$$

which is analogous to (16) and proves the second equation of (18) and, as a consequence, completes the proof of the theorem. □

4.2. Input-output stability analysis

Now, we present our results on the input-output properties of the ReO for MIMO systems. We use the previously defined performance evaluation output $\xi_l \in \mathbb{R}^{L_2}$ which will be used to estimate the $\mathcal{L}_2$ gain of the MIMO ReO in a similar way than [24].

Theorem 2. *For given A_η, B_η, C_η, C_{η_L} and A_{Ri} the augmented error dynamics shown in (13) with m outputs, $i = 1\ldots\beta$ reset conditions computed according to (6), the collections of index sets $\mathcal{A}$, $\mathcal{B}_i$ and C_i, and $j = 1\ldots m$ matrices M_j defined according to (7) are quadratically stable and has a $\mathcal{L}_2$ gain from w to ξ_L which is smaller than γ, if there exist a matrix $P = P^T > 0$ and scalars $\tau_{Fj} \ge 0\ \forall j \in \mathcal{A}$, $\tau_{Jj_i} \ge 0\ \forall j \in \mathcal{B}_i$ and $\tau_{Jj_i} \ge 0\ \forall j \in C_i$ and $\gamma > 0$ subject to*

$$\begin{aligned} \begin{bmatrix} A_\eta^T P + PA_\eta + C_{\eta_L}^T C_{\eta_L} + \sum_{j\in\mathcal{A}} \tau_{Fj} M_j & PB_\eta \\ B_\eta^T P & -\gamma^2 I \end{bmatrix} &< 0, \\ A_{Ri}^T P A_{Ri} - P - \sum_{j\in\mathcal{B}_i} \tau_{Jj_i} M_j + \sum_{j\in C_i} \tau_{Jj_i} M_j &\le 0, \;\; i = 1\ldots\beta \end{aligned} \tag{26}$$

which is a linear matrix inequality problem in the variables P, τ_{Fj}, τ_{Jj_i} and γ.

Proof. According to Definition 4, to prove the stability of our proposed reset observer and that the $\mathcal{L}_2$ gain from w to ξ_L is smaller than γ, we have to check that:

$$\begin{array}{ll} \dot{V}(\eta) < \gamma^2 w^T w - \xi_L^T \xi_L & \eta \in \mathcal{F} \\ V(\eta_i^+) \le V(\eta) & \eta \in \mathcal{J}_i, \;\; i = 1\ldots\beta \end{array} \tag{27}$$

The first equation of (27) relays on Lemma 1, and the second equation of (27) is equal to the second equation of (18) which has been already proved. Then, let us concentrate on the first equation of (27). Again, since $\mathcal{F} := \bigcup_{j\in\mathcal{A}} \mathcal{S}'_j = \eta^T M_1\, \eta \ge 0 \wedge \ldots \wedge \eta^T M_m \eta \ge 0$ and employing the S-procedure, the first term of (27) is equivalent to the existence of $\tau_{Fj} \ge 0\ \forall j \in \mathcal{A}$ such that

$$\dot{V}(\eta) < \gamma^2 w^T w - \xi_L^T \xi_L - \sum_{j\in\mathcal{A}} \eta^T \tau_{Fj} M_j \eta \tag{28}$$

In this case, the time derivative of (17) is

$$\begin{aligned}\dot{V}(\eta) &= \dot{\eta}^T P\eta + \eta^T P\dot{\eta} \\ &= \eta^T A_\eta^T P\eta + w^T B_\eta^T P\eta + \eta^T PA_\eta\eta + \eta^T PB_\eta w \\ &= \eta^T (A_\eta^T P + PA_\eta)\eta + w^T B_\eta^T P\eta + \eta^T PB_\eta w\end{aligned} \tag{29}$$

Rearranging terms of equations (28) and (29), the first term of (27) holds if the following inequality is satisfied

$$\begin{aligned}&\eta^T (A_\eta^T P + PA_\eta)\eta + w^T B_\eta^T P\eta + \eta^T PB_\eta w \\ &+\xi_L^T\xi_L + \sum_{j\in\mathcal{A}} \eta^T \tau_{Fj} M_j\eta - \gamma^2 w^T w < 0\end{aligned} \tag{30}$$

Since $\xi_L^T\xi_L = \eta^T C_{\eta_L}^T C_{\eta_L}\eta$, (30) can also be rearranged as an equivalent LMI problem in the variables $P > 0$ and $\tau_{Fj} \geq 0$ as follows

$$\begin{bmatrix} A_\eta^T P + PA_\eta + C_{\eta_L}^T C_{\eta_L} + \sum_{j\in\mathcal{A}} \tau_{Fj} M_j & PB_\eta \\ B_\eta^T P & -\gamma^2 I \end{bmatrix} < 0, \tag{31}$$

which is analogous to (26) and proves the first equation of (27) and, as a consequence, completes the proof of the theorem. □

5. Simulation Results

It was shown in [8] how a properly designed ReO could decrease the settling time as well as the overshooting compared with traditional observers dealing with SISO systems. Now, an example is presented in order to show that we can obtain the same improvement for MIMO systems by using the presented proposal. In order to highlight the effect of the reset elements on the estimation process, we compare the simulation results obtained by our proposed ReO with two PIOs. On the one hand, the first PIO will be tuned to minimize the overshooting and, as a consequence, it provides a smooth response. On the other hand, the latter PIO will be designed to minimize the rising time, and hence, it gives an oscillating and faster response. The next simulation example will show that our proposed ReO can achieve both requirements (i.e. a smooth and quick response) simultaneously.

Let us consider the following forth-order noise-corrupted LTI system:

$$\begin{aligned}\dot{x}_1 &= -1.5x_1 - 1x_2 + 0.5x_3 + u_1 + 0.5w \\ \dot{x}_2 &= 0.5x_1 - 2x_2 + 2u_1 - u_2 + 0.5w \\ \dot{x}_3 &= -0.5x_1 - 2x_3 + x_4 - 1u_1 + u_2 + 0.5w \\ \dot{x}_4 &= -1.5x_1 - x_4 + 2u_2 + 0.5w \\ y_1 &= x_1 \\ y_2 &= x_3 \\ y_L &= x_1\end{aligned} \tag{32}$$

with $x(t=0)=[-2.5,1.5,-1.5,-2]^T$, $u(t)=[sin(4t),t/(t+1)]^T$ and $w(t)=sin(15t)$. The aim is to develop an state observer for the system described by (32) which satisfies that the state estimation error tends to zero without overshooting as fast as possible. According to (1), (32) has the following parameters:

$$A=\begin{bmatrix} -1.5 & -1 & 0.5 & 0 \\ 0.5 & -2 & 0 & 0 \\ -0.5 & 0 & -2 & 1 \\ -1.5 & 0 & 0 & -1 \end{bmatrix},\ B=\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 1 \\ 0 & 2 \end{bmatrix},\ B_w=\begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix},$$

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ C_L=\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

The system (32) has $m=2$ outputs and by using (6) it has $\beta=3$ different reset conditions. Additionally, we can also define the $m=2$ auxiliary sets $\mathcal{S}_1 := \eta\ M_1\ \eta \leq 0 \Rightarrow \tilde{y}_1\zeta_1 \leq 0$ and $\mathcal{S}_2 := \eta\ M_2\ \eta \leq 0 \Rightarrow \tilde{y}_2\zeta_2 \leq 0$ and their corresponding complementary sets $\mathcal{S}_1^{'} \Rightarrow \tilde{y}_1\zeta_1 \geq 0$ and $\mathcal{S}_2^{'} \Rightarrow \tilde{y}_2\zeta_2 \geq 0$. According to (7), M_1 and M_2 are defined as follows:

$$M_1=\begin{bmatrix} 0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0 \end{bmatrix},\ M_2=\begin{bmatrix} 0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&1&0&0&0 \end{bmatrix}.$$

Moreover, the different index sets would be $\mathcal{A}=[1,2]$, $\mathcal{B}_1=[1,2]$ and $\mathcal{C}_1=\emptyset$, $\mathcal{B}_2=[1]$ and $\mathcal{C}_2=[2]$, $\mathcal{B}_3=[2]$ and $\mathcal{C}_3=[1]$. Using these auxiliary sets and the index sets, the $i=1\ldots\beta$ reset regions could be finally defined as $\mathcal{J}_1 := \mathcal{S}_1 \cup \mathcal{S}_2$, $\mathcal{J}_2 := \mathcal{S}_1 \cup \mathcal{S}_2^{'}$ and $\mathcal{J}_3 := \mathcal{S}_2 \cup \mathcal{S}_1^{'}$, whereas the flow set would be $\mathcal{F} := \mathcal{S}_1^{'} \cup \mathcal{S}_2^{'}$. Finally according to (12), we can define the $\beta=3$ different reset matrices A_{ri} as $A_{r1}=\text{diag}(0,0)$, $A_{r2}=\text{diag}(0,1)$ and $A_{r3}=\text{diag}(1,0)$.

The other tuning parameters have been obtained following the guidelines given [8]. Firstly, we have designed the conservative PIO in such a manner that its rising time is roughly equal to 0.6 seconds without overshooting. After that, to design the oscillating PIO we have increased the K_I gain until its rising time is around 0.2 seconds, that implies an oscillating estimation process. Finally, to make the results more comparable, the ReO has the same K_I and K_P than the oscillating PIO.

On the one hand, the parameters of the conservative PIO are $\hat{x}(t=0)=[0,0,0,0]^T$, $z(t=0)=[0,0]^T$, $A_z=\text{diag}(-0.1,-0.1)$, $B_z=[1,1]^T$, $K_P=[3.19,-1.31,-0.21,-0.82; 0.65,-0.20,4.49,7.36]^T$, and $K_I=[0.83,-0.11,-0.06,-0.25; 0.19,-0.06,0.45,0.17]^T$, whereas the parameters of the oscillating PIO are $\hat{x}(t=0)=[0,0,0,0]^T$, $z(t=0)=[0,0]^T$, $A_z=\text{diag}(-0.1,-0.1)$, $B_z=[1,1]^T$, $K_P=[3.19,-1.31,-0.21,-0.82; 0.65,-0.20,4.49,7.36]^T$, and $K_I=[25.55,-10.50,-1.67,-6.59; 5.21,-1.60,35.94,58.88]^T$.

On the other hand, the ReO for the system (32) has been designed according to (2)-(10) and it has the following tuning parameters: $\hat{x}(t=0)=[0,0,0,0]^T$, $\zeta(t=$

$0) = [0,0]^T$, $A_\zeta = \text{diag}(-0.1,-0.1)$, $B_\zeta = [1,1]^T$, $K_P = [3.19,-1.31,-0.21,-0.82; 0.65,-0.20,4.49,7.36]^T$, $K_I = [25.55,-10.50,-1.67,-6.59; 5.21,-1.60,35.94,58.88]^T$. Notice that the K_P and K_I gains of the ReO are equal to the gains of the oscillating PIO. For a further discussion on the structure of PIOs and PIAOs the reader is referred to [7] and [8].

The state estimation error $\tilde{x}(t) = [\tilde{x}_1(t), \tilde{x}_2(t), \tilde{x}_3(t), \tilde{x}_4(t)]^T$ of each observer is shown in Figs. 2-3. It is evident that our proposed ReO has a better performance compared with the traditional PIOs, since it has a response as quick as the oscillating PIO but without overshooting.

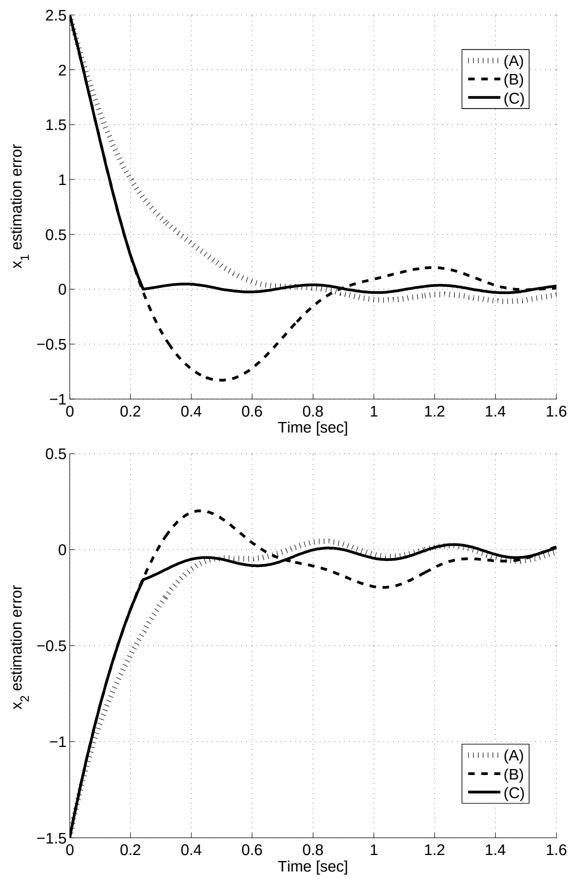

Figure 2: State estimation error $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ for each observer. (A) Dotted lines have been obtained by using the conservative PIO. (B) Dasehd lines have been obtained by using the oscillating PIO. (C) Solid lines have been obtained by using the ReO.

Additionally, the integral absolute error (IAE) and the integral time absolute error ($ITAE$) are computed to numerically evaluate the performance of our proposal, com-

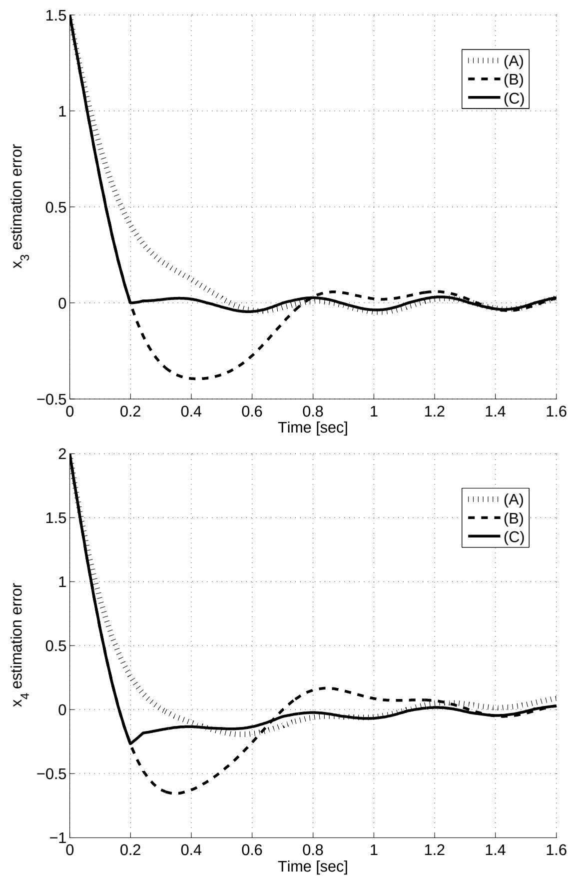

Figure 3: State estimation error $\tilde{x}_3(t)$ and $\tilde{x}_4(t)$ for each observer. (A) Dotted lines have been obtained by using the conservative PIO. (B) Dasehd lines have been obtained by using the oscillating PIO. (C) Solid lines have been obtained by using the ReO.

Table 3: Performance indexes obtained by each observer

Observer	IAE	$ITAE$
Conservative PIO	439.87	135.89
Oscillating PIO	560.69	242.05
ReO	272.56	63.44

pared with traditional state observers. They are defined as follows

$$\begin{aligned} IAE &= \sum_{i=1}^{n} \left(\int_0^{\infty} |\tilde{x}_i(t)| \, dt \right), \\ ITAE &= \sum_{i=1}^{n} \left(\int_0^{\infty} |t \cdot \tilde{x}_i(t)| \, dt \right), \end{aligned} \tag{33}$$

which should be as small as possible. Table 3 summarizes the performance indexes obtained by each observer. These results underline the potential benefit of using reset elements. From Figs. 2-3 and Table 3, we can conclude that ReOs can be used to improve the transitory response of some estimation processes. The rationale behind this is that ReOs remove almost completely the oscillating behavior of the estimated variables by resetting their corresponding integral term.

Finally, it can be interesting to estimate an upper bound of the $\mathcal{L}_2$ gain of ReO for performance purposes. Indeed, it can be done by solving the LMI problem showed in (2) which obtains $\mathcal{L}_2 = 0.22$ as an estimation of the upper bound of the $\mathcal{L}_2$ gain of the ReO.

6. Conclusion

So far, the application of reset elements for control and observation purposes has been limited to SISO systems. Since in the SISO case there is only one input and one output, the reset condition is automatically obtained. On the other hand, in the MIMO framework there are many possible reset conditions and as a consequence a proper formulation is far from trivial.

In this paper, we have extended our previous results on reset observers to the MIMO framework. According to the authors knowledge, this contribution can be regarded as the first result about how to cope with MIMO systems with multiple reset conditions within the reset system framework. Moreover, an algorithm to compute a estimated $\mathcal{L}_2$ gain of the MIMO ReO for performance proposal is also given. Additionally, a simulation example has been given to check the effectiveness of the proposed MIMO ReO, compared with traditional state observers.

However, the research on ReO is still an open problem. One topic for future research would be the application of the proposed methodology to nonlinear systems. As it shown in [26], the LTI system formulation can be modified through a change of coordinates in order to handle certain class of nonlinear systems. Another topic would be the development of a method to automatically determine the optimal ReO tuning parameters. Recent results in the adaptive observer framework [7], show that it could be done by solving the L2 gain minimization problem, which can be rewritten as an equivalent LMI problem.

References

[1] K. S. Narendra, A. M. Annaswamy, Stable Adaptive Systems, Dover Publications, 2005.

[2] P. A. Ioannou, J. Sun, Robust Adaptive Control, Prentice Hall, 1996.

[3] B. Shafai, R. L. Carroll, Design of proportional-integral observer for linear time-varying multivariable systems, in: Proc. of the IEEE Conference on Decision and Control, 1985, pp. 597–599.

[4] B. Shafai, S. Beale, H. H. Niemannv, J. Stoustrup, LTR design of discrete-time proportional integral observer, IEEE Trans. on Automatic Control 41 (7) (1996) 1056–1062.

[5] B. Shafai, C. Pi, S. Nork, S. Linder, Proportional integral adaptive observer for parameter and disturbance estimations, in: Proc. of the IEEE Conference of Decision and Control, 2002, pp. 4694–4699.

[6] E. Vahedforough, B. Shafai, Design of proportional integral adaptive observers, in: Proc. of the IEEE American Control Conference, 2008, pp. 3683–3688.

[7] J. Jung, J. Hwang, K. Huh, Optimal proportional-integral adaptive observer design for a class of uncertain nonlinear systems, in: Proc. of the IEEE American Control Conference, 2007, pp. 1931–1936.

[8] D. Paesa, C. Franco, S. Llorente, G. Lopez-Nicolas, C. Sagues, Reset adaptive observers and stability properties, in: Proc. of the 18th IEEE Mediterranean Control Conference, 2010, pp. 1435–1440.

[9] S. Studener, K. Habaieb, B. Lohmann, R. Wolf, Estimation of process parameters on a moving horizon for a class of distributed parameter systems, Journal of Process Control 20 (1) (2010) 58–62.

[10] M. Sourander, M. Vermasvuori, D. Sauter, T. Liikala, S.-L. Jamsa-Jounela, Fault tolerant control for a dearomatisation process, Journal of Process Control 19 (7) (2009) 1091–1102.

[11] J. Clegg, A nonlinear integrator for servomechanisms, Trans. of the A.I.E.E. 77 (Part-II) (1958) 41–42.

[12] I. Horowitz, P. Rosenbaum, Nonlinear design for cost of feedback reduction in systems with large parameter uncertainty, International Journal of Control 21 (6) (1975) 977–1001.

[13] O. Beker, C. Hollot, Y. Chait, Plant with integrator: An example of reset control overcoming limitations of linear feedback, IEEE Trans. on Automatic Control 46 (11) (2001) 1797–1799.

[14] O. Beker, C. Hollot, Y. Chait, H. Han, Fundamental properties of reset control systems, Automatica 40 (6) (2004) 905–915.

[15] A. Baños, A. Barreiro, Delay-independent stability of reset systems, IEEE Trans. on Automatic Control 52 (2) (2009) 341–346.

[16] A. Barreiro, A. Baños, Delay-dependent stability of reset systems, Automatica 46 (1) (2010) 216–221.

[17] A. Vidal, A. Baños, Stability of reset control systems with variable reset: application to PI+CI compensation, in: Proc. of the European Control Conference, 2009, pp. 4913–4918.

[18] D. Nesic, L. Zaccarian, A. R. Teel, Stability properties of reset systems, Automatica 44 (8) (2008) 2019–2026.

[19] Q. Zhang, Adaptive observer for multiple-input multiple-output (MIMO) linear time-varying systems, IEEE Trans. on Automatic Control 47 (3) (2002) 525–529.

[20] H. Lin, P. J. Antsaklis, Stability and stabilizability of switched linear systems: A survey of recent results, IEEE Trans. on Automatic Control 52 (2) (2009) 308–322.

[21] R. Goebel, R. G. Sanfelice, A. R. Teel, Hybrid dynamical systems, IEEE Control Systems Magazine 29 (2) (2009) 28–93.

[22] W. H. T. M. Aangenet, G. Witvoet, W. P. M. H. Heemels, M. J. G. van de Molengraft, M. Steinbuch, Performance analysis of reset control systems, International Journal of Robust and Nonlinear Control 20 (11) (2009) 1213–1233.

[23] S. Boyd, L. E. Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM - Society for Industrial Applied Mathematics, 1994.

[24] D. Peaucelle, A. Fradkov, Robust adaptive $\mathcal{L}_2$-gain control of polytopic MIMO LTI systems - LMI results, Systems and Control Letters 57 (11) (2008) 881–887.

[25] W. H. T. M. Aangenet, G. Witvoet, W. P. M. H. Heemels, M. J. G. van de Molengraft, M. Steinbuch, An LMI-based $\mathcal{L}_2$ gain performance analysis for reset control systems, in: Proc. of the IEEE American Control Conference, 2008, pp. 2248–2253.

[26] R. Marino, P. Tomei, Adaptive observers with arbitrary exponential rate of convergence for nonlinear systems, IEEE Trans. on Automatic Control 40 (7) (1995) 1300–1304.