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# Formation control synthesis in local frames under communication delays and switching topology: An LMI approach

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*Abstract*— **This paper presents a formation control synthesis method for a multiagent system to reach a prescribed rigid formation under communication delays and time-varying switching communication topology. The proposed control scheme only requires the knowledge of relative measurements of some neighbor agents, expressed in each agent's local frame, to be implemented. The presence of communication delays and switching topology are critical factors in the control design that could lead the system to slow convergence or even instability. To cope with this problem, we give sufficient conditions based on Linear Matrix Inequalities (LMI) and convex sum relaxation techniques which allow finding the control parameters that maximize the worst-case delay whilst keeping a minimum speed of convergence. Finally, simulation results are provided to show the effectiveness of the proposed approach.**

## I. Introduction

Formation control for groups of autonomous mobile agents has received much attention due to its high potential in a large variety of research areas, for example: unmanned aerial vehicle (UAV) formation [1], search and rescue missions [2], cooperative transport [3], etc. In this domain, the key problem is how to design a distributed control strategy for the multiagent system to achieve a geometrical formation shape [4]. Depending on how the target formation the agents must reach is defined, different formation control strategies have been investigated, including distance-based formation [5] and position-based formation, in terms of absolute [6] and relative positions [7], [8]. In particular, it is interesting to consider that only relative position measurements, expressed in each local agent's frame, are available. Under these premises, it is not necessary to share a global coordinate reference frame by all the agents. Therefore, the *coordinate-free* property offers important advantages in terms of flexibility and autonomy. For instance, they can operate in a GPS-denied environment by using the locally referred information coming from their independent onboard sensors. A coordinate-free position based control strategy was proposed in [9] with the advantage of minimizing a cost function based on the sum of distances. However, as pointed out in [10], the price to pay is that the relative position measurements from all other agents should be available to each individual agent, which implies that the coordinate-free formation control in [9] cannot be strictly considered as a distributed control strategy. Further works investigated a distributed implementation of coordinate-free vector-position based formation control [11], but limited to rigid topologies.

On the other hand, it is well known that the presence of time delays has a negative effect on the stability of the closed-loop control systems if they are not taken into account in the control design [12]. In the context of multiagent systems, the impact of time delays was extensively analyzed in related fields like the consensus problem with switching topologies (see [13]–[15] and references therein). More specifically, the effect of time delays on the stability of coordinate-free position-based formation control was first investigated in [9] and further extended to formation control synthesis under time-varying delays and sensor failures in [16], and nonholonomic systems in [17]. Nevertheless, switching communication topologies were not considered in these works. Therefore, to the best authors' knowledge, the design of coordinate-free formation control strategy aimed at maximizing the worst-case time delay whilst guaranteeing a minimum speed of convergence under switching topology has not been fully investigated, and deserves further research.

In this paper, we propose a coordinate-free formation control synthesis method, where all the agents find an agreed global reference frame by consensus using their available relative misalignment angles, whilst the formation control is executed using the relative position vectors. By applying Lyapunov-Krasovskii approaches and some convex sum properties, we obtain sufficient conditions based on LMI to ascertain the stability of the multiagent system. Moreover, the control parameters that maximize the worst-case delay whilst guaranteeing a minimum speed of convergence are obtained using numerical efficient algorithms implemented in commercially available software (SEDUMI [18], Matlab LMI Toolbox, etc.).

## II. Preliminaries and problem formulation

### A. Preliminaries

The following notations are used: Given matrices $A_1, \cdots, A_n$, the notation $diag(A_1, ..., A_n)$ stands for a block diagonal matrix. The symbol $\otimes$ stands for the Kronecker product. We denote the set of positive integers as $\mathcal{N} = \{1, 2, \cdots\}$. Given $m \times n$ scalars $t_{11}, t_{12}, ..., t_{mn}$, we define $[t_{ij}]_{m\times n}$ as the corresponding $m \times n$ matrix. Conversely, given a matrix $T = [t_{ij}]_{m\times n}$, we denote $col(T)$ as the column vector obtained by joining the column vectors $[t_{11}, \cdots, t_{1n}]$, $[t_{21}, \cdots, t_{2n}]$, ..., $[t_{m1}, \cdots, t_{mn}]$. For any

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integer $n > 1$ and $1 \leq p \leq n$, the symbol $\mathcal{I}_n^p$ defines a $n-1 \times n$ matrix which is built from the identity matrix $I_n$ by removing its $p_{th}$ row.

**Lemma 1.** *[17] Given two square matrices A and B, the following equivalence holds:*

$$\mathcal{P}^T (A \otimes B) \mathcal{P} = B \otimes A, \tag{1}$$

*where $\mathcal{P}$ is some regular permutation matrix.*

**Lemma 2.** *(Projection lemma) [19] Let $X = X^T, M, Y, N$ be matrices of appropriate dimensions. The following two conditions are equivalent:*

- *(i) $X + MYN + (MYN)^T < 0$*
- *(ii) $\left(M^{\perp}\right)^T XM^{\perp} < 0$ and $\left(N^{\perp}\right)^T XN^{\perp} < 0$,*

*where $M^{\perp}$ and $N^{\perp}$ are right orthogonal complements of $M$ and $N$ respectively.*

**Lemma 3.** *(Sum Relaxation Lemma) [20] Given arbitrary matrices $\Upsilon_{st}$, where $[s,t] \in [1,2,\ldots,p] \times [1,2,\ldots,p]$, and some arbitrary scalars $\lambda_1, ..., \lambda_p$ satisfying $0 \leq \lambda_s \leq 1$, $\forall s = 1, ..., p$, and $\sum_{s=1}^{p} \lambda_s = 1$, the inequality $\sum_{s=1}^{p}\sum_{t=1}^{p} \lambda_s \lambda_t \Upsilon_{st} < 0$ is fulfilled if the following conditions hold, $\forall s, t$:*

$$\Upsilon_{ss} < 0, \qquad \frac{2}{p-1}\Upsilon_{ss} + \Upsilon_{st} + \Upsilon_{ts} < 0, \quad s \neq t. \tag{2}$$

### B. Problem formulation

Consider a multiagent system formed by $N$ agents, where the motion model of each agent is:

$$\dot{q}_i = u_i, \tag{3}$$

where $q_i$ is the position vector of each agent, referred to any arbitrary reference frame, and $u_i$ is the control action.

The following assumptions on the multiagent system (3) are considered regarding the communication topology, the reference system available, and the nature of communication delays in order to define the particular problem addressed:

**Assumption 1.** *The communication topology is switching time-varying, where the time-dependent adjacency matrix $\mathcal{A}(t) \in (A_1, \cdots, A_p)$ can be directed or undirected, being $p$ the maximum number of possible communication topologies. Moreover, each communication topology $A_i$, $i = 1, ..., p$ is assumed to be weakly connected, and does not contains self-loops.*

**Assumption 2.** *The agents do not share a global reference frame.*

**Assumption 3.** *The communication links between two agents $i$ and $j$ are affected by time delays $\delta_{ji}$.*

From Assumption 1, we can write

$$\mathcal{A}(t) = \sum_{s=1}^{p} \lambda_s(t) A_s, \quad \lambda_s(t) = \begin{cases} 1, & \text{if } \mathcal{A}(t) = A_s \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Let $a_{ji}(t) = \sum_{s=1}^{p} a_{ji}^{(s)}$ the $j,i$ entry of matrix $\mathcal{A}(t)$, and $a_{ji}^{(s)}$ the $j,i$ entry of matrix $A_s$, $\forall s = 1, ..., p$. Let $d_{i,s} = \sum_{m=1}^{N} a_{mi}^{(s)}$ be the sum of incoming links in each agent $i$, and $D_s = diag\,(d_{1,s}, ..., d_{N,s})$.

Consistently with Assumption 1 and 2, each agent $i = 1, ..., N$ can obtain from agents $j$ satisfying $a_{ji}(t) = 1$ at instant $t$ the following relative measurements:

- the relative position vector $q_{ji} = q_j - q_i$, expressed in its local frame: $q_{ji}^L = R(\phi_i)\,(q_j - q_i)$, where $R(\phi_i)$ is the rotation matrix $R(\phi_i) = (cos(\phi_i)\ -sin(\phi_i); sin(\phi_i)\ cos(\phi_i))$, being $\phi_i$ the agent's orientation angle, also expressed in some arbitrary reference frame.
- the relative misalignment angle: $\phi_{ji} = \phi_j - \phi_i$.

Let $c_{ji}$ be the set of relative interagent position vectors, which describes the prescribed target formation. Note that, for any arbitrary $\alpha$, the formation control objective is achieved when the formation error $q_{ji} - R(\alpha)c_{ji} = 0$, $\forall i, j$.

### C. Coordinate-free control strategy

First, we consider that the control is time-triggered, that is to say, it is executed at each sampling instants $t_k = kT_s, k = 0, 1, 2, ...$, where $T_s$ is the sampling period. For simplicity, we define the notation $q_{ji,k} = q_{ji}(t_k), a_{ji,k} = a_{ji}(t_k)$, etc). The proposed control law renders:

$$u_{i,k} = K \sum_{i=1}^{N} a_{ji,k} \left( q_{ji,k-\tau_{ji}} - R(\hat{\phi}_{i,k}) c_{ji} \right), \tag{5}$$

where $\tau_{ji} = ceil(\delta_{ji}/T_s)$, being $\delta_{ji}$ the communication delay expressed in units of time, $\tau_{ji}$ the delay expressed in number of sampling periods $T_s$, and $ceil(.)$ a function which rounds a real number to the nearest integer towards infinity. The parameter $\hat{\phi}_{i,k}$ is the rotation angle consensus of the reference frame, obtained from the available measurements of the misalignment $\phi_{ji} = \phi_j - \phi_i$ (assumed to be time-constant) by the following Jacobi Over-Relaxation (JOR) consensus law:

$$\hat{\phi}_{i,k+1} = h\hat{\phi}_{i,k} + (1-h) d_{i,k}^{-1} \sum_{j=1}^{N} a_{ji,k} \left( \hat{\phi}_{j,k-\tau_{ji}} - \phi_{ji} \right), \tag{6}$$

where $d_{i,k} = \sum_{j=1}^{N} a_{ji,k}$ is the sum of incoming links at time instant $kT_s$ in each agent $i$. The parameter $0 < h < 1$ in (6) should be therefore designed so that $lim_{k\to\infty}\hat{\phi}_{i,k} = \phi_i + \alpha$, being $\alpha$ the agreed value for the rotation angle of the reference frame.

Our objective is therefore to design the parameters $h$ and $K$ of the coordinate-free control (5) and the consensus law (6) such that the multiagent system (3) converges to the prescribed target formation $c_{ji}$ with a minimum guaranteed speed of convergence and maximum worst-case delay $\bar{\tau} = max(\tau_{ij})$.

## III. Control synthesis

In the sequel, we say that a system is $\beta-$stable if it converges with some decay rate $0 < \beta < 1$, that is: $||x_k|| \leq ||x_0||\beta^{-k}$. This section gives sufficient conditions for the $\beta-$stability of the multiagent system (3) with the proposed control law (5). Define respectively the rotation angle consensus error $\tilde{\phi}_{i,k}$ and the formation error $\varepsilon_{ji,k}$ as:

$$\tilde{\phi}_{i,k} = \hat{\phi}_{i,k} - \phi_i, \qquad (7)$$
$$\varepsilon_{ji,k} = q_{ji,k} - R\left(\hat{\phi}_{i,k}\right) c_{ji}.$$

To address the control synthesis, the following two conditions must be satisfied to ensure the convergence of the formation control system:

- *Condition (i)*: All the consensus errors $\tilde{\phi}_{i,k}$ must converge with decay rate $\beta_1$ to the same value $\alpha$, that is to say: $\tilde{\phi}_{i,k} \to \alpha, \forall i = 1, ...N$.
- *Condition (ii)*: All the formation error vectors $\varepsilon_{ji,k}$ must converge to zero with decay rate $\beta_2$, $\forall [i,j] \in [1,...,N] \times [1,...,N]$, when $k \to \infty$.

Prior to describe the control design method, we introduce the following theorems (Theorem 1 and 2), which are sufficient conditions for (i) and (ii), respectively:

**Theorem 1.** *Given some scalar $h$, $\bar{\tau}$ and $\beta_1$, the above condition (i) is satisfied if there exist matrices $P, Q, Z > 0$ and $M$ such that the LMIs* (2) *hold $\forall [s,t] \in [1, \cdots, p] \times [1, \cdots, p]$, where:*

$$\Upsilon_{st} = \Xi_0 + J_{st}\Xi_1 + \left(J_{st}\Xi_1\right)^T, \qquad (8)$$

*and*

$$\Xi_0 = \begin{bmatrix} P + \bar{\tau}^2 Z & 0 & 0 \\ (*) & -\beta_1^2 P + Q - \beta_1^{2\bar{\tau}} Z & \beta_1^{2\bar{\tau}} Z \\ (*) & (*) & -\beta_1^{2\bar{\tau}} Q - \beta_1^{2\bar{\tau}} Z \end{bmatrix}, \qquad (9)$$

$$\Xi_1^T = \begin{bmatrix} M & 0 & 0 \end{bmatrix},$$
$$J_{st} = \begin{bmatrix} -E_{st} & F_{st} & G_{st} \end{bmatrix},$$
$$E_{st} = \left(w_s \cdot 1_N\right) D_t,$$
$$F_{st} = h\left(w_s \cdot 1_N\right) D_t - h \cdot 1_N \cdot \left(w_s \cdot D_t\right),$$
$$G_{st} = (1-h)\left(w_s \cdot 1_N\right) A_t - (1-h) \cdot 1_N \cdot \left(w_s \cdot D_t\right),$$

*where $D_t = diag\left(d_{1,t}, ..., d_{N,t}\right)$, $A_t$ are the adjacency matrices defined in Assumption 1 with $t = 1, ..., p$, and $w_s$ are vectors satisfying ($\forall s = 1, ..., p$):*

$$w_s^T \left(h I_N + (1-h) D_s^{-1} A_s\right) = w_s^T. \qquad (10)$$

*Proof.* From $\tilde{\phi}_{i,k} = \hat{\phi}_{i,k} - \phi_i$ and $\phi_{ji} = \phi_j - \phi_i$, the observer (6) can be written as:

$$\tilde{\phi}_{i,k+1} = h\tilde{\phi}_{i,k} + (1-h) d_{i,k}^{-1} \sum_{j=1}^{N} a_{ji,k}\ \tilde{\phi}_{i,k-\tau_{ji}}. \qquad (11)$$

Let $\tau_{ji} \equiv \bar{\tau}$. The above system can be written in compact form as:

$$\tilde{\Phi}_{k+1} = h\tilde{\Phi}_k + (1-h)\mathcal{D}_k^{-1}\mathcal{A}_k \tilde{\Phi}_{i,k-\bar{\tau}}\ , \qquad (12)$$

where

$$\tilde{\Phi}_k^T = \begin{bmatrix} \tilde{\phi}_{1,k} & \cdots & \tilde{\phi}_{N,k} \end{bmatrix}, \qquad (13)$$
$$\mathcal{D}_k = diag\left(d_{1,k}, \cdots, d_{N,k}\right).$$

If the graph $\mathcal{A}_k$ is connected $\forall k \geq 0$, we have that $lim_{k\to\infty}\tilde{\phi}_{i,k} = \alpha, \forall i = 1, ..., N$, where $\alpha$ is the weighted average consensus value.

Note that we are only interested in proving the convergence. Therefore, we will obtain an equivalent system in which all the states converge to 0 by removing the eigenvalue 1 on the augmented system matrix of system (12) with dimensions $N\left(\bar{\tau}+1\right)$:

$$\begin{bmatrix} hI_N & 0 & ... & 0 & (1-h)\mathcal{D}_k^{-1}\mathcal{A}_k \\ I_N & 0 & ... & 0 & 0 \\ 0 & I_N & ... & 0 & 0 \\ 0 & 0 & ... & I_N & 0 \end{bmatrix} \qquad (14)$$

by the new matrix with the same dimensions:

$$\begin{bmatrix} h\Gamma_{1,k} & 0 & ... & 0 & (1-h)\Gamma_{2,k} \\ I_N & 0 & ... & 0 & 0 \\ 0 & I_N & ... & 0 & 0 \\ 0 & 0 & ... & I_N & 0 \end{bmatrix}, \qquad (15)$$

where

$$\Gamma_{1,k} = I_N - 1_N \cdot w_k^T \left(w_k^T \cdot 1_N\right)^{-1}, \qquad (16)$$
$$\Gamma_{2,k} = \mathcal{D}_k^{-1}\mathcal{A}_k - 1_N \cdot w_k^T \left(w_k^T \cdot 1_N\right)^{-1}\ ,$$

and $w_k$ is the left eigenvector at instant $k$ associated to the eigenvalue 1 on the matrix $\Gamma_k :\ w_k^T \Gamma_k = w_k^T,\ \Gamma_k = h\Gamma_{1,k} + (1-h)\Gamma_{2,k}$ (Note that the eigenvalues of $\Gamma_k^* = \Gamma_k - 1_N \cdot w_k^T \left(w_k^T \cdot 1_N\right)^{-1}$ are those of $\Gamma_k$, except for the eigenvalue 1, which is 0 in $\Gamma_k^*$). Therefore, by defining $\tilde{\Phi}_k^* = \tilde{\Phi}_k - \alpha 1_N$, we can reformulate system (12) as:

$$\tilde{\Phi}_{k+1}^* = h\Gamma_{1,k}\tilde{\Phi}_k^* + (1-h)\Gamma_{2,k}\tilde{\Phi}_{k-\bar{\tau}}^*\ . \qquad (17)$$

To eliminate the time-dependent fractional terms $\mathcal{D}_k^{-1}$ and $\left(w_k^T \cdot 1_N\right)^{-1}$ in the above expression (17), we pre-multiply both sides by the term $\left(w_k^T \cdot 1_N\right)\mathcal{D}_k$, obtaining:

$$\mathcal{E}_k\tilde{\Phi}_{k+1}^* = \mathcal{F}_k\tilde{\Phi}_k^* + \mathcal{G}_k\tilde{\Phi}_{k-\bar{\tau}}^*\ , \qquad (18)$$

where

$$\mathcal{E}_k = \left(w_k^T \cdot 1_N\right)\mathcal{D}_k, \qquad (19)$$
$$\mathcal{F}_k = h\left(w_k^T \cdot 1_N\right)\mathcal{D}_k - h \cdot 1_N \cdot \left(w_k^T \cdot \mathcal{D}_k\right),$$
$$\mathcal{G}_k = (1-h)\left(w_k^T \cdot 1_N\right)\mathcal{A}_k - (1-h) \cdot 1_N \cdot \left(w_k^T \cdot \mathcal{D}_k\right)\ .$$

Now, consider the discrete-time Lyapunov-Krasovskii function:

$$V_k = \tilde{\Phi}_k^{*T}\left(\mathcal{E}_k^T P \mathcal{E}_k\right)\tilde{\Phi}_k^* + \sum_{m=k-\bar{\tau}}^{k-1} \beta_1^{2(k-m-1)}\bar{\eta}_m^T Q \bar{\eta}_m$$
$$+ \bar{\tau}\sum_{j=-\bar{\tau}}^{-1}\sum_{m=k+j}^{k-1} \beta_1^{2(k-m-1)}\bar{\eta}_m^T Z \bar{\eta}_m\ , \qquad (20)$$

where $\bar{\eta}_k = \tilde{\Phi}_{k+1}^* - \tilde{\Phi}_k^*$. The difference $\Delta_k^V = V_{k+1} - \beta_1^2 V_k$ yields:

$$\Delta_k^V = \tilde{\Phi}_{k+1}^{*T} P \tilde{\Phi}_{k+1}^* - \beta_1^2\tilde{\Phi}_k^{*T} P \tilde{\Phi}_k^* + \tilde{\Phi}_k^{*T} Q \tilde{\Phi}_k^* \qquad (21)$$
$$- \beta_1^2\tilde{\Phi}_{k-\bar{\tau}}^{*T} Q \tilde{\Phi}_{k-\bar{\tau}}^* + \bar{\tau}^2\bar{\eta}_k^T Z \bar{\eta}_k - \bar{\tau}\sum_{m=k-\bar{\tau}}^{k-1} \beta_1^{2m}\bar{\eta}_m^T Z \bar{\eta}_m\ .$$

Taking into account that $0 < \beta_1 \leq 1$ and applying Jensen's inequality, we have that:

$$- \bar{\tau} \sum_{m=k-\bar{\tau}}^{k-1} \beta_1^{2m} \bar{\eta}_m^T Z \bar{\eta}_m \leq -\bar{\tau} \beta_1^{2\bar{\tau}} \sum_{m=k-\bar{\tau}}^{k-1} \bar{\eta}_m^T Z \bar{\eta}_m \quad (22)$$

$$\leq -\beta_1^{2\bar{\tau}} \left( \sum_{m=k-\bar{\tau}}^{k-1} \bar{\eta}_m^T \right)^T Z \left( \sum_{m=k-\bar{\tau}}^{k-1} \bar{\eta}_m^T \right)$$

$$= -\beta_1^{2\bar{\tau}} \left( \tilde{\Phi}_k^* - \tilde{\Phi}_{k-\bar{\tau}}^* \right)^T Z \left( \tilde{\Phi}_k^* - \tilde{\Phi}_{k-\bar{\tau}}^* \right) .$$

Therefore, the exponential convergence of $\tilde{\Phi}_k^*$ with decay rate $\beta_1$ is fulfilled if $\Delta_k^V < 0$. Note that the inequality (22) can also be written in matricial form as:

$$\Delta_k^V \leq \nu_k^T \left( \Xi_0 \otimes I_2 \right) \nu_k < 0 \ , \quad (23)$$

where $\nu_k^T = \begin{bmatrix} \tilde{\Phi}_{k+1}^{*T} & \tilde{\Phi}_k^{*T} & \tilde{\Phi}_{k-\bar{\tau}}^{*T} \end{bmatrix}$. Taking into account from (18) that $\mathcal{J}_k \nu_k = 0$, where $\mathcal{J}_k = \left[ \mathcal{E}_k, \mathcal{F}_k, \mathcal{G}_k \right]$, we have by Lemma 2 that the inequality (23) is true if

$$\left( \Xi_0 + \mathcal{J}_k \Xi_1 + (\mathcal{J}_k \Xi_1)^T \right) \otimes I_2 < 0 \ . \quad (24)$$

Note that (24) can be written as:

$$\sum_{s=1}^{p} \sum_{t=1}^{p} \lambda_{s,k} \lambda_{t,k} \left( \left( \Xi_0 + J_{st} \Xi_1 + (J_{st} \Xi_1)^T \right) \otimes I_2 \right) < 0, \quad (25)$$

$$\lambda_{s,k} = \begin{cases} 1, & \text{if } \mathcal{A}_k = A_s \\ 0, & \text{otherwise} \end{cases}, \quad \lambda_{t,k} = \begin{cases} 1, & \text{if } \mathcal{A}_k = A_t \\ 0, & \text{otherwise.} \end{cases}$$

From the fact that the scalar functions $\sum_{s=1}^{p} \lambda_{s,k} = 1$ and $0 \leq \lambda_{s,k} \leq 1, \quad \forall s = 1, 2, ..., p$ and applying the convex sum relaxation lemma (Lemma 3), the inequality (25) is satisfied $\forall [s,t] \in [1,...,p] \times [1,...,p]$ if:

$$\left( \Xi_0 + J_{st} \Xi_1 + (J_{st} \Xi_1)^T \right) \otimes I_2 < 0 \ . \quad (26)$$

Applying Lemma 1, we have by congruence that the above inequalities are equivalent to:

$$\mathcal{P}^T \left( I_2 \otimes \left( \Xi_0 + J_{st} \Xi_1 + (J_{st} \Xi_1)^T \right) \right) \mathcal{P} < 0, \quad (27)$$

where $\mathcal{P}$ is a permutation matrix. Finally, the above inequality and (8) can easily be found to be equivalent, concluding the proof. □

**Theorem 2.** *Given some $K$ and $\bar{\tau}$, the multiagent system* (3) *with the proposed control law* (5) *expontentially converges with decay rate $\beta_2$ to the prescribed formation if LMIs* (8) *are satisfied, and there exists a scalar $\mu > 0$ such that the following LMIs hold, $\forall s = 1, ..., p$:*

$$\Omega_s < 0, \quad (28)$$

*where*

$$\Omega_s = \begin{bmatrix} -\left(\beta_2^2 + \mu \beta_2^{2\bar{\tau}}\right) I_{N-1} & \mu \beta_2^{2\bar{\tau}} & I_{N-1} & 0 \\ (*) & -\mu \beta_2^{2\bar{\tau}} & K \Pi_s^T & \mu \bar{\tau}^2 K \Pi_s^T \\ (*) & (*) & -I_{N-1} & 0_{N-1} \\ (*) & (*) & (*) & -\mu \bar{\tau}^2 I_{N-1} \end{bmatrix}, \quad (29)$$

*and*

$$\begin{aligned} &\Pi_s = \mathcal{T}^+ M_s \mathcal{T}, \\ &\mathcal{T}^+ = \begin{bmatrix} I_{N-1} & 0_{N-1 \times \bar{N}-N+1} \end{bmatrix}, \qquad \bar{N} = N(N-1), \\ &\mathcal{T} = (Q_1 - Q_2) \begin{bmatrix} 0_{N-1 \times 1} & I_{N-1} \end{bmatrix}^T, \qquad (30) \\ &M_s = T_s \left( (Q_1 - Q_2) \otimes 1_{1 \times N-1} \right) \cdot diag \left( Q_3 \ col(A_s) \right), \\ &Q_1^T = \left[ \left(\mathcal{I}_N^1\right)^T, \cdots, \left(\mathcal{I}_N^N\right)^T \right], \\ &Q_2 = I_N \otimes 1_{N-1 \times 1}, \\ &Q_3 = diag \left( \mathcal{I}_N^1, \cdots, \mathcal{I}_N^N \right). \end{aligned}$$

*Proof.* Let us write system (3) in discrete-time with sampling period $T_s$ as:

$$q_{i,k+1} = q_{i,k} + T_s u_{i,k} \ . \quad (31)$$

From $q_{ji} = q_j - q_i$, we have that

$$q_{ji,k+1} = q_{ji,k} + T_s \left( u_{j,k} - u_{i,k} \right). \quad (32)$$

On the other hand, the one-step ahead of the formation error $\varepsilon_{ji,k}$ defined in (7) yields:

$$\varepsilon_{ji,k+1} = q_{ji,k+1} - R\left( \hat{\phi}_{i,k+1} \right) c_{ji} \ . \quad (33)$$

Applying the Mean Value Theorem, we have that:

$$\begin{aligned} R\left(\tilde{\phi}_{i,k+1}\right) c_{ji} &= R\left(\hat{\phi}_{i,k}\right) c_{ji} \qquad (34) \\ &+ \left( \frac{d}{d\hat{\phi}_{i,k}} R(\xi_k) \right) c_{ji} \left( \hat{\phi}_{i,k+1} - \hat{\phi}_{i,k} \right), \end{aligned}$$

where $\xi_k$ is some unknown time-varying parameter satisfying $\hat{\phi}_{i,k} \leq \xi_k \leq \hat{\phi}_{i,k+1}$. From (33) and (34), we obtain:

$$\begin{aligned} \varepsilon_{ji,k+1} &= \varepsilon_{ji,k} + T_s \left( u_{j,k} - u_{i,k} \right) \qquad (35) \\ &- \left( \frac{d}{d\hat{\phi}_{i,k}} R(\xi_k) \right) c_{ji} \left( \hat{\phi}_{i,k+1} - \hat{\phi}_{i,k} \right). \end{aligned}$$

Replacing $u_{i,k}$ from (5) into the above expression and applying the definition of the formation error $\varepsilon_{ji,k}$, we obtain:

$$\begin{aligned} \varepsilon_{ji,k+1} &= \varepsilon_{ji,k} + KT_s \sum_{m=1}^{N} a_{mj,k} \ \varepsilon_{mj,k-\tau_{mj}} \\ &- KT_s \sum_{m=1}^{N} a_{mi,k} \ \varepsilon_{mi,k-\tau_{mi}} - KT_s \omega_{ji,k}, \qquad (36) \end{aligned}$$

where

$$\begin{aligned} \omega_{ji,k} &= \left( \frac{d}{d\hat{\phi}_{i,k}} R(\xi_k) \right) c_{ji} \left( \hat{\phi}_{i,k+1} - \hat{\phi}_{i,k} \right) \qquad (37) \\ &+ \sum_{m=1}^{N} a_{mj,k} \left( R(\hat{\phi}_{j,k}) - R(\hat{\phi}_{j,k-\tau_{mj}}) \right) c_{mj} \\ &- \sum_{m=1}^{N} a_{mi,k} \left( R(\hat{\phi}_{i,k}) - R(\hat{\phi}_{i,k-\tau_{mj}}) \right) c_{mi}. \end{aligned}$$

Assume that Condition (i) holds. Then, we have that $\hat{\phi}_{i,k+1} - \hat{\phi}_{i,k} \to 0$ when $k \to \infty$, leading to $lim_{k \to \infty} \omega_{ji,k} = 0$, where $\omega_{ji,k}$ is defined in (37). Therefore, the term $\omega_{ji,k}$ can be safely neglected in the stability analysis. Thus, to ascertain the convergence of (36) we can consider:

$$\varepsilon_{ji,k+1} = \varepsilon_{ji,k} + KT_s \sum_{m=1}^{N} a_{mj,k}\, \varepsilon_{mj,k-\tau_{mj}} - KT_s \sum_{m=1}^{N} a_{mi,k} \varepsilon_{mi,k-\tau_{mi}}. \quad (38)$$

The above system can be written in compact form as:

$$\bar{e}_{k+1} = \bar{e}_k + K\left(\mathcal{M}_k \otimes I_2\right) \bar{e}_{k-\bar{\tau}}, \quad (39)$$

where

$$\bar{e}_k^T = \begin{bmatrix}\varepsilon_{21}^T & \cdots & \varepsilon_{N1}^T, & e_{12}^T & \cdots & \varepsilon_{N2}^T & \cdots & \varepsilon_{N-1,N}^T\end{bmatrix}, \quad (40)$$

and

$$\mathcal{M}_k = T_s\left((Q_1 - Q_2) \otimes 1_{1\times N-1}\right) \cdot diag\left(Q_3\ col(A_k)\right) \quad (41)$$

On the other hand, let us write the formation error $\varepsilon_{ji,k}$ defined in (7) as

$$\varepsilon_{ji,k} = q_{j1,k} + q_{i1,k} + R(\alpha)\left(c_{j1,k} + c_{i1,k}\right) + \omega_{R,k}, \quad (42)$$

where $\omega_{R,k} = \left(R(\hat{\phi}_{i,k}) - R(\alpha)\right) c_{ji}$. It can be deduced from Condition (i) that the above term $\omega_{R,k}$ vanishes when $k \to \infty$. From this fact, we can safely assume that $\varepsilon_{ji,k} = \varepsilon_{j1,k} + \varepsilon_{i1,k}$. Let $\bar{\varepsilon}_k^T = \begin{bmatrix}\varepsilon_{21,k}^T & \cdots & \varepsilon_{N1,k}^T\end{bmatrix}$. Note the equivalences $\bar{\varepsilon}_k = \mathcal{T}^+ \bar{e}_k$ and $\bar{e}_k = \mathcal{T}\bar{\varepsilon}_k$. Taking into account that all the vectors $q_{ji,k}$ can be obtained as a linear combination of $q_{j1,k}, j = 2, ..., N$, from (39) we can obtain the reduced system:

$$\bar{\varepsilon}_{k+1} = \bar{\varepsilon}_k + K\left(\mathcal{A}_k \otimes I_2\right) \bar{\varepsilon}_{k-\bar{\tau}}, \quad (43)$$

where $\mathcal{A}_k = \mathcal{T}^+ \mathcal{M}_k \mathcal{T}$. From (43) and the Lyapunov-Krasovskii function candidate:

$$V_k = \bar{\varepsilon}_k^T \bar{\varepsilon}_k + \mu\bar{\tau} \sum_{j=-\bar{\tau}}^{-1} \sum_{m=k+j}^{k-1} \beta_2^{2(k-m-1)} \bar{\eta}_m^T \bar{\eta}_m, \quad (44)$$

where $\bar{\eta}_m = \bar{\varepsilon}_{m+1} - \bar{\varepsilon}_m$, the rest of the proof can be outlined following the baseline given in proof of Theorem 1.

□

**Remark 1.** *The control parameters $K$ and $h$ defined in* (6) *and* (5) *that maximize $\bar{\tau}$ can easily be found by dichotomic search algorithms using Theorem 1 and Theorem 2 (as illustrated later through an example in Fig. 2).*

## IV. Simulation results

In this section, simulation examples are provided to show the performance of the coordinate-free multiagent formation control system in presence of time delays and time-varying switching communication topologies.

Consider a multiagent system formed by $N = 12$ agents. We show the prescribed target formation (solid blue line) in Fig. 1. A sampling period $T_s = 0.05s$ is assumed. In Fig 1 we depict the three possible cases of communication topologies (dashed red line) corresponding to the adjacency matrices $A_s,\ s = 1, 2, 3$. The control gains $h = 0.9$ and $K = 0.15$ are designed using Theorem 1 and 2 together with Remark 1 to maximize the worst-case delay $\bar{\tau}$ whilst keeping a minimum decay rate $\beta_1 = \beta_2 = 0.99$, leading to a maximum worst-case delay $\bar{\tau} = 168$ and $\bar{\tau} = 9$, respectively. Therefore, the convergence is demonstrated for delays up to 9 sampling periods $T_s$, which is equivalent to $0.45s$.

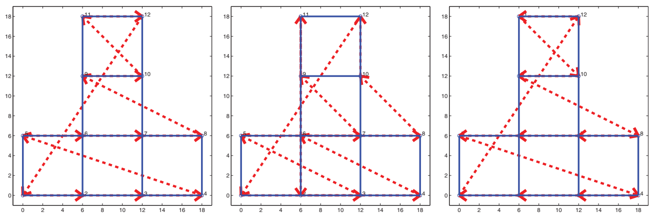

Fig. 1. Target formation (solid blue lines) and the different communication topologies (dashed red lines) $A_s,\ s = 1, 2, 3$.

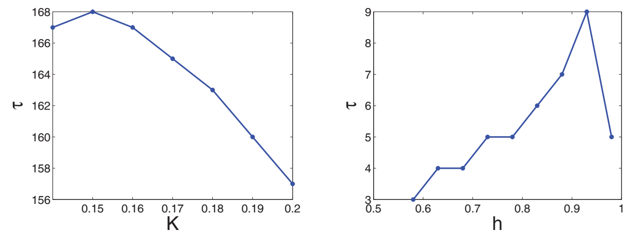


Fig. 2. Maximum worst-case delay $\bar{\tau}$ for different values of $K$ (left) and $h$ (right) with minimum decay rate $\beta_1 = \beta_2 = 0.99$.

The simulation results depicted in the middle-column of Fig. 3 and Fig. 4 correspond to the designed control parameters $K = 0.15$ and $h = 0.9$ (middle-column) assuming a worst-case delay $\bar{\tau} = 9$ and switching topology defined in Fig. 1. It can be appreciated that the multiagent system converges to the desired formation with the prescribed decay rate (see fourth row in Fig. 3). Note that smaller choices for $K$ lead to slower convergence (see left-column in Fig. 3), and greater choices for $K$ are close to the limit of instability (the agents' trajectories are visibly degraded, as shown in the upper-right corner in Fig. 3). Also, different values for $h$ (see left and right columns in Fig. 4) lead to slower convergence than $h = 0.9$. Consistently with Fig. 2, it can be deduced that the best choice for $K$ and $h$ are the designed values using the proposed control synthesis method. For a fair comparison, the same initial conditions for each agent's position and the same time-varying pattern for the switching topology $\mathcal{A}(t) \in [A_1, A_2, A_3]$ have been used.

## V. Conclusions and perspectives

This paper has presented a coordinate-free control synthesis algorithm for systems with communication delays and switching communication topology. Through numerical efficient algorithms based on LMI, the control gains can be designed to maximize the worst-case delay by keeping a decay rate performance. Appealing extensions of this work could be the implementation of event-triggered mechanisms

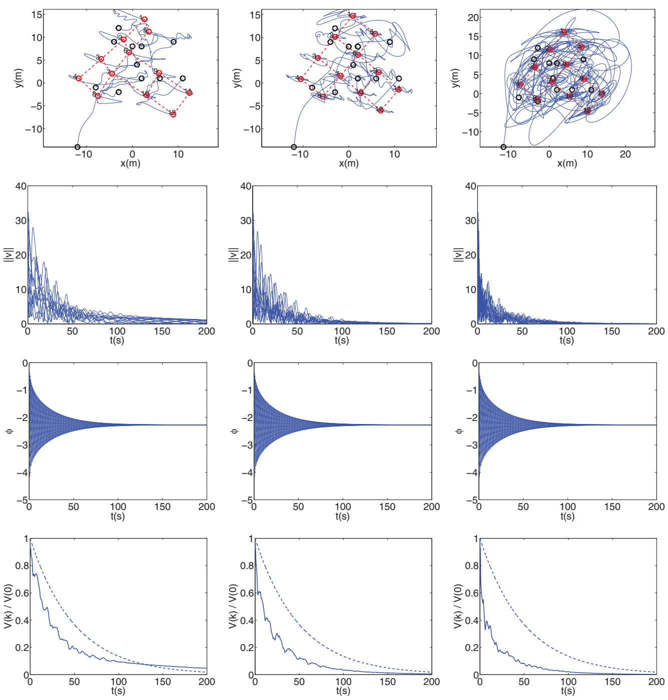


Fig. 3. Simulation results for different values of $K$ with $h = 0.9$: left-column ($K = 0.05$), middle-column ($K = 0.15$) and right-column ($K = 0.85$). First row: trajectories followed by each agent. Second row: Velocity norms of each agent $||v_i||$. Third row: Estimation of the angle $\hat{\phi}_i$, and Fourth row: Normalized cost function (solid-line) vs decay rate (dashed-line)

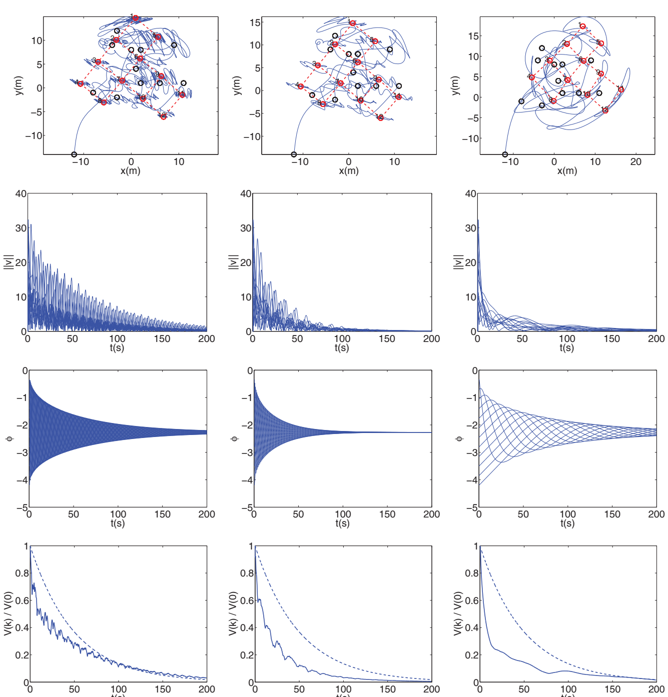


Fig. 4. Simulation results for different values of $h$ with $K = 0.15$: left-column ($h = 0.75$), middle-column ($h = 0.9$) and right-column ($h = 0.98$). First row: trajectories followed by each agent. Second row: Velocity norms of each agent $||v_i||$. Third row: Estimation of the angle $\hat{\phi}_i$, and Fourth row: Normalized cost function (solid-line) vs decay rate (dashed-line)

in the formation control strategy with the objective of reducing the bandwidth usage and energy consumption. Another extension is the analysis of the conditions that ensures collision avoidance and smooth trajectories of the agents with prescribed maximum curvatures.

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